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Internal Assesment Test – II August 2022

Sub:	Complex Analysis, Probability and Statistical Methods							Code:	18MAT41
Date:	03/08/2022	Duration:	90 mins	Max Marks:	50	Sem:	4	Branch:	ALL (Except for EEE B)

Question 1 is compulsory and answer any 6 from the remaining questions.

		Marks	OBE	
			CO	RBT
1	Discuss the transformation $\boldsymbol{w = z + \frac{1}{z}, z \neq 0}$.	[8]	CO2	L3
2	Discuss the transformation $\boldsymbol{w = e^z}$.	[7]	CO2	L3
3	Find the bilinear transformation which maps the points $\boldsymbol{z = 1, i, -1}$ onto the points $\boldsymbol{w = 2, i, -2}$.	[7]	CO2	L3
4	Find the bilinear transformation which maps the points $\boldsymbol{z = \infty, i, 0}$ onto the points $\boldsymbol{w = -1, -i, 1}$. Also find the invariant points of this transformation.	[7]	CO2	L3

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5	Evaluate $\int_{(0,1)}^{(2,5)}(3x+y)dx+(2y-x)dy$ along (i) the curve $y=x^2+1$, (ii) the line joining $(0,1)$ and $(2,5)$.	[7]	CO2	L3
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7	Evaluate $\int_C \frac{e^{2z}}{z+i\pi}dz$ over (i) $\lvert z\rvert=2\pi$ and (ii) $\lvert z-1\rvert=1$.	[7]	CO2	L3
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(5)

1. Discussion of Transformation $w = z + \frac{1}{z}$, $z \neq 0$

consider $w = z + \frac{1}{z}$, put $z = re^{i\theta}$ & $w = u + iv$

$$u + iv = re^{i\theta} + \frac{1}{r} e^{-i\theta} \qquad \Big/ \because \frac{1}{z} = \frac{1}{e^{i\theta}} = e^{-i\theta}$$

$$= r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$u + iv = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$$

Equating real and imaginary parts

$$u = \left(r + \frac{1}{r}\right)\cos\theta, \quad v = \left(r - \frac{1}{r}\right)\sin\theta \quad \text{—— (1)}$$

we shall eliminate r & θ separately from (1)

From (1), $\frac{u}{\left(r + \frac{1}{r}\right)} = \cos\theta$, $\frac{v}{\left(r - \frac{1}{r}\right)} = \sin\theta$

Squaring and adding b.s.

$$\frac{u^2}{\left(r + \frac{1}{r}\right)^2} + \frac{v^2}{\left(r - \frac{1}{r}\right)^2} = 1, \quad r \neq 1 \quad \text{—— (2)}$$

Also, from (1), $\frac{u}{\cos\theta} = \left(r + \frac{1}{r}\right)$, $\frac{v}{\sin\theta} = \left(r - \frac{1}{r}\right)$

Squaring & subtracting above equations, we get

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = \left(r + \frac{1}{r}\right)^2 - \left(r - \frac{1}{r}\right)^2$$

$$= \cancel{r^2} + \cancel{\frac{1}{r^2}} + 2 - \left[\cancel{r^2} + \cancel{\frac{1}{r^2}} - 2\right]$$

$$= 4$$

$$\therefore \frac{u^2}{(2\cos\theta)^2} - \frac{v^2}{(2\sin\theta)^2} = 1 \quad \text{—— (3)}$$

Since $z = re^{i\theta}$, $|z| = r$, amplitude of $z = \text{amp}\, z = \theta$

$|z| = r \Rightarrow |x + iy| = r \Rightarrow \sqrt{x^2 + y^2} = r \Rightarrow x^2 + y^2 = r^2$

$\therefore |z| = r$, represents a circle with centre at origin and radius r in z-plane.

