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Internal Assessment Test III– Aug' 2022

Sub:	Advanced Calculus and Numerical Methods					Sub Code:	21MAT21		
Date:	23/08/2022	Duration:	90 mins	Max.marks	50	Sem/Sec:	A to G (CHE CYCLE)	OBE	

ATTEMPT ANY FIVE

		MARKS	CO	RBT
1.	Evaluate $\iint xy(x+y)dydx$ taken over the area between $y = x^2$ and $y = x$	[10]	CO1	L3
2.	Evaluate the double integral $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dxdy$ by changing the order of integration.	[10]	CO1	L3
3.	Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \; dxdydz$.	[10]	CO1	L3

4. Show that $\vec{F} = (siny + z)i + (xcosy - z)j + (x - y)k$ is irrotational. Also, find scalar function φ such that $\vec{F} = \nabla\varphi$ [10] CO2 L3

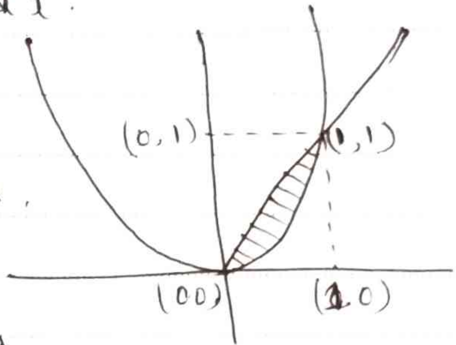
5. Find $Div\ \vec{F}$ and $curl\ \vec{F}$ where $\vec{F} = grad(x^3 + y^3 + z^3 - 3xyz)$ [10] CO2 L3

6. Show that $\vec{F} = (y^2 - z^2 + 3yz - 2x)i + (3xz + 2xy)j + (3xy - 2xz + 2z)k$ is both solenoidal and irrotational. [10] CO2 L3

7. Find the directional derivative of $\varphi = xy^2 + yz^3$ at the point (2,-1,1) in the direction of the vector $i + 2j + 2k$. [10] CO2 L3

① Evaluate $\iint xy(x+y)\,dy\,dx$ taken over the area b/w $y=x^2$ and $y=x$.

Sol^n: $I = \iint_R xy(x+y)\,dy\,dx$.



$$I = \int_{x=0}^{1}\int_{y=x^2}^{x} xy(x+y)\,dy\,dx.$$

$$= \int_0^1\int_{x^2}^{x} (x^2y + xy^2)\,dy\,dx.$$

$$= \int_0^1 \left\{ x^2\left[\frac{y^2}{2}\right]_{y=x^2}^{x} + x\left[\frac{y^3}{3}\right]_{y=x^2}^{x} \right\} dx.$$

$$= \int_0^1 \left(\frac{x^4}{2} - \frac{x^6}{2} + \frac{x^4}{3} - \frac{x^7}{3}\right) dx.$$

$$= \left.\frac{x^5}{10} - \frac{x^7}{14} + \frac{x^5}{15} - \frac{x^8}{24}\right]_{x=0}^{1}$$

$$= \frac{1}{10} - \frac{1}{14} + \frac{1}{15} - \frac{1}{24} = \frac{3}{56}$$

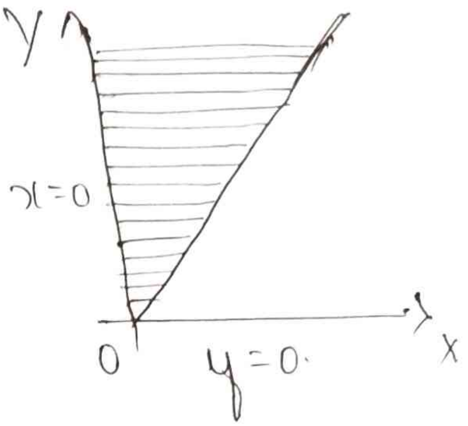
$$\boxed{I = \frac{3}{56}}$$

I.A.T-3

(2) Evaluate the double integral by changing in $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dx\,dy$ by changing the order of integration.

Soln:

$$I = \int_{x=0}^{\infty} \int_{y=x}^{\infty} \frac{e^{-y}}{y}\,dy\,dx.$$



On changing the order, we have.

$$I = \int_{y=0}^{\infty} \int_{x=0}^{y} \frac{e^{-y}}{y}\,dx\,dy$$

$$= \int_0^\infty \frac{e^{-y}}{y}\, x\Big]_0^y\, dy$$

$$= \int_0^\infty \frac{e^{-y}}{y}\, y\, dy$$

$$= \int_0^\infty e^{-y}\,dy$$

$$= -e^{-y}\Big]_0^\infty$$

$$= -\left[e^{-\infty} - e^{0}\right]$$

$$I = 1$$

③. Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \, dz \, dy \, dx$.

