CMR INSTITUTE OF TECHNOLOGY

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Internal Assesment Test - II								
Sub:	ELECTROMAGNETIC FIELD THEORY						Code:	18EE45
Date:	05/08/2022	Duration:	90 mins	Max Marks:	50	Sem: 4	Branch:	EEE
Answer FIVE FULL Questions. Mention units wherever necessary.								

		Marks	OBE CO	RBT
1 (a)	Define current and current density. Derive the expression for equation of continuity of current.	[08]	CO2	L2
1 (b)	Determine whether or not the given potential field satisfy the Laplace equation: $V = 2x^2 - 3y^2 + z^2$.	[02]	CO3	L3
2	Obtain the boundary conditions at the interface between a Conductor and a Dielectric.	[10]	CO2	L3
3 (a)	Derive the expression for capacitance of a parallel plate capacitor with multiple dielectrics.	[07]	CO2	L2
3 (b)	Determine the capacitance of a capacitor consisting of the parallel plates $30\ cm \times 30\ cm$ surface area separated by $5\ mm$ in air.	[03]	CO2	L3
4 (a)	Starting from the Gauss's law deduce Poisson's and Laplace's equations.	[05]	CO3	L3
4 (b)	Find V at P (2, 1, 3) for the field of two infinite radial conducting planes with V = 50V at $\phi = 10°$ and V = 30V at $\phi = 30°$.	[05]	CO3	L4
5	Derive the expression for capacitance of coaxial cable using Laplace's equation. Consider radius of inner conductor 'a' and outer conductor 'b'.	[10]	CO3	L3
6 (a)	State and prove the Uniqueness theorem.	[08]	CO3	L2
6 (b)	Given potential field $V = (A\rho^4 + B\rho^{-4}) volts$. Show that $\nabla^2 V$=0, where A and B are constants.	[02]	CO3	L3
7	State and explain Ampere's circuital law and Prove $\nabla \times \boldsymbol{H} = \boldsymbol{J}$. Also obtain the expression for Stokes' theorem.	[10]	CO3	L2
8 (a)	Discuss the scalar and vector magnetic potentials.	[07]	CO3	L2
8 (b)	At a point P(x,y,z) the components of vector magnetic potential are given as $A_x = 4x + 3y + 2z, A_y = 5x + 6y + 3z, A_z = 2x + 3y + 5z$. Determine B at point P.	[03]	CO3	L3

CI CCI HoD

1.a.

Current and Current Density

The current is defined as a rate of movement of charge passing a given reference point (or crossing a given reference plane) of one coulomb per second. Current is symbolized by I.

$$\boxed{I = \frac{dQ}{dt}} \quad A$$

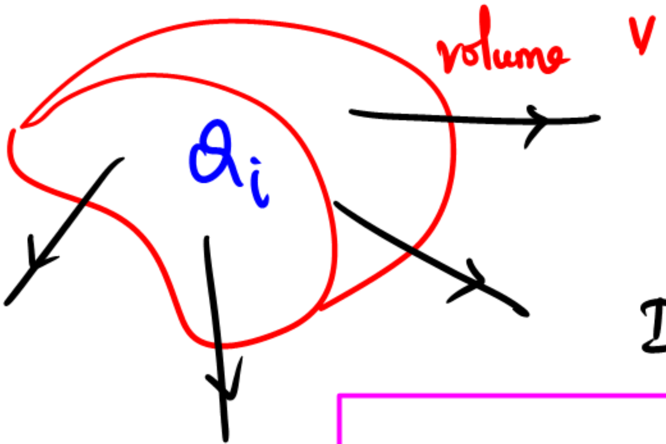
The current density, measured in amperes per square meter, is a vector flux density represented by J. It is defined as the current per unit cross sectional area.

$$\boxed{|\vec{J}| = \frac{dI}{ds}} \quad A/m^2$$

$$dI = \vec{J} \cdot \vec{ds}$$

$$\boxed{I = \iint_s \vec{J} \cdot \vec{ds}}$$

Equation of Continuity of Current



$$Q_i = \iiint_V \rho_v \, dv$$

$$I = \oint\!\!\oint_s \vec{J} \cdot \vec{ds}$$

$$\boxed{I = -\frac{dQ_i}{dt}}$$

$$I = -\frac{d}{dt}\left[\iiint_V \rho_v \, dv\right]$$

$$\oint_S \vec{J} \cdot \vec{ds} = -\iiint_V \frac{\partial \rho_v}{\partial t} \, dv$$

Divergence theorem:

$$\oint_S \vec{J} \cdot \vec{ds} = \iiint_V \vec{\nabla} \cdot \vec{J} \, dv$$

$$\iiint_V \vec{\nabla} \cdot \vec{J} \, dv = -\iiint_V \frac{\partial \rho_v}{\partial t} \, dv$$

$$\boxed{\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t}}$$

Continuity equation of current

1. b.

1.b) $V = 2x^2 - 3y^2 + z^2$

Laplace equation: $\nabla^2 V = 0$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$\frac{\partial^2}{\partial x^2}(2x^2 - 3y^2 + z^2) + \frac{\partial^2}{\partial y^2}(2x^2 - 3y^2 + z^2) + \frac{\partial^2}{\partial z^2}(2x^2 - 3y^2 + z^2) = 0$$

$$\frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-6y) + \frac{\partial}{\partial z}(2z) = 0$$

$$4 - 6 + 2 = 0$$

$$\boxed{\nabla^2 V = 0 = 0}$$

Given V satisfies Laplace's equation

2. a.

BOUNDARY CONDITIONS:

If the field exists in a region consisting of two different media, the conditions that the field must satisfy at the interface separating the media are called boundary conditions.

These conditions are helpful in determining the field on one side of the boundary if the field on the other side is known.

