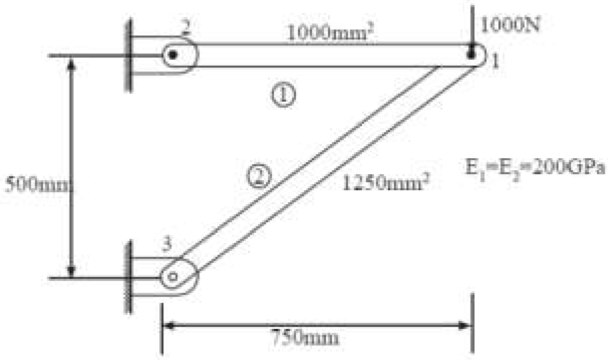
Internal Assessment Test II – June 2022

Sub:	Finite Element Methods	**Code:**	18ME61
Date: 08/ 06 / 2022	**Duration:** 90 mins	**Max Marks:** 50	**Sem:** VI
		Branch:	MECH

Note: Answer all questions.

		Marks	OBE CO	RBT
1	Derive the stiffness matrix for truss element in terms of direction cosines.	**08**	CO4	L3
2	For the Pin Jointed Configuration shown in below Figure. Formulate the stiffness matrix also determine the nodal displacements.	**16**	CO4	L3



		Marks	CO	RBT
3	Derive Hermite shape function for beams.	**12**	CO3	L2

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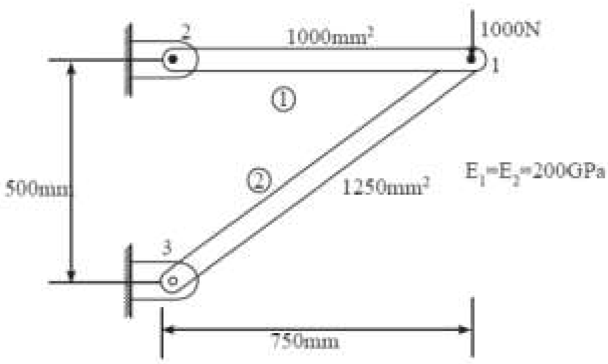

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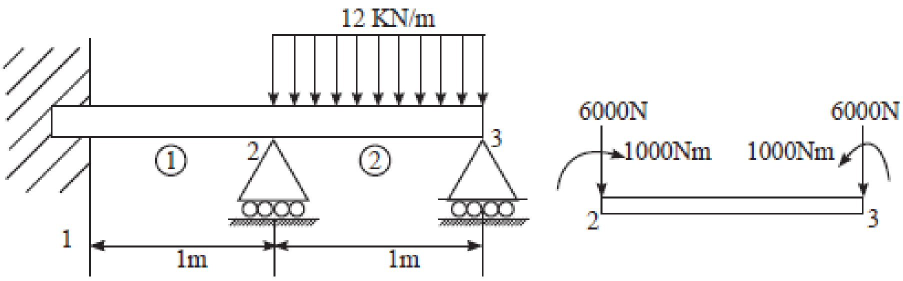
4 For the beam and loading shown in Figure, determine **14** CO4 L3

i. Slopes at 2 and 3

ii. The vertical deflection at the midpoint of the distributed load.

Take E = 200 GPa, I = 4×10^6 mm^4.

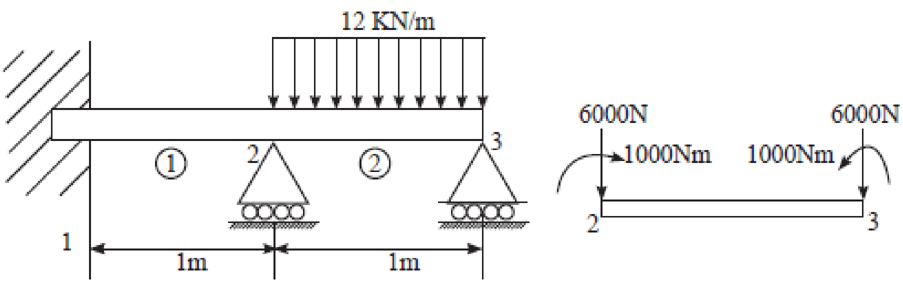


4 For the beam and loading shown in Figure 1, determine **14** CO4 L3

i. Slopes at 2 and 3

ii. The vertical deflection at the midpoint of the distributed load.