$\text{amp } z = \theta = \tan^{-1}(y/x)$

$\Rightarrow y/x = \tan\theta \Rightarrow y = x(\tan\theta)$

This represents a straight line in z-plane (with θ = constant)

Case(1): Let r = constant

Equation (2) ~~of~~ becomes.

$$\frac{u^2}{(r+\frac{1}{r})^2} + \frac{v^2}{(r-\frac{1}{r})^2} = 1 \Rightarrow \frac{u^2}{A^2} + \frac{v^2}{B^2} = 1$$

where $A = r + \frac{1}{r}$, $B = r - \frac{1}{r}$

This represents an ellipse in the w-plane with foci (plural of focus) $[\pm\sqrt{A^2-B^2}, 0] = (\pm 2, 0)$

$\left[\because A^2 - B^2 = (r+\frac{1}{r})^2 - (r-\frac{1}{r})^2 = 4\right]$

$\therefore$ The circle $|z| = r$ in the z-plane maps onto an ellipse in the w-plane with foci $(\pm 2; 0)$

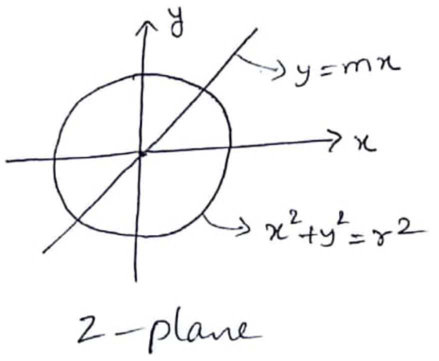
Case(2): Let θ = constant

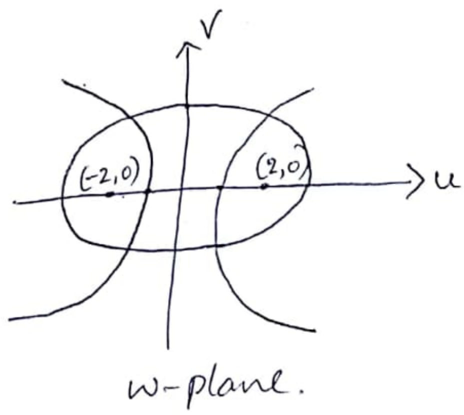
Eqn (3) can be written as $\frac{u^2}{A^2} - \frac{v^2}{B^2} = 1$

where $A = 2\cos\theta$, $B = 2\sin\theta$

This represents a hyperbola in the w-plane with foci $(\pm 2, 0)$

Hence the straight line passing through the origin in the z-plane maps onto a hyperbola in the w-plane, with foci $(\pm 2, 0)$.





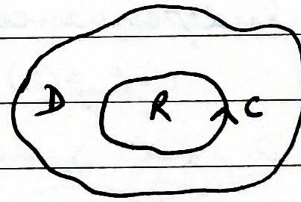
integral or Cauchy's theorem -

⇒ Cauchy's theorem — If $f(z)$ is analytic in a simply connected domain D then

$$\oint_C f(z)dz = 0$$ for any simple closed curve C lying entirely with in D.

Proof - Consider

$$\oint_C f(z)dz = \oint_C (u(x,y)+iv(x,y))(dx+idy)$$



$$= \oint_C (udx - vdy) + i\oint_C (udy+vdx) = I_1 + I_2$$

Given $f(z)$ is analytic so u and v have continuous partial derivative in D. (and f' is assumed to be continuous)

Apply Green's theorem in plane of I_1 and I_2

Green's theorem $\boxed{\int_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy}$

$$\therefore\ I_1 = \oint_C (udx - vdy) = \iint_R \left(\underbrace{\frac{-\partial v}{\partial x} - \frac{\partial u}{\partial y}}_{(v_y - u_y)}\right) dx\,dy = 0$$

(by C-R eqⁿ.)

Sly,

$$I_2 = \oint_C (udy+vdx) = \iint_R \left(\underbrace{\frac{\partial u}{\partial x}}_{v_y} - \frac{\partial v}{\partial y}\right) dx\,dy = 0$$

by C-R eqⁿ

$$\therefore\ \oint_C f(z)dz = I_1 + I_2 = 0$$

Date. Page.