The boundary conditions at an interface separating
- **Dielectric εr1 and dielectric εr2**
- **Conductor and dielectric**
- **Conductor and free space**

To determine the boundary conditions, we need to use Maxwell's equations:

Work done around a closed path is Zero,

$$W = -Q\oint_L \vec{E}\cdot\vec{dl} = 0$$

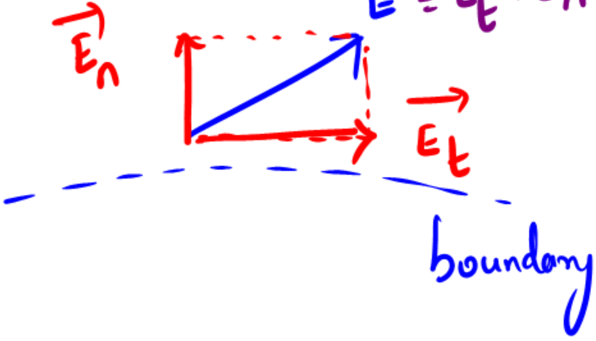
$$\boxed{\oint_L \mathbf{E}\cdot d\mathbf{l} = 0} \longrightarrow (1)$$

Gauss's Law,

$$\boxed{\oint_S \mathbf{D}\cdot d\mathbf{S} = Q_{\text{enc}}} \longrightarrow (2)$$

Also we need to decompose the electric field intensity E into two orthogonal components:

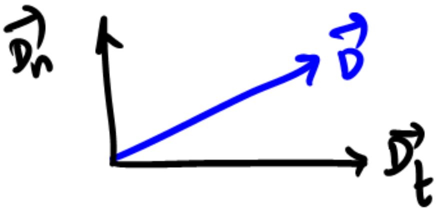
$$\vec{E} = \vec{E}_t + \vec{E}_n$$



$$\boxed{\vec{\mathbf{E}} = \vec{\mathbf{E}}_t + \vec{\mathbf{E}}_n}$$

where Et and En are respectively the tangential and normal components of E to the interface (boundary)

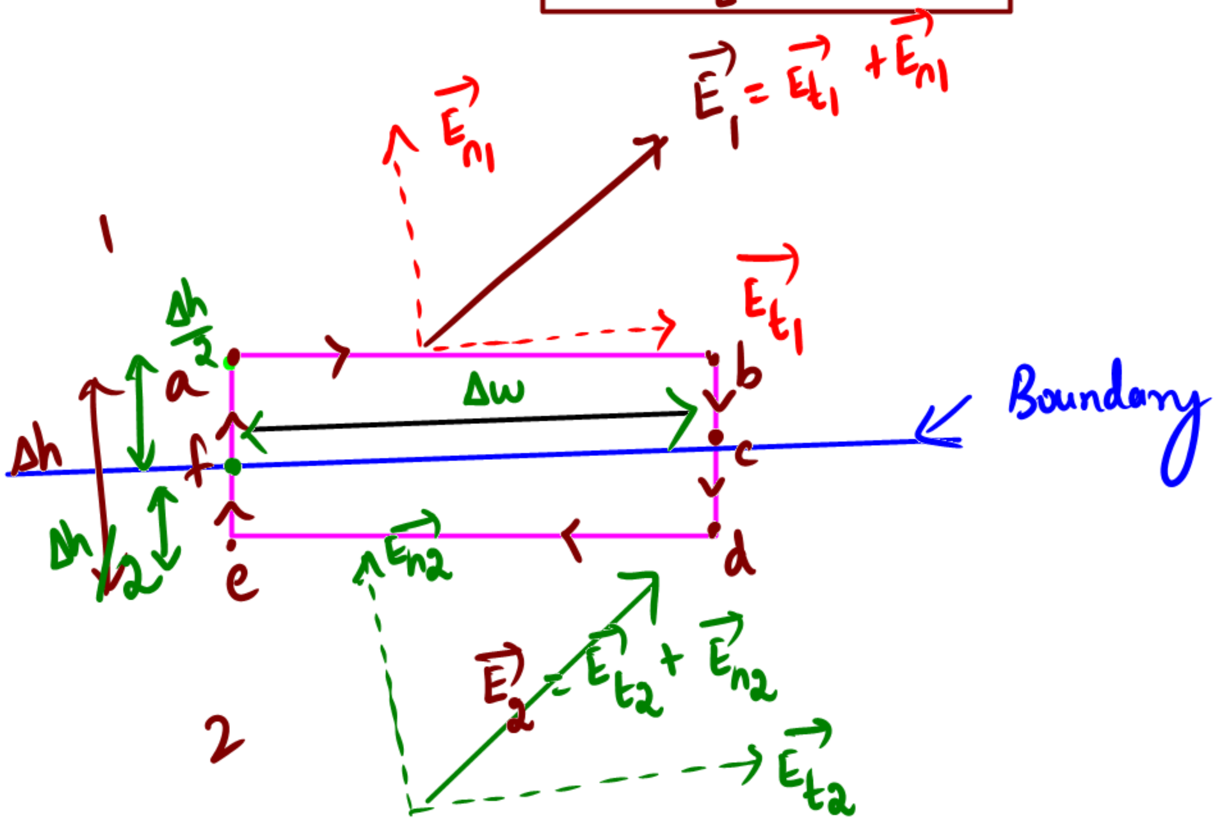
A similar decomposition can be done for the electric flux density D.



$$\vec{D} = \vec{D_t} + \vec{D_n}$$

Boundary Conditions

$$\boxed{① \oint_L \vec{E}\cdot\vec{dl} = 0}$$



$$\oint_L \vec{E}\cdot\vec{dl} = 0$$

$$\int_a^b + \int_b^c + \int_c^d + \int_d^e + \int_e^f + \int_f^a \vec{E}\cdot\vec{dl} = 0$$

$$E_{t_1}\cdot\Delta w - \underbrace{E_{n_1}\frac{\Delta h}{2}}_{\to 0} - \cancel{E_{n_2}\frac{\Delta h}{2}}_{\to 0} - E_{t_2}\Delta w$$

$$+ \cancel{E_{n_2}\frac{\Delta h}{2}}_{\to 0} + \cancel{E_{n_1}\frac{\Delta h}{2}}_{\to 0} = 0$$

To obtain conditions at the boundary,

$$\Delta h \to 0$$

$$E_{t_1} \Delta w - E_{t_2} \Delta w = 0$$

$$\boxed{E_{t_1} = E_{t_2}}$$

$$\boxed{\vec{E}_{t_1} = \vec{E}_{t_2}} \rightarrow (i)$$

$$\boxed{\frac{\vec{D}_{t_1}}{\varepsilon_0 \varepsilon_{r_1}} = \frac{\vec{D}_{t_2}}{\varepsilon_0 \varepsilon_{r_2}}} \rightarrow (ii)$$