Take E = 200 GPa, I = 4×10^6 mm^4.



Subject: FEM
Code: 18ME61

DERIVATION OF ELEMENTAL STIFFNESS MATRIX FOR A TRUSS ELEMENT

Relation b/w nodal displacement of a truss element in local coordinates & global coordinate is expressed as

$$q' = L q$$

Where $q' = \begin{bmatrix} q_1' \\ q_2' \end{bmatrix}$; $L = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$; $q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$

A truss element in local coordinate is equivalent to 1D bar element having the stiffness matrix

$$K' = \frac{EA}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Strain energy for a truss element in local coordinate is given by

$$U_e = \frac{1}{2} q'^T K' q' \rightarrow (1)$$

It is required to determine strain energy of truss element in global coordinates.

$$q' = L q$$

Sub. this in eqn (1) we get

$$U_e = \frac{1}{2} q^T L^T K' L q$$

$$U_e = \frac{1}{2} q^T \left(L^T K' L \right) q$$

$$U_e = \frac{1}{2} q^T K_e q$$

Where K_e - elemental stiffness matrix in global coordinates

$$K_e = L^T K' L$$

$$L = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix} \quad ; \quad L^T = \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix} \quad ; \quad K' = \frac{EA}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K_e = \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix} \frac{EA}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$$

$$K_e = \frac{EA}{l_e} \begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

Where l & m are direction cosines.

Nodal Data.

Node	x	y
1	750	500
2	0	500
3	0	0

Element connectivity table.

Element	Initial Node.	Final Node.	Length of element.	Direction cosines	
				l	m
1	1	2	750.	-1	0
2	1	3	901.38.	-0.83	-0.55.

$$l_e = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

$$= \sqrt{(0-750)^2 + (0-500)^2}$$

$$= 901.38 \text{ mm}$$

$$l = \frac{x_2 - x_1}{l_e} = \frac{0-750}{750} \; ; \; \frac{0-750}{901.38} = -0.83$$

$$m = \frac{y_2 - y_1}{l_e} = \frac{500-500}{750} = 0 \; ; \; \frac{0-500}{901.38} = -0.55.$$

Elemental Stiffness Matrix.

$$k = \frac{EA}{le}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

$$k_1 = \frac{200\times10^3\times1000}{750}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= 10^5 \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 2.66 & 0 & -2.66 & 0 \\ 0 & 0 & 0 & 0 \\ -2.66 & 0 & 2.66 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}$$

$$k_2 = \frac{200\times10^3\times1250}{901.38}\begin{bmatrix} 0.688 & 0.456. & -0.688 & -0.456 \\ 0.456 & 0.302 & -0.456 & -0.202 \\ -0.688 & -0.456 & 0.688 & 0.456. \\ -0.456 & -0.302 & 0.456 & 0.302. \end{bmatrix}$$

$$= 10^5 \begin{array}{c} \begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} \\ \begin{bmatrix} 1.90 & 1.26 & -1.90 & -1.26 \\ 1.26 & 0.836 & -1.26 & -0.836 \\ -1.90 & -1.26 & 1.90 & 1.26. \\ -1.26 & -0.836 & 1.26 & 0.836 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 5 \\ 6. \end{array}$$

Global Stiffness Matrix.

$$K = 10^5 \begin{array}{c} \left[\begin{matrix} 4.56 & 1.26 & -2.66 & 0 & -1.90 & -1.26 \\ 1.26 & 0.836 & 0 & 0 & -1.26 & -0.836 \\ -2.66 & 0 & 2.66 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1.90 & -1.26 & 0 & 0 & 1.90 & 1.26 \\ -1.26 & -0.836 & 0 & 0 & 1.26 & 0.836 \end{matrix}\right] \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

Equlibrium Eq.