Sol^n:-

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \, dz\, dy\, dx.$$

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \left. xy \frac{z^2}{2} \right]_0^{\sqrt{1-x^2-y^2}} dy\, dx.$$

$$I = \frac{1}{2} \int_0^1 \int_0^{\sqrt{1-x^2}} xy(1-x^2-y^2)\, dy\, dx.$$

$$= \frac{1}{2} \int_0^1 \left. \frac{xy^2}{2} - \frac{x^3y^2}{2} - \frac{xy^4}{4} \right]_{y=0}^{\sqrt{1-x^2}} dx.$$

$$= \frac{1}{8} \int_0^1 \left[2x(1-x^2) - 2x^3(1-x^2) - x(1-x^2)^2 \right] dx.$$

$$= \frac{1}{8} \int_0^1 (2x - 2x^3 - 2x^3 + 2x^5 - x + 2x^3 - x^5)\, dx.$$

$$= \frac{1}{8} \int_0^1 (x^5 - 2x^3 + x.)\, dx.$$

$$= \frac{1}{8} \left[\frac{x^6}{6} - \frac{x^4}{2} + \frac{x^2}{2} \right]_{x=0}^{1}$$

$$= \frac{1}{8} \left[\frac{1}{6} - \frac{1}{2} + \frac{1}{2} \right] = \frac{1}{48}$$

$$I = \frac{1}{48}.$$

4 S.T $\bar{F} = (\sin y + z)i + (x\cos y - z)j + (x-y)k$

Sol^n,

$$\nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y + z & x\cos y - z & x-y \end{vmatrix}.$$

$$= i\left[\frac{\partial}{\partial y}(x-y) - \frac{\partial}{\partial z}(x\cos y - z)\right] - j\left[\frac{\partial}{\partial x}(x-y) - \frac{\partial}{\partial z}[\sin y + z]\right]$$
$$+ k\left[\frac{\partial}{\partial x}[x\cos y - z] - \frac{\partial}{\partial y}[\sin y + z]\right]$$

$$= i[-1+1] - j[1-1] + k[\cos y - \cos y]$$

$\nabla \times \bar{F} = 0.$

$\Rightarrow$ $\bar{F}$ is irrotational.

Consider. $\nabla\phi = \bar{F}$

i.e $\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k = (\sin y + z)i + (x\cos y - z)j + (x-y)k.$

$\Rightarrow \frac{\partial\phi}{\partial x} = \sin y + z. \Rightarrow \phi = \int(\sin y + z)\,dx$
$= x(\sin y + z) + f_1(y,z)$ — (1)

$\Rightarrow \frac{\partial\phi}{\partial y} = x\cos y - z \Rightarrow \phi = \int(x\cos y - z)\,dy$
$= x\sin y - zy + f_2(x,z)$ — (2)

$\Rightarrow \frac{\partial\phi}{\partial z}$

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$\Rightarrow \frac{\partial \phi}{\partial z} = x-y \Rightarrow \phi = \int (x-y)dz$

$= xz - yz + f_3(x,y)$ — ③

Let $f_1(y,z) = -yz$, $f_2(x,z) = xz$, $f_3(x,y) = x\sin y$

from ①, ② & ③

$\therefore \boxed{\phi = x\sin y + xz - yz}$ is the required scalar potential

5. Find Div $\vec{F}$ and Curl $\vec{F}$ where $\vec{F} =$
$\vec{F} = \text{grad}(x^3+y^3+z^3-3xyz$.

Sol: given $\vec{F} = \nabla(x^3+y^3+z^3-3xyz)$

$\vec{F} = \frac{\partial}{\partial x}(x^3+y^3+z^3-3xyz)i + \frac{\partial}{\partial y}(x^3+y^3+z^3-3xyz)j + \frac{\partial}{\partial z}(x^3+y^3+z^3-3xyz)k$

$\vec{F} = (3x^2-3yz)i + (3y^2-3xz)j + (3z^2-3xy)k$

Now $\text{div}\vec{F} = \nabla\cdot\vec{F}$

$= \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)\cdot\left((3x^2-3yz)i + (3y^2-3xz)j + (3z^2-3xy)k\right)$