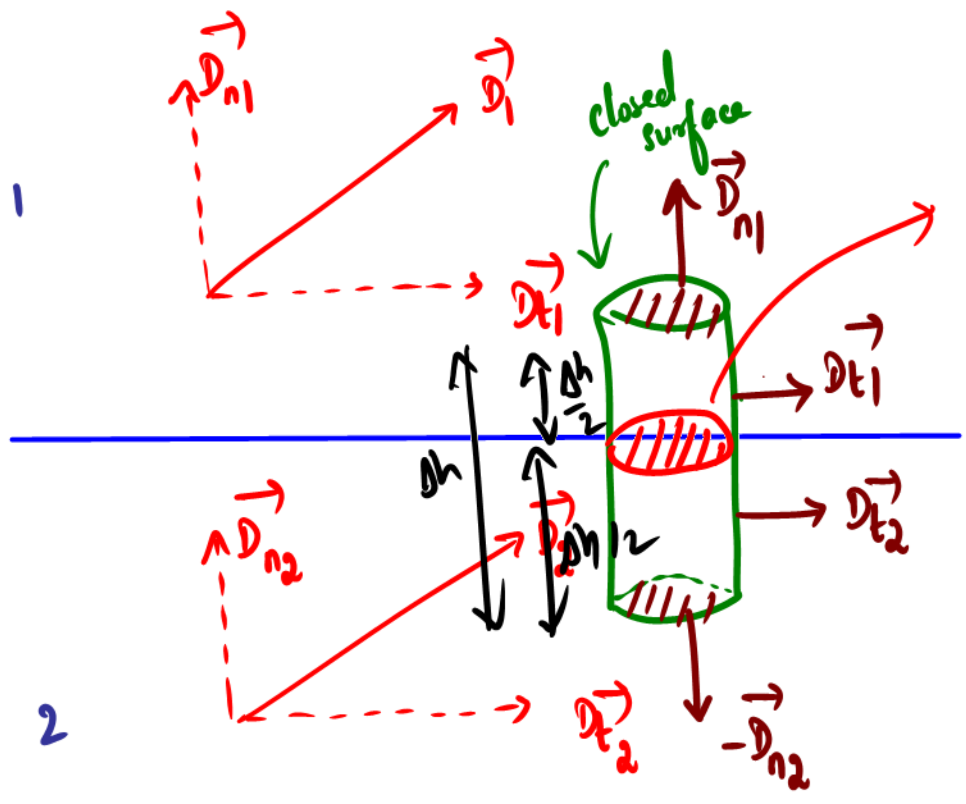
w.k.t

$$\vec{D}_1 = \varepsilon_0 \varepsilon_{r_1} \vec{E}_1$$

$$\vec{D}_2 = \varepsilon_0 \varepsilon_{r_2} \vec{E}_2$$

2

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q_{\text{enc}}$$



$$\Delta S = \pi r^2$$

$$Q_{enc} = \rho_s \cdot \Delta S$$

Gauss's law

$$\oint_S \vec{D}\cdot\vec{ds} = Q_{enc}$$

$$\iint_{top_1} + \iint_{bottom_2} + \iint_{side_1} + \iint_{side_2} \vec{D}\cdot\vec{ds} = \rho_s \cdot \pi r^2$$

$$D_{n_1}\pi r^2 - D_{n_2}\pi r^2 + D_{t_1}\,2\pi r\,\cancel{\frac{\Delta h}{2}}^{\,0} + D_{t_2}\cdot 2\pi r\,\cancel{\frac{\Delta h}{2}}^{\,0} = \rho_s \cdot \pi r^2$$

To obtain conditions at the boundary

$\Delta h \to 0$

$$D_{n_1}\pi r^2 - D_{n_2}\pi r^2 = \rho_s \cdot \pi r^2$$

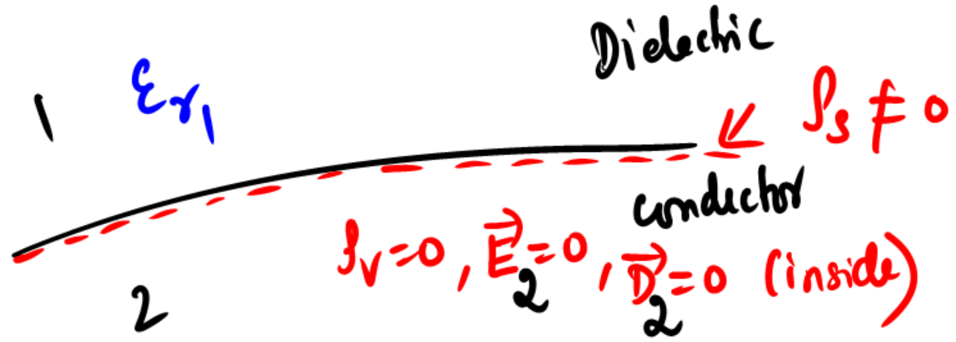
$$\boxed{D_{n_1} - D_{n_2} = \rho_s}$$

$$\boxed{(\vec{D_{n1}} - \vec{D_{n2}})\cdot\vec{a_n} = \rho_s} \rightarrow \text{(iii)}$$

$$\boxed{(\varepsilon_0\varepsilon_{r1}\vec{E_{n1}} - \varepsilon_0\varepsilon_{r2}\vec{E_{n2}})\cdot\vec{a_n} = \rho_s} \rightarrow \text{(iv)}$$

Conductor Dielectric Boundary:

The Surface of the conductor has charges.
Also charge is zero inside the conductor and therefore,
electric field inside the conductor is zero.



$$\vec{E}_{t1} = 0 \implies \boxed{\vec{E}_{t1} = 0}$$

$$\frac{\vec{D}_{t1}}{\varepsilon_0 \varepsilon_{r1}} = 0 \implies \boxed{\vec{D}_{t1} = 0}$$

$$(\vec{D}_{n1} - 0) \cdot \vec{a}_n = \rho_s \implies \boxed{\vec{D}_{n1} \cdot \vec{a}_n = \rho_s}$$

$$(\varepsilon_0 \varepsilon_{r1} \vec{E}_{n1} - 0) \cdot \vec{a}_n = \rho_s$$

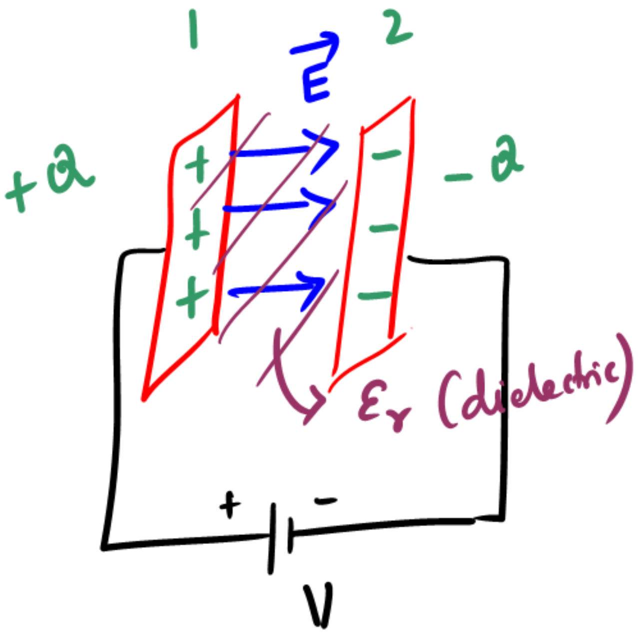
$$\hookrightarrow \boxed{\vec{E}_{n1} \cdot \vec{a}_n = \frac{\rho_s}{\varepsilon_0 \varepsilon_{r1}}}$$

3.a.