$[K][q] = [F]$.

$$10^5 \begin{bmatrix} 4.56 & 1.26 & -2.66 & 0 & -1.90 & -1.26 \\ 1.26 & 0.836 & 0 & 0 & -1.26 & -0.836 \\ -2.66 & 0 & 2.66 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1.90 & -1.26 & 0 & 0 & 1.90 & 1.26 \\ -1.26 & -0.836 & 0 & 0 & 1.26 & 0.836 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix} = \begin{bmatrix} 0 \\ -1000 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying Boundary conditions.

$$10^5 \begin{bmatrix} 4.56 & 1.26 & -2.66 & 0 & -1.90 & -1.26 \\ 1.26 & 0.836 & 0 & 0 & -1.26 & -0.836 \\ -2.66 & 0 & 2.66 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1.90 & 1.26 & 0 & 0 & 1.90 & 1.26 \\ -1.26 & -0.836 & 0 & 0 & 1.26 & 0.836 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix} = \begin{bmatrix} 0 \\ -1000 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$q_1 = 5.06 \times 10^{-3}$ mm ; $q_2 = -0.020$ mm.

3 Hermite Shape Function

DERIVATION OF SHAPE FUNCTION [HERMITE SHAPE FUNCTION]

Let us consider the shape function H_i as

$$H_i = a_i + b_i\xi + c_i\xi^2 + d_i\xi^3$$

	H_1	H_1'	H_2	H_2'	H_3	H_3'	H_4	H_4'
$\xi = -1$ (Node 1)	1	0	0	1	0	0	0	0
$\xi = 1$ (Node 2)	0	0	0	0	1	0	0	1

$$H_i' = \frac{dH}{d\xi}$$

Let shape function H_1 be

$$H_1 = a_1 + b_1\xi + c_1\xi^2 + d_1\xi^3$$

$$H_1' = \frac{dH_1}{d\xi} = b_1 + 2c_1\xi + 3d_1\xi^2$$

At node 1, $H_1 = 1$, $\xi = -1$.

$$1 = a_1 - b_1 + c_1 - d_1 \rightarrow (1)$$

At node 1, $H_1' = 0$, $\xi = -1$

$$0 = b_1 - 2c_1 + 3d_1 \rightarrow (2)$$

At node 2, $H_1 = 0$, $\xi = 1$.

$$0 = a_1 + b_1 + c_1 + d_1 \rightarrow (3)$$

At node 2, $H_1' = 0$, $\xi = 1$.

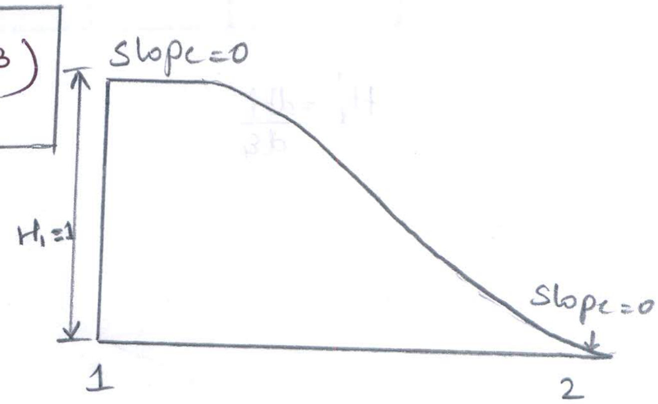
$$0 = b_1 + 2c_1 + 3d_1 \rightarrow (4)$$