• 2. Discussion of e^z.

Consider $w = e^z \Rightarrow u + iv = e^{x+iy}$

$\Rightarrow u = e^x \cos y$ —(1) & $v = e^x \sin y$ —(2)

Case-1 Consider $x =$ const.

From eqⁿ (1) & (2) & $\frac{u}{v} = \frac{1}{\tan y} \Rightarrow \frac{v}{u} = \tan y$,

$$u^2 + v^2 = e^{2x}$$

$$\Rightarrow u^2 + v^2 = e^{2c_1} = \text{cons.} = a = r^2 \text{ (say)}$$

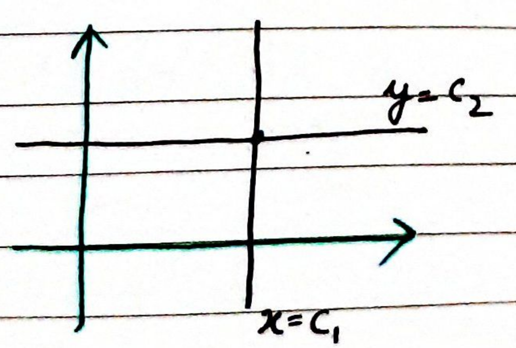
Which represent a circle with centre orig and radius r in w-plane

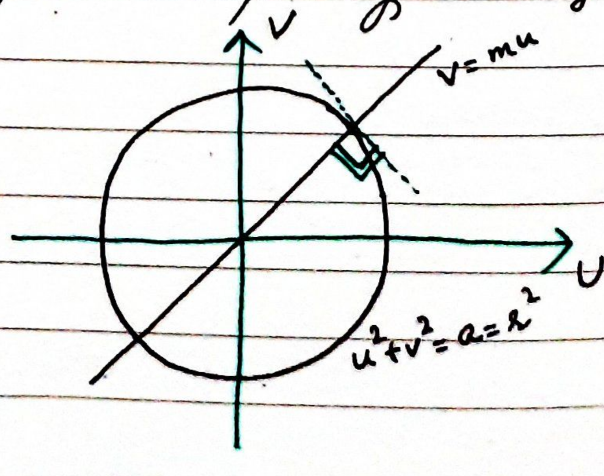
Case-2 Consider ~~u/v = tan y~~ $y = c_2$

$$\frac{v}{u} \cancel{\frac{y}{v}} = \tan y = \tan c_2 = m \text{ (say)}$$

$$\therefore v = mu$$

Which represents a straight line passing through the origin in w-plane.





(3) Find the bilinear transformation which maps the points $z = 1, i, -1$ into $w = 2, i, -2$. Also find the invariant points of the transformation. (or fixed points)

Soln :- Let $w = \dfrac{az+b}{cz+d}$ be the required BLT —— (1)

Put $z=1$, $w=2$ in (1) we get $2 = \dfrac{a+b}{c+d}$

$$a+b-2c-2d=0 \quad \text{---} \quad (2)$$

put $z=i$, $w=i$ in eqn (1), $i=\dfrac{ai+d}{ci+d}$.

$$ai+b+c-di=0 \quad \text{---} \quad (3)$$

put $z=-1$, $w=-2$ in eqn (1), $-2=\dfrac{-a+b}{-c+d}$

$$-a+b-2c+2d=0 \quad \text{---} \quad (4)$$

eqn (2) + (4) $\Rightarrow$ $2b-4c=0$ (or) $b-2c=0$ (5)

eqn (3) + $i\times$(4) $\Rightarrow$ $(1+i)b+(1-2i)c+id=0$ --- (6)