Capacitance:

The capacitance of a two conductor system is defined as the ratio of the magnitude of the total charge on either conductor to the magnitude of the potential difference between conductors.

$$C = \frac{Q}{V_d} \quad (F)$$



From the boundary conditions for the conductor - dielectric interface,

$$\vec{E}_{t1} = 0$$

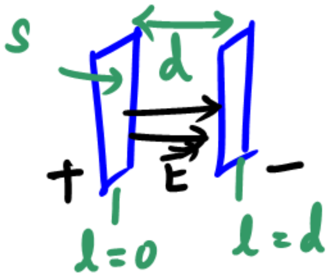
$$\vec{E}_{N1} = \frac{\rho_s}{\varepsilon_0 \varepsilon_r}$$

$$\boxed{\vec{E}_1 = \frac{\rho_s}{\varepsilon_0 \varepsilon_{r1}}}$$

$$Q = \oiint_S \vec{D} \cdot \vec{ds} = \oiint_S \varepsilon_0 \varepsilon_r \vec{E} \cdot \vec{ds}$$

$$Q = \varepsilon_0 \varepsilon_r \frac{\rho_s}{\varepsilon_0 \varepsilon_r} \underbrace{\left[\iint_S ds \right]}_{\text{'S' area}}$$

$$\boxed{Q = \rho_s \cdot S}$$



$$V = -\int_L \vec{E}\cdot\vec{dl}$$

$$V = -\frac{\rho_s}{\varepsilon_0\varepsilon_r}\int_{l=d}^{l=0} dl$$

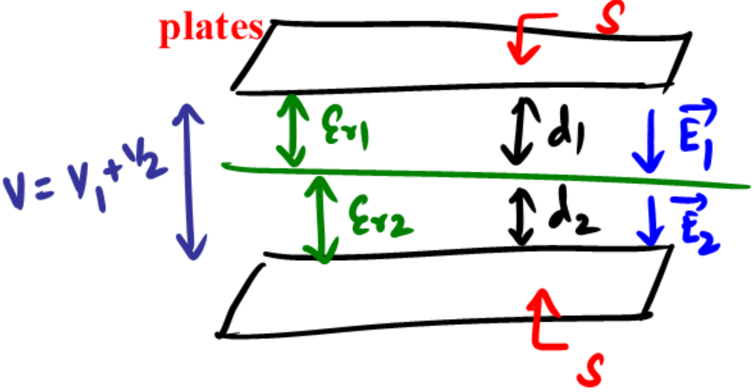
$$\boxed{V = +\frac{\rho_s}{\varepsilon_0\varepsilon_r}d}$$

Capacitance of a parallel plate capacitor:

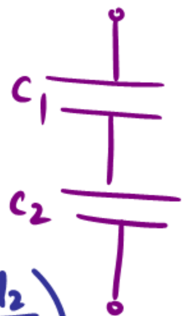
$$\boxed{C = \frac{Q}{V} = \frac{\rho_s . S}{\frac{\rho_s . d}{\varepsilon_0\varepsilon_r}} = \frac{\varepsilon_0\varepsilon_r S}{d}}$$

Capacitance of a parallel plate capacitor with multiple dielectrics:

Dielectric interface parallel to the capacitor plates



$$C = \frac{Q}{V}$$



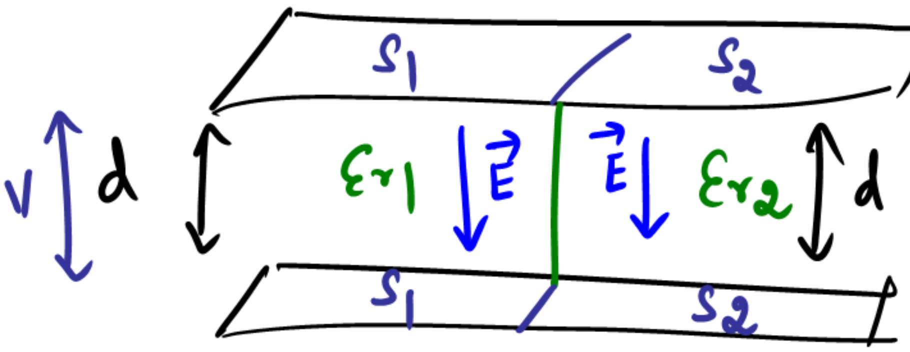
$$C = \frac{\rho_s S}{\left(\frac{\rho_s . d_1}{\varepsilon_0\varepsilon_{r1}}\right) + \left(\frac{\rho_s d_2}{\varepsilon_0\varepsilon_{r2}}\right)}$$

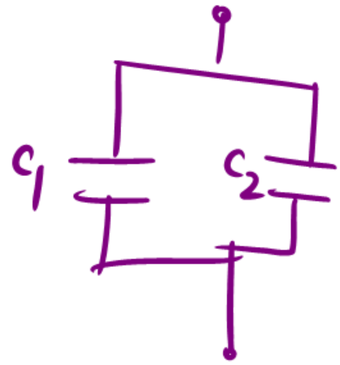
$$C_1 = \frac{\varepsilon_0\varepsilon_{r1} S}{d_1}$$

$$C_2 = \frac{\varepsilon_0\varepsilon_{r2} S}{d_2}$$

$$\boxed{C = \frac{1}{\frac{d_1}{\varepsilon_0\varepsilon_{r1} S} + \frac{d_2}{\varepsilon_0\varepsilon_{r2} S}} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}}$$

Dielectric interface perpendicular to the capacitor plates





$$C = \frac{Q}{V}$$

$$C = \frac{\rho_{s_1} S_1 + \rho_{s_2} S_2}{V}$$

$$V = \frac{\rho_s d}{\varepsilon_0 \varepsilon_r} = \frac{\rho_{s_1} d_1}{\varepsilon_0 \varepsilon_{r1}} = \frac{\rho_{s_2} d_2}{\varepsilon_0 \varepsilon_{r2}}$$

$$C = \frac{\rho_{s_1} S_1}{\frac{\rho_{s_1} d_1}{\varepsilon_0 \varepsilon_{r1}}} + \frac{\rho_{s_2} S_2}{\frac{\rho_{s_2} d_2}{\varepsilon_0 \varepsilon_{r2}}}$$

$$C = \frac{\varepsilon_0 \varepsilon_{r1} d_1}{S_1} + \frac{\varepsilon_0 \varepsilon_{r2} d_2}{S_2}$$

$$\boxed{C = C_1 + C_2}$$

3.b.