Solving (1) (2) (3) & (4) we get

$$a_1 = \frac{1}{2} \; ; \quad b_1 = \frac{-3}{4} \; ; \quad c_1 = 0 \; ; \quad d_1 = \frac{1}{4}$$

$$H_1 = \frac{1}{2} - \frac{3}{4}\xi + \frac{1}{4}\xi^3$$

$$\boxed{H_1 = \frac{1}{4}\left(2 - 3\xi + \xi^3\right)}$$



Let the shape function H_2 be

$$H_2 = a_2 + b_2 \xi + c_2 \xi^2 + d_2 \xi^3$$

$$H_2' = b_2 + 2c_2 \xi + 3d_2 \xi^2$$

At node 1, $H_2 = 0$, $\xi = -1$

$$0 = a_2 - b_2 + c_2 - d_2 \rightarrow (5)$$

At node 1, $H_2' = 1$, $\xi = -1$

$$1 = b_2 - 2c_2 + 3d_2 \rightarrow (6)$$

At node 2, $H_2 = 0$, $\xi = 1$.

$$0 = a_2 + b_2 + c_2 + d_2 \rightarrow (7)$$

At node 2, $H_2' = 0$, $\xi = 1$.

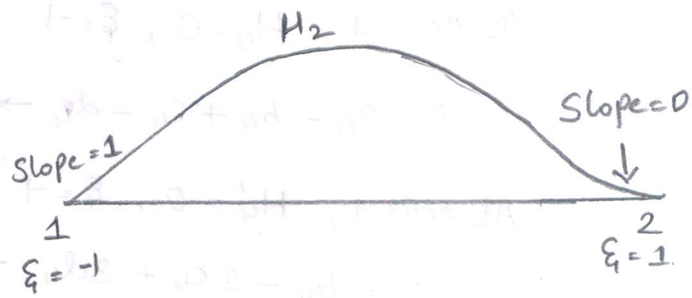
$$0 = b_2 + 2c_2 + 3d_2 \rightarrow (8)$$

Solving 5, 6, 7 & 8

$a_2 = \frac{1}{4}$, $b_2 = -\frac{1}{4}$; $c_2 = -\frac{1}{4}$; $d_2 = \frac{1}{4}$

$$H_2 = \frac{1}{4} - \frac{1}{4}\xi - \frac{1}{4}\xi^2 + \frac{1}{4}\xi^3$$

$$\boxed{H_2 = \frac{1}{4}\left(1 - \xi - \xi^2 + \xi^3\right)}$$



Let the shape function H_3 be

$$H_3 = a_3 + b_3 \xi + c_3 \xi^2 + d_3 \xi^3$$

$$H_3' = b_3 + 2c_3 \xi + 3d_3 \xi^2$$

At node 1, $H_3 = 0$, $\xi = -1$

$$0 = a_3 - b_3 + c_3 - d_3 \rightarrow (9)$$

At node 1, $H_3' = 0$ $\xi = -1$

$$0 = b_3 - 2c_3 + 3d_3 \rightarrow (10)$$

At node 2, $H_3 = 1$, $\xi = 1$.

$$1 = a_3 + b_3 + c_3 + d_3 \rightarrow (11)$$

At node 2, $H_3 = 0$, $\xi = 1$.

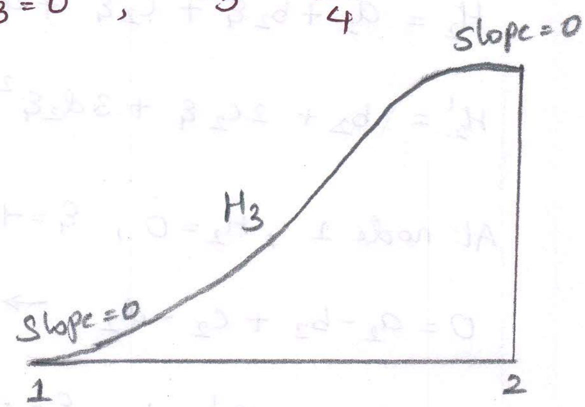
$$0 = b_3 + 2c_3 + 3d_3 \rightarrow (12)$$