Now eqn (5) & (6) can be written as

(5) & (6) contains three constants b, c, d

$$1b-2c+0d=0$$

$$(1+i)b+(1-2i)c+id=0$$

Applying the rule of cross multiplication

$$\frac{b}{\begin{vmatrix} -2 & 0 \\ 1-2i & i \end{vmatrix}} = \frac{-c}{\begin{vmatrix} 1 & 0 \\ 1+i & i \end{vmatrix}} = \frac{d}{\begin{vmatrix} 1 & -2 \\ 1+i & 1-2i \end{vmatrix}}$$

$$\frac{b}{-2i-0} = \frac{-c}{i-0} = \frac{d}{1-2i-[-2(1+i)]}$$

$$\frac{b}{-2i} = \frac{-c}{i} = \frac{d}{3}$$

$\therefore$ $\boxed{b=-2i\,,\ c=-i\,,\ d=3}$

Substituting this in equation (4), $-a-\cancel{2i}+\cancel{2i}+6=0$

$\therefore$ $\boxed{a=6}$

Substitute a, b, c, d, we get

$$\boxed{w=\frac{6z-2i}{-iz+3}}$$

4) Find BLT which maps $z = \infty, i, 0$ to $w = -1, -i, 1$
Also find fixed pts of the transformation

Sn

Let $w = \dfrac{az+b}{cz+d}$ be the req BLT

$z = \infty$ & $w = -1$ $\Rightarrow$ $w = \dfrac{\cancel{z}(a+b/z)}{\cancel{z}(c+d/z)} = \dfrac{a+b/z}{c+d/z}$

$\therefore$ $-1 = \dfrac{a+0}{c+0}$ $\Rightarrow$ $a+c=0$ — (1)

$z = i$ & $w = -i$ $\Rightarrow$ $-i = \dfrac{ai+b}{ci+d}$

$\Rightarrow$ $ai+b-c+di=0$ — (2)

$z=0$ & $w=1$ $\Rightarrow$ $1 = \dfrac{0+b}{0+d}$ $\Rightarrow$ $b-d=0$ — (3)

(1) + (2) give $a(1+i)+b+id=0$ — (4)

Solving (3) & (4)

$0a + 1b - 1d = 0$

$a(1+i) + 1b + id = 0$

$$\frac{a}{\begin{vmatrix} 1 & -1 \\ 1 & i \end{vmatrix}} = \frac{-b}{\begin{vmatrix} 0 & -1 \\ 1+i & i \end{vmatrix}} = \frac{d}{\begin{vmatrix} 0 & 1 \\ 1+i & 1 \end{vmatrix}} \Rightarrow \frac{a}{i+1} = \frac{-b}{1+i} = \frac{d}{-(1+i)}$$

$$\Rightarrow \frac{a}{1} = \frac{b}{-1} = \frac{d}{-1}$$

$a=1$ & $b=-1$
& $d=-1$

$\therefore$ ~~c = a~~

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From ① $c = -a \Rightarrow \boxed{c = -1}$

$\therefore w = \dfrac{1z - 1}{-1z - 1} = \dfrac{1 - z}{1 + z}$

Invariant pts are obtained by taking $w = z$

$$z = \frac{1-z}{1+z} \Rightarrow z + z^2 = 1 - z$$

$$z^2 + 2z - 1 = 0$$

$$z = \frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}$$

$-1 + \sqrt{2}$ & $-1 - \sqrt{2}$ are invariant pts.

5. i) $y = x^2+1 \Rightarrow dy = 2x\,dx$ and x varies from 0 to 2.

$$\int_{(0,1)}^{(2,5)} (3x+y)\,dx + (2y-x)\,dy$$

$$= \int_{x=0}^{2} \left\{(3x+x^2+1)\,dx + (2x^2+2-x)\,2x\,dx\right\}$$

$$= \int_0^2 (4x^3-x^2+7x+1)\,dx = \left[x^4 - \frac{x^3}{3} + \frac{7x^2}{2} + x\right]_0^2$$

$$= 16 - \frac{8}{3} + 14 + 2 = \frac{88}{3}.$$

ii) Equation of the line joining (0,1) and (2,5) is

$$\frac{y-1}{x-0} = \frac{1-5}{0-2} \Rightarrow y = 2x+1$$

$$\Rightarrow dy = 2\,dx$$

$$\int_{x=0}^{2} (3x+2x+1)\,dx + (4x+2-x)\,2\,dx$$

$$= \int_0^2 (11x+5)\,dx = \left(11\frac{x^2}{2} + 5x\right)_0^2 = 32.$$

7.