3. b)

$$S = 30\,cm \times 30\,cm = 30\times10^{-2} \times 30\times10^{-2}$$

$$d = 5\,mm = 5\times10^{-3}$$

$$\varepsilon_r = 1 \text{ (air)}$$

$$C = \frac{\varepsilon_0 \varepsilon_r S}{d} = \frac{8.854\times10^{-12}\times1\times30\times30\times10^{-4}}{5\times10^{-3}} = 159.372\,pF$$

$$\boxed{C = 159.372\,pF}$$

4.a. Poisson's and Laplace's Equations :

Gauss's law (Point form) / Maxwell's first eqn of electrostatics:

$$\vec{\nabla}\cdot\vec{D} = \rho_V$$

w.k.t

$$\vec{D} = \varepsilon_0 \varepsilon_r \vec{E}$$

$$\vec{E} = -\vec{\nabla} V$$

$$\vec{D} = \varepsilon_0 \varepsilon_r (-\vec{\nabla} V)$$

$$\boxed{\vec{D} = -\varepsilon_0 \varepsilon_r \vec{\nabla} V}$$

$$\vec{\nabla}\cdot(-\varepsilon_0 \varepsilon_r \vec{\nabla} V) = \rho_V$$

For a homogenous, isotropic medium,

ε_r is constant

$$-\varepsilon_0 \varepsilon_r \vec{\nabla}\cdot\vec{\nabla} V = \rho_V$$

$$\vec{\nabla}\cdot\vec{\nabla} V = -\frac{\rho_V}{\varepsilon_0 \varepsilon_r}$$

Also

$$\vec{\nabla}\cdot\vec{\nabla} = \nabla^2$$

Laplacian Operator

$$\boxed{\nabla^2 V = -\frac{\rho_V}{\varepsilon_0 \varepsilon_r}} \rightarrow \text{Poisson's equation}$$

Laplace's equation is a special case of Poisson's equation, where the region is free of charges $\rightarrow \rho_V = 0$

$$\boxed{\nabla^2 V = 0} \rightarrow \text{Laplace's equation}$$

4. b.

4) b) Boundary conditions:

$V = 50V$ at $\phi = 10^\circ = \pi/18 \rightarrow$ (i)

$V = 30V$ at $\phi = 30^\circ = \pi/6 \rightarrow$ (ii)

$$\boxed{V(\phi) = C_1\phi + C_2} \rightarrow (1)$$

(i) in (1) $\Rightarrow 50 = C_1\left(\frac{\pi}{18}\right) + C_2$

(ii) in (1) $\Rightarrow 30 = C_1\left(\pi/6\right) + C_2$

$C_1 = -57.29$

$C_2 = 60$

$P(2,1,3)$

$$\boxed{V(\phi) = -57.29\,\phi + 60}$$

$\phi = \tan^{-1}(y/x) = \tan^{-1}(1/2) = 0.4636$ rad

$$\boxed{\phi = 26.56^\circ}$$

$$\boxed{\phi = 0.4636 \text{ rad}}$$

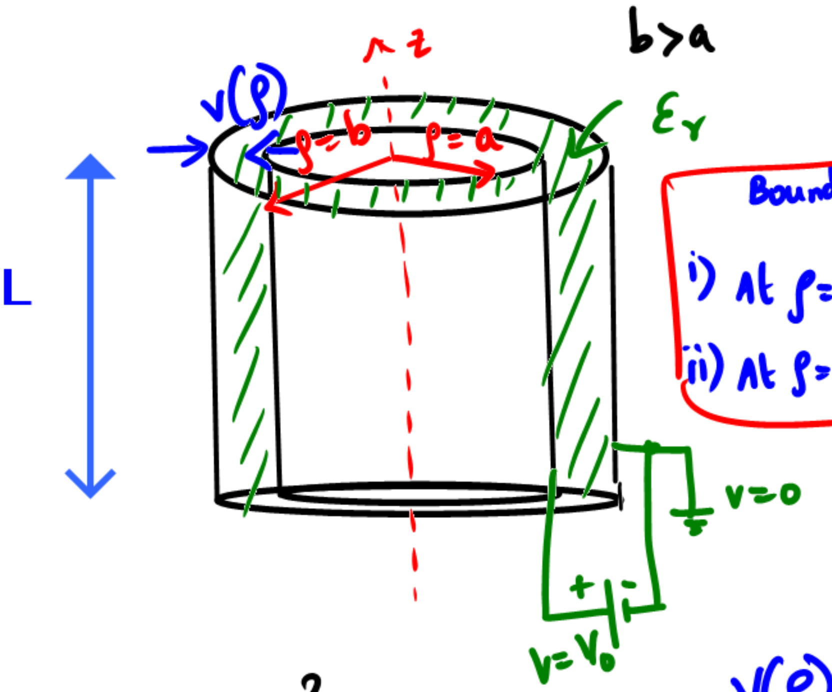
V at $P(2,1,3)$

$= -26.55 + 60$

$$\boxed{V\text{at}(2,1,3) = 33.44\,V}$$

5.

Coaxial cylindrical Capacitor:



Boundary condition

i) At $\rho = a$, $V = V_0$

ii) At $\rho = b$, $V = 0$

$$\nabla^2 V = 0 \qquad V(\rho)$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\cancel{\frac{\partial^2 V}{\partial \phi^2}}_{\to 0} + \cancel{\frac{\partial^2 V}{\partial z^2}}_{\to 0} = 0$$

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{dV}{d\rho}\right) = 0$$

$$\frac{d}{d\rho}\left(\rho \frac{dV}{d\rho}\right) = 0$$

Integrate w.r.t 'ρ'

$$\int \frac{d}{d\rho}\left(\rho \frac{dv}{d\rho}\right) d\rho = \int 0 \, d\rho$$

$$\rho \frac{dv}{d\rho} = 0 + c_1$$

$$\frac{dv}{d\rho} = \frac{c_1}{\rho}$$

Integrate w.r.t. ρ

$$\int \frac{dv}{d\rho} d\rho = \int \frac{c_1}{\rho} d\rho$$

$$\boxed{V = c_1 \ln\rho + c_2}$$

$c_1 = ?$

$c_2 = ?$

Apply the following boundary conditions in the above equation

(i) At $\rho = a$, $V = V_0$

(ii) At $\rho = b$, $V = 0$

(i) $\rightarrow$ $V_0 = c_1 \ln a + c_2$

(ii) $\rightarrow$ $0 = c_1 \ln b + c_2$

Solving the two equations give the values of constants c1 and c2

$$V_0 = c_1 \ln a - c_1 \ln b$$

$$-V_0 = c_1 (\ln b - \ln a)$$

$$-V_0 = c_1 \ln (b/a)$$

$$\boxed{c_1 = -\frac{V_0}{\ln (b/a)}}$$

$$c_2 = -c_1 \ln b = \frac{V_0}{\ln (b/a)} \cdot \ln b$$

$$\boxed{c_2 = \frac{V_0 \ln b}{\ln (b/a)}}$$

Substituting c1 and c2 in the equation of V gives the following expression

$$\boxed{V(\rho) = -\frac{V_0}{\ln (b/a)} \ln \rho + \frac{V_0 \ln b}{\ln (b/a)}}$$