Solving 9, 10, 11 & 12 we get

$a_3 = \frac{1}{2}$; $b_3 = \frac{3}{4}$; $c_3 = 0$; $d_3 = -\frac{1}{4}$

$$H_3 = \frac{1}{2} + \frac{3}{4}\xi - \frac{1}{4}\xi^3$$

$$\boxed{H_3 = \frac{1}{4}\left(2 + 3\xi - \xi^3\right)}$$



Let the Shape function H_4 be

$$H_4 = a_4 + b_4\xi + c_4\xi^2 + d_4\xi^3$$

$$H_4' = b_4 + 2c_4\xi + 3d_4\xi^2$$

At node 1, $H_4 = 0$, $\xi = -1$

$$0 = a_4 - b_4 + c_4 - d_4 \rightarrow 13$$

At node 1, $H_4' = 0$, $\xi = -1$

$$0 = b_4 - 2c_4 + 3d_4 \rightarrow 14$$

At node 2, $H_4 = 0$, $\xi = 1$

$$0 = b_4 + 2c_4 + 3d_4 \rightarrow 15$$

At node 2, $H_4 = 01$ $\xi = 1$.

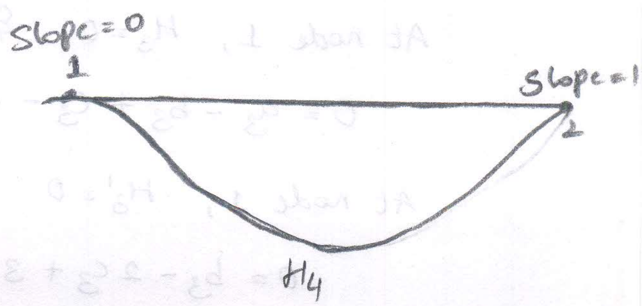
$$01 = b_4 + 2c_4 + 3d_4 \rightarrow 16$$

Solving 13 14 15 & 16 we get

$a_4 = \frac{-1}{4}$; $b_4 = -\frac{1}{4}$; $c_4 = \frac{1}{4}$; $d_4 = \frac{1}{4}$

$$H_4 = -\frac{1}{4} - \frac{1}{4}\xi + \frac{1}{4}\xi^2 + \frac{1}{4}\xi^3$$

$$\boxed{H_4 = \frac{1}{4}\left(-1 - \xi + \xi^2 + \xi^3\right)}$$



4 Elemental Stiffness matrix

$$K = \frac{EI}{l_e^3}\begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}$$

$$k_1 = \frac{200\times10^9 \times 4\times10^{-6}}{1^3}\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}$$

$$k_2 = 8\times10^5\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}$$

Global Stiffness

$$K = 8\times10^5\begin{bmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix}$$

Load vector

$$f_e = \begin{bmatrix} 0 \\ 0 \\ -6\times10^3 \\ -1\times10^3 \\ -6\times10^3 \\ +1\times10^3 \end{bmatrix}$$

Equilibrium eqn

$$[K][q] = [F]$$

$$8\times10^5 \begin{bmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -6\times10^3 \\ -1\times10^3 \\ -6\times10^3 \\ 1\times10^3 \end{bmatrix}$$

Slope at node 2 ; $\theta_2 = -2.67\times10^{-4}$ rad

Slope at node 3 ; $\theta_3 = 4.46\times10^{-4}$ rad

Max' deflection

$$y = H \cdot q$$

$$y = H_1 \cdot v_2 + H_2 \cdot \frac{l_e}{2} \cdot \theta_2 + H_3 \cdot v_3 + H_4 \cdot \frac{l_e}{2}\,\theta_3$$

$\xi = 0$ at center

$$H_1 = \frac{1}{4}\left[2 - 3\xi + \xi^3\right] = \frac{1}{2}$$

$$H_2 = \frac{1}{4}\left[1 - \xi - \xi^2 + \xi^3\right] = \frac{1}{4}$$

$$H_3 = \frac{1}{4}\left[2 + 3\xi - \xi^3\right] = \frac{1}{2}$$

$$H_4 = \frac{1}{4}\left(-1 - \xi - \xi^2 + \xi^3\right) = -\frac{1}{4}$$

$$y = 8.9\times10^{-5}\ \text{m}$$