Soln: we have $\int_C \frac{f(z)}{z-a}\,dz$

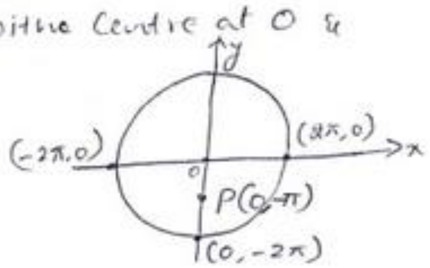
Given integral can be written as $\int \frac{e^z}{z-(-i\pi)}\,dz$

$f(z) = e^z$, $a = -i\pi$ This is the point $P(\overset{x}{0}, \overset{y}{-\pi})$

(i)

(a) $|z| = 2\pi$ represents a circle with the centre at O & radius 2π.

The point $z = a = -i\pi$ is a point $P(0, -\pi)$, lies within the circle $|z| = 2\pi$



we have cauchy's integral formula

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

we have $f(z) = e^z$, $a = -i\pi$

$$\therefore \int_C \frac{e^z}{z+i\pi}\,dz = 2\pi i\, f(-i\pi) = 2\pi i\, e^{-i\pi} = 2\pi i(\cos\pi - i\sin\pi)$$

$$= -2\pi i$$

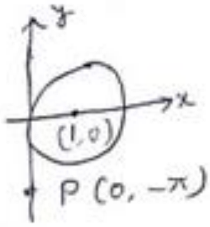
$\cos\pi = -1$
$\sin\pi = 0$

$$\therefore \boxed{\int_C \frac{e^z}{z+i\pi}\,dz = 2\pi i}$$

(ii)

(c) $|\overset{\text{centre}}{z-1}| = \overset{\text{radius}}{1}$ is a circle with centre at $z = a = 1$ & radius 1

i.e It is a circle with centre (1,0) and radius 1



The point $P(0, -\pi)$ lies outside the circle $|z-1| = 1$.

Hence by cauchy's theorem

$$\boxed{\int_C \frac{e^z}{z+i\pi}\,dz = 0}$$ where C is $|z-1| = 1$

8)

$$\int \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)} dz = \int_c \frac{f(z)}{(z-1)(z-2)} dz$$

Now $\frac{1}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$

$\Rightarrow 1 = A(z-2) + B(z-1)$

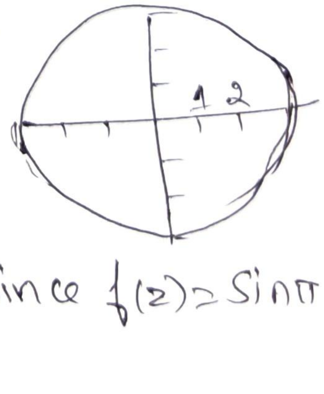
$z=1 \Rightarrow A = -1, \quad z=2 \Rightarrow B = 1$

$\therefore \frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$

$\int_c \frac{f(z)}{(z-a)} dz = 2\pi i \, f(a)$, a is a pt lies inside

Given $|z|=3$ is a circle with center 0 & radius 3

Both points 1 & 2 lies inside



& $f(1) = \sin\pi + \cos\pi = -1$

$f(2) = \sin 4\pi + \cos 4\pi = 1$ Since $f(z) = \sin\pi z^2 + \cos\pi z^2$

$$\therefore \int_c \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)} dz = -2\pi i \, f(1) + 2\pi i \, f(2)$$

$$= -2\pi i(-1) + 2\pi i(1)$$

$$= 4\pi i$$