$$\vec{E} = -\vec{\nabla} V$$

$$= -\left[\frac{\partial V}{\partial \rho}\vec{a_\rho} + \underbrace{\cancel{\frac{1}{\rho}\frac{\partial V}{\partial \phi}\vec{a_\phi}}}_{0} + \underbrace{\cancel{\frac{\partial V}{\partial z}\vec{a_z}}}_{0} \right]$$

$$= -\left[\frac{-V_0}{\ln(b/a)} \frac{d(\ln\rho)}{d\rho} \vec{a_\rho} \right]$$

$$\boxed{\vec{E} = \frac{V_0}{\rho \ln(b/a)} \vec{a_\rho}}$$

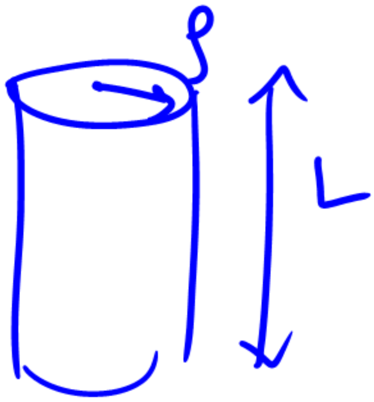
$$\boxed{\vec{D} = \varepsilon_0 \varepsilon_r \vec{E} = \frac{V_0 \varepsilon_0 \varepsilon_r}{\rho \ln(b/a)} \cdot \vec{a_\rho}}$$

$$Q = \oiint_S \vec{D} \cdot \vec{ds}$$

$$Q = \iint_S \frac{V_0 \varepsilon_0 \varepsilon_r \vec{a_\rho}}{\rho \ln(b/a)} \cdot \vec{ds}$$

$$Q = \frac{V_0 \varepsilon_0 \varepsilon_r}{\rho \ln(b/a)} \left[\underbrace{\iint_S ds}_{\text{surface area of cylinder}} \right]$$

$$Q = \frac{V_0 \varepsilon_0 \varepsilon_r \, 2\pi \cancel{\rho} L}{\cancel{\rho} \ln(b/a)}$$



$$\boxed{Q = \frac{V_0 \varepsilon_0 \varepsilon_r \, 2\pi L}{\ln(b/a)}}$$

$$S = 2\pi\rho L$$

$$C = \frac{Q}{V_d}$$

The difference in potential between two cylindrical plates is Vd = V0 - 0 = V0

$$C = \frac{\dfrac{\cancel{V_0} \varepsilon_0 \varepsilon_r \, 2\pi L}{\ln(b/a)}}{\cancel{V_0}}$$

$$\boxed{C = \frac{\varepsilon_0 \varepsilon_r \, 2\pi L}{\ln(b/a)}}$$

6.a.

Uniqueness Theorem:

If a solution to Laplace's (or Poisson's) equation can be found that satisfies the boundary conditions, then the solution is unique. This is known as the uniqueness theorem

The theorem applies to any solution of Poisson's or Laplace's equation in a given region or closed surface.

The theorem is proved by contradiction.
We assume that there are two solutions V1 and V2 of Laplace's (or Poisson's) equation both of which satisfy the prescribed boundary conditions.

V_1 & V_2 are fns of x, y, z

Case (i)

Poisson's eqn:

$$\nabla^2 V_1 = -\frac{\rho_v}{\varepsilon_0 \varepsilon_r} \rightarrow (i)$$

$$\nabla^2 V_2 = -\frac{\rho_v}{\varepsilon_0 \varepsilon_r} \rightarrow (ii)$$

(i) − (ii)

$$\boxed{\nabla^2 V_1 - \nabla^2 V_2 = 0}$$

Case (ii)

Laplace's eqn:

$$\nabla^2 V_1 = 0 \rightarrow (iii)$$

$$\nabla^2 V_2 = 0 \rightarrow (iv)$$

(iii) − (iv)

$$\boxed{\nabla^2 V_1 - \nabla^2 V_2 = 0}$$

$V_1(x, y, z)$

$V_2(x, y, z)$

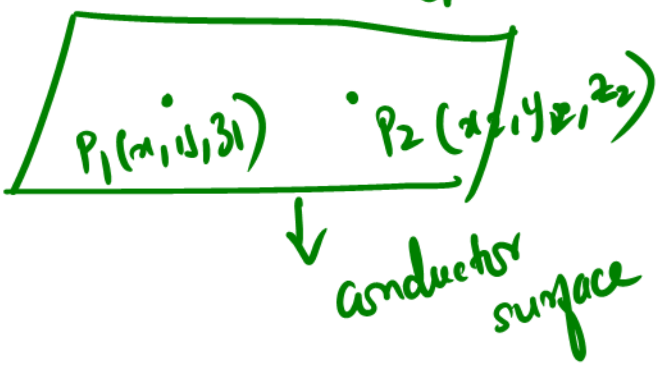
$$\boxed{\nabla^2 (V_1 - V_2) = 0}$$

$$V_d(x,y,z) = V_1(x,y,z) - V_2(x,y,z)$$

$$\boxed{\nabla^2 V_d = 0} \rightarrow (1)$$

In all the electrostatic field problems, atleast one conductor surface exists.

$V_{1b} = V_{2b}$

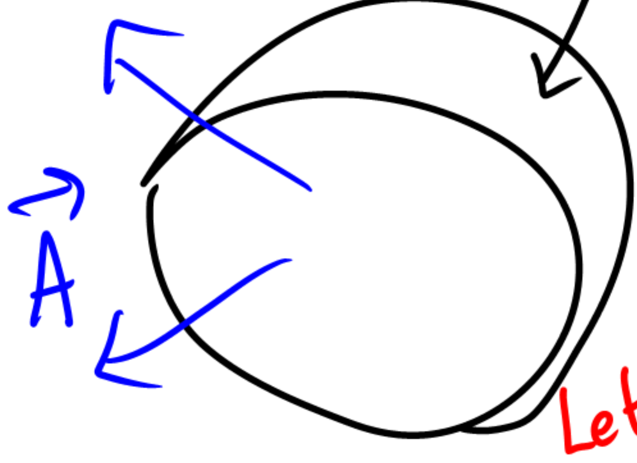


i) $\rho_s \neq 0$

ii) $\rho_v = 0$, $\vec{E} = 0$ (inside)