$= \frac{\partial}{\partial x}(3x^2-3yz) + \frac{\partial}{\partial y}(3y^2-3xz) + \frac{\partial}{\partial z}(3z^2-3xy)$

$\therefore \text{div}\vec{F} = 6x+6y+6z$

$= 6(x+y+z)$

I.A.T-3

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}.$$

$$= i\left(\frac{\partial}{\partial y}(3z^2-3xy) - \frac{\partial}{\partial z}(3y^2-3xz)\right)$$
$$-j\left(\frac{\partial}{\partial x}(3z^2-3xy) - \frac{\partial}{\partial z}(3x^2-3yz)\right)$$
$$+k\left(\frac{\partial}{\partial x}(3y^2-3xz) - \frac{\partial}{\partial y}(3x^2-3yz)\right)$$

$$= i(-3x-(-3x)) - j(-3y-(-3y)) + k(-3z-(-3z))$$

$$= 0$$

$\text{Curl } \vec{F} = 0$

6. S.T. $\vec{F} = (y^2-z^2+3yz-2x)i + (3xz+2xy)j + (3xy-2xz+2z)k$ is both solenoidal and irrotational

Sol^n: given $\vec{F} = (y^2-z^2+3yz-2x)i + (3xz+2xy)j + (3xy-2xz+2z)k$.

$$\text{div } \vec{F} = \nabla \cdot \vec{F}$$
$$= \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right) \cdot \left((y^2-z^2+3yz-2x)i + (3xz+2xy)j + (3xy-2xz+2z)k\right)$$

$$= \frac{\partial}{\partial x}(y^2-z^2+3yz-2x) + \frac{\partial}{\partial y}(3xz+2xy) + \frac{\partial}{\partial z}(3xy-2xz+2z)$$

$$= -2 + 2x - 2x + 2$$

$$= 0 \quad \Rightarrow \nabla \cdot \vec{F} = 0$$

$\Rightarrow \vec{F}$ is solenoidal

Curl $\bar{F} = \nabla \times \bar{F}$

$$\nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2-z^2+3yz-2x & 3xz+2xy & 3xy-2xz+2z \end{vmatrix}.$$

$$\nabla \times \bar{F} = i\left[\frac{\partial}{\partial y}(3xy-2xz+2z)-\frac{\partial}{\partial z}(3xz+2xy)\right]$$
$$-j\left[\frac{\partial}{\partial x}(3xy-2xz+2z)-\frac{\partial}{\partial z}(y^2-z^2+3yz-2x)\right]$$
$$+k\left[\frac{\partial}{\partial x}(3xz+2xy)-\frac{\partial}{\partial y}(y^2-z^2+3yz-2x)\right]$$

$$\nabla \times \bar{F} = i\left[3x-3x\right]-j\left[3y-2z+2z-3y\right]$$
$$+k\left[3z+2y-2y-3z\right]$$

$$= i0 - j0 + k\cdot 0$$

$\nabla \times \bar{F} = 0$

$\Rightarrow$ $\bar{F}$ is irrotational

7. Find the directional derivative of $\phi = xy^2+yz^3$ at the point $(2,-1,1)$ in the direction of the vector $i+2j+2k$.

Sol^n :- The Directional derivative is $\nabla\phi \cdot \hat{n}$

$$\hat{n} = \frac{\bar{d}}{|\bar{d}|}$$

given. $\phi = xy^2+yz^3$ $\quad \bar{d} = i+2j+2k$

$$\nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k.$$

$$\nabla\phi = \frac{\partial}{\partial x}(xy^2+yz^3)i+\frac{\partial}{\partial y}(xy^2+yz^3)j+\frac{\partial}{\partial z}(xy^2+yz^3)k.$$

$$\nabla\phi = y^2 i+(2xy+z^3)j+3yz^2 k.$$

$$\hat{n} = \frac{\vec{d}}{|\vec{d}|} = \frac{i+2j+2k}{\sqrt{1+4+4}} = \frac{i+2j+2k}{3}$$

$$\nabla\phi \cdot \hat{n} = y^2$$

$$\nabla\phi_{(2,-1,1)} = 1i+(-4+1)j+(-3)k$$

$$\nabla\phi_{(2,-1,1)} = i-3j-3k$$

$$\nabla\phi \cdot \hat{n} = (i-3j-3k)\cdot\frac{(i+2j+2k)}{3}$$

$$= 1-6-6$$

$$= -\frac{11}{3}$$