(iii) Equipotential surface

$$\boxed{V_{d_b} = 0} \longrightarrow (2) \quad \text{Boundary conditions}$$



volume V

Let $\vec{A} = V_d \vec{\nabla} V_d$

Divergence theorem

$$\boxed{\oiint_S \vec{A}\cdot d\vec{s} = \iiint_V (\vec{\nabla}\cdot\vec{A})\, dV}$$

$$\oiint_S v_d \vec{\nabla} v_d \cdot d\vec{s} = \iiint_V (\vec{\nabla}\cdot v_d \vec{\nabla} v_d)\, dv$$

$$\left\{ \begin{array}{l} \alpha = v_d \\ \vec{A} = \vec{\nabla} v_d \end{array} \right\}$$

Vector identity: $\vec{\nabla}\cdot\alpha\vec{A} = \alpha\,\vec{\nabla}\cdot\vec{A} + \vec{A}\cdot\vec{\nabla}\alpha$

$$\vec{\nabla}\cdot v_d \vec{\nabla} v_d = v_d\,\vec{\nabla}\cdot\vec{\nabla} v_d + \vec{\nabla} v_d\cdot\vec{\nabla} v_d$$

$$\oiint_S v_d \vec{\nabla} v_d \cdot d\vec{s} = \iiint_V v_d\, \vec{\nabla}\cdot\vec{\nabla} v_d\, dv + \iiint_V \vec{\nabla} v_d\cdot\vec{\nabla} v_d\, dv$$

w.k.t

$\vec{A}\cdot\vec{A} = |\vec{A}|^2$

$\rightarrow$ ③

$$\boxed{\oiint_s v_d \vec{\nabla} v_d \cdot d\vec{s} = \iiint_V v_d\, \cancel{\nabla^2 v_d}^{\,0}\, dv + \iiint_V |\vec{\nabla} v_d|^2\, dv}$$

Apply ① in ③

$$\nabla^2 V_d = 0$$

$$V_{db} = 0$$

$$\Rightarrow \oint\!\!\oint_S \cancel{V_d}\, \vec{\nabla} V_d \cdot \vec{ds} = \iiint_V |\vec{\nabla} V_d|^2 \, dv$$

0

Apply the boundary condition given by ②

$$\Rightarrow \quad 0 = \iiint_V |\vec{\nabla} V_d|^2 \, dv$$

$$|\vec{\nabla} V_d|^2 = 0$$

$\boxed{\vec{\nabla} V_d = 0}$ only if V_d is constant

V_d at the boundary is constant

$$V_{db} = 0$$

$$V_1 - V_2 = 0$$

$$\boxed{V_1 = V_2}$$

6.b.

③ Given potential field $V = (A\rho^4 + B\rho^{-4})\sin 4\phi$. show that $\nabla^2 V = 0$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\cdot\frac{\partial}{\partial\rho}(A\rho^4 + B\rho^{-4})\sin 4\phi\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}(\sin 4\phi)(A\rho^4 + B\rho^{-4}) + 0$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\cdot(4A\rho^3 - 4B\rho^{-5})\sin 4\phi\right) + \frac{1}{\rho^2}\frac{\partial}{\partial\phi}(\cos 4\phi)\,4\,(A\rho^4 + B\rho^{-4})$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}(4A\rho^4 - 4B\rho^{-4})\sin 4\phi + \frac{4}{\rho^2}\frac{\partial}{\partial\phi}(\cos 4\phi)(A\rho^4 + B\rho^{-4})$$

$$= \cancel{\frac{4\sin 4\phi}{\rho}\cdot(16A\rho^3 + 16B\rho^{-5}) - \frac{16}{\rho^2}\sin 4\phi\,(A\rho^3 + B\rho^{-5})}$$

$$\boxed{\nabla^2 V = 0}$$

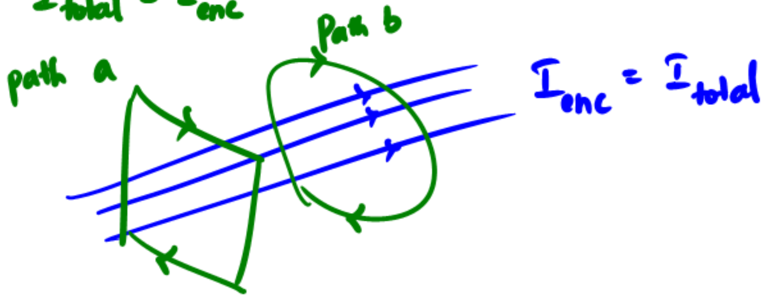
7.

Ampere's Circuital Law

Ampère's circuit law states that the line integral of **H** around a *closed* path is the same as the net current I_{enc} enclosed by the path.

$$\boxed{\oint_L \vec{H} \cdot \vec{dl} = I_{enc}}$$

$$\oint_L \vec{H} \cdot \vec{dl} = I_{total} = I_{enc}$$



CURL AND STOKES' THEOREM

Curl of $\vec{H}$:

It is an axial (rotational) vector whose magnitude is the maximum circulation of $\vec{H}$ per unit area as the area tends to zero and whose direction is the normal direction of the area when the area is oriented as to make the circulation maximum.

$$\underbrace{\text{curl } \vec{H} = \vec{\nabla} \times \vec{H} = \underbrace{\left(\lim_{\Delta s \to 0} \frac{\overbrace{\boxed{\oint_L \vec{H} \cdot \vec{dl}}}^{\text{circulation of } \vec{H}}}{\Delta s} \right)_{max}}_{\text{magnitude of curl } \vec{H}} \cdot \underbrace{\vec{a_n}}_{\text{dxn of curl } \vec{H}}}_{\text{curl } \vec{H} \text{ is a vector}}$$

By Ampere's Circuital Law

$$\boxed{\oint_L \vec{H} \cdot \vec{dl} = I_{enc}}$$

Integral form of A.C.L

$$\text{curl } \vec{H} = \vec{\nabla} \times \vec{H} = \left(\lim_{\Delta S \to 0} \frac{I_{enc}}{\Delta S} \right)_{max} \cdot \vec{a_n}$$

$$\text{curl } \vec{H} = \vec{\nabla} \times \vec{H} = J \cdot \vec{a_n}$$

$$\boxed{\vec{\nabla} \times \vec{H} = \vec{J}}$$

Point form of A.C.L

Stokes Theorem from Ampere's Circuital Law

$$\oint_L \vec{H} \cdot \vec{dl} = I = \iint_S \vec{J} \cdot \vec{ds}$$

$$\boxed{\vec{\nabla} \times \vec{H} = \vec{J}}$$

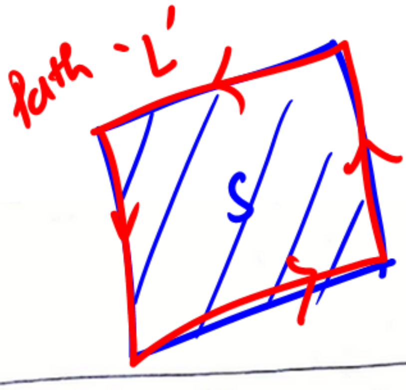
$$\oint_L \vec{H} \cdot \vec{dl} = \iint_S \vec{J} \cdot \vec{ds}$$

$$\boxed{\oint_L \vec{H} \cdot \vec{dl} = \iint_S (\vec{\nabla} \times \vec{H}) \cdot \vec{ds}}$$

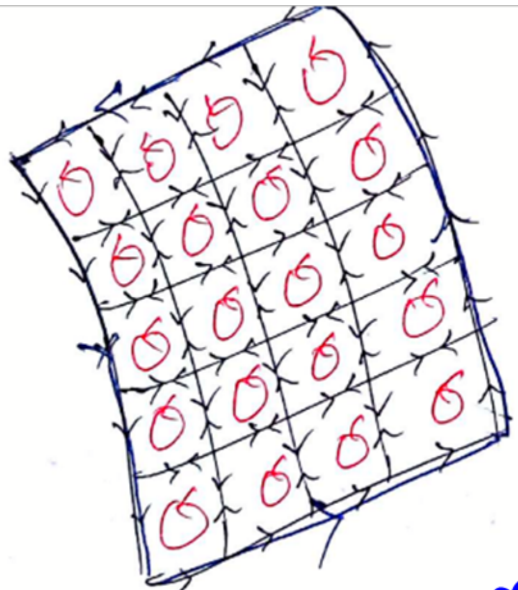
Stokes theorem

Proof of STOKES' THEOREM:

$$\boxed{\oint_L \vec{H} \cdot \vec{dl} = \iint_S (\vec{\nabla} \times \vec{H}) \cdot \vec{ds}}$$



The Circulation of a vector field $\vec{H}$ around a closed path L is equal to surface integral of curl of $\vec{H}$ over the open surface S bounded by L, provided $\vec{H}$ & $\vec{\nabla} \times \vec{H}$ are continuous on S.



$\Delta S_k \to 0$; $K \to \infty$

$$\oint_L \vec{H} \cdot \vec{dl} = \sum_K \oint_{L_k} \vec{H} \cdot \vec{dl}$$

$$\oint_L \vec{H} \cdot \vec{dl} = \sum_K \frac{\oint_{L_k} \vec{H} \cdot \vec{dl}}{\Delta S_k} \cdot \Delta S_k .$$

If $\Delta S_k \to 0$

$$= \sum_{K \to \infty} \left(\lim_{\Delta S_k \to 0} \frac{\oint_{L_k} \vec{H} \cdot \vec{dl}}{\Delta S_k} \right) \cdot \Delta S_k .$$

$$\boxed{\oint_L \vec{H} \cdot \vec{dl} = \iint_S (\vec{\nabla} \times \vec{H}) \cdot \vec{ds}}$$

8. a.

Magnetic Scalar Potential:

We can define the magnetic scalar potential measured in Amperes (A)

$$\boxed{\vec{H} = -\vec{\nabla} V_m} \rightarrow (1) \Rightarrow \boxed{V_m = -\int_a^b \vec{H} \cdot \vec{dl}} \rightarrow (2) \text{ (path specific)}$$

$(\because V_m$ – magnetic scalar potential$)$

From Ampere's Circuital law of magnetostatics,

$$\boxed{\vec{\nabla} \times \vec{H} = \vec{J}} \rightarrow (3)$$

Vector Identities:

(i) $\vec{\nabla} \times \vec{\nabla} \alpha = \vec{0}$

(ii) $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$

Sub (1) in (3) $\Rightarrow \vec{\nabla} \times (-\vec{\nabla} V_m) = \vec{J}$

From vector identity (i)

$$\vec{\nabla} \times \vec{\nabla} V_m = 0$$

$$\therefore \boxed{\vec{J} = 0}$$

Therefore, magnetic scalar potential V_m given by

$\vec{H} = -\vec{\nabla} V_m$ and $V_m = -\int_a^b \vec{H} \cdot \vec{dl}$ (path specific)

can be defined only if $\vec{J} = 0$

Magnetic Vector Potential:

Magnetic scalar potential can be defined only if current density is zero.

We can define magnetic vector potential $\vec{A}$ in Wb/m.
The vector magnetic potential may be used in regions where the current density is zero or nonzero, and we will also be able to extend it to the time-varying cases.

$$\boxed{\vec{\nabla} \cdot \vec{B} = 0}$$

Differential form of Gauss's Law for magnetic fields

$$\updownarrow$$

$$\boxed{\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0}$$

Vector Identities:

1) $\vec{\nabla} \times \vec{\nabla} \alpha = \vec{0}$

2) $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$

Comparing the above two equations, we can write Magnetic flux density in terms of vector potential

$$\boxed{\vec{B} = \vec{\nabla} \times \vec{A}}$$

where

$\vec{A}$ is magnetic vector potential

The expression for magnetic vector potential can be given as follows:

$$\vec{B} = \mu_0 \vec{H}$$

$$\boxed{\mu_0 \vec{H} = \vec{\nabla} \times \vec{A}}$$

Solution of the above equation yields the expression for Magnetic vector potential as given below:

Line current $\boxed{\vec{A} = \oint_L \frac{\mu_0 I \vec{dl}}{4\pi R}}$ $\frac{HA}{m}$ (or Wb/m

Sheet current $\boxed{\vec{A} = \iint_S \frac{\mu_0 \vec{K} ds}{4\pi R}}$

Volume current $\boxed{\vec{A} = \iiint_V \frac{\mu_0 \vec{J} dv}{4\pi R}}$

8.b.

8. b)

$A_x = 4x + 3y + 2z$

$A_y = 5x + 6y + 3z$

$A_z = 2x + 3y + 5z$

$$\vec{B} = \vec{\nabla} \times \vec{A} = \begin{vmatrix} \vec{a_x} & \vec{a_y} & \vec{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\vec{B} = \vec{a_x}\left[\frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y\right] - \vec{a_y}\left[\frac{\partial}{\partial x}A_z - \frac{\partial}{\partial z}A_x\right] + \vec{a_z}\left[\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x\right]$$

$$\vec{B} = \vec{a_x}[3 - 3] - \vec{a_y}[2 - 2] + \vec{a_z}[5 - 3]$$

$$\boxed{\vec{B} = 2\vec{a_z}} \ \text{Wb/m}^2 \text{ (or) } T$$