CMR INSTITUTE OF TECHNOLOGY	USN

Internal Assessment Test – III FEB 2023

Sub:	Transform Calculus, Fourier Series and Numerical Techniques							Code:	21MAT31
Date:	6-2-2023	Duration:	90 mins	Max Marks:	50	Sem:	III	Branch:	All

	Question 1 is compulsory and Answer any 6 from the remaining questions	Marks	OBE	
			CO	RBT
1	Solve the difference equation $u_{n+2} + 6u_{n+1} + 9u_n = 2^n$, with $u_0 = 0 = u_1$ using Z-transform	[8]	CO3	L3
2	Find the Z-transform of $\cos(\frac{n\pi}{2} + \frac{\pi}{4})$	[7]	CO3	L3
3	Find Z transform of $\frac{1}{n!}$ *and hence find* $z_T \frac{1}{(n+1)!}$ *and* $z_T \frac{1}{(n+2)!}$	[7]	CO3	L3
4	Define Geodesics and prove that geodesics on a plane are straight lines	7	CO5	L3

<table>
<tr><td>5</td><td>Find the extremal of the functional $\int_{x1}^{x2} (y^2 + y'^2 + 2ye^x)dx$</td><td>7</td><td>CO5</td><td>L3</td></tr>
<tr><td>6</td><td>Given $y''= y^3, y(0) = 10, y'(0) = 5$. Evaluate $y(0.1)$ using Runge-Kutta method of order 4</td><td>7</td><td>CO5</td><td>L3</td></tr>
<tr><td>7</td><td>Apply Milne's method to find y(0.4) given $y''+xy'+y = 0$ and the following table of initial values.
<table>
<tr><td>x</td><td>0</td><td>0.1</td><td>0.2</td><td>0.3</td></tr>
<tr><td>y</td><td>1</td><td>0.995</td><td>0.9801</td><td>0.956</td></tr>
<tr><td>y'</td><td>0</td><td>-0.0995</td><td>-0.196</td><td>-0.2867</td></tr>
</table></td><td>[7]</td><td>CO5</td><td>L3</td></tr>
<tr><td>8</td><td>Solve the elliptic equation $\nabla^2 u = 0$ in square region bounded by coordinate axis and the lines x=4, y=4 with boundary conditions given by
$(i) u(0, y) = 0 \; for \; 0 \leq y \leq 4$ $(ii) \; u(4, y) = 12 + y \; for \; 0 \leq y \leq 4$
$(iii) \; u(x, 0) = 3x \; for \; 0 \leq x \leq 4$ (iv) $u(x, 4) = x^2 \; for \; 0 \leq x \leq 4$</td><td>7</td><td>CO4</td><td>L3</td></tr>
</table>

[73] *Solve the difference equation* $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$ *with* $y_0 = y_1 = 0$ *using Z-transforms.* **[Dec 2018]**

☞ Taking Z-transforms on both sides of the given equation we have,

$$Z_T(y_{n+2}) + 6Z_T(y_{n+1}) + 9Z_T(y_n) = Z_T(2^n)$$

ie., $$z^2[\bar{y}(z) - y_0 - y_1 z^{-1}] + 6z[\bar{y}(z) - y_0] + 9\bar{y}(z) = \frac{z}{z-2}$$

ie., $[z^2 + 6z + 9]\,\bar{y}(z) = \dfrac{z}{z-2}$, by using the initial values.

or $$\bar{y}(z) = \frac{z}{(z-2)(z+3)^2}$$

Let $$\frac{z}{(z-2)(z+3)^2} = A \cdot \frac{z}{z-2} + B \cdot \frac{z}{z+3} + C \cdot \frac{z}{(z+3)^2}$$

or $$1 = A(z+3)^2 + B(z-2)(z+3) + C(z-2)$$

Put $z = 2$: $1 = A(25)$ $\therefore$ $A = 1/25$

Put $z = -3$: $1 = C(-5)$ $\therefore$ $C = -1/5$

Equating the coefficient of z^2 on both sides we get, $0 = A + B$ $\therefore$ $B = -1/25$

Hence, $$\bar{y}(z) = \frac{1}{25} \cdot \frac{z}{z-2} - \frac{1}{25}\frac{z}{z+3} - \frac{1}{5}\frac{z}{(z+3)^2}$$

or $$\bar{y}(z) = \frac{1}{25} \cdot \frac{z}{z-2} - \frac{1}{25} \cdot \frac{z}{z+3} - \frac{1}{5} \cdot \frac{1}{-3} \frac{-3z}{(z+3)^2}$$

$$\Rightarrow \quad Z_T^{-1}[\bar{y}(z)] = \frac{1}{25} Z_T^{-1}\left[\frac{z}{z-2}\right] - \frac{1}{25} Z_T^{-1}\left[\frac{z}{z+3}\right] + \frac{1}{15} Z_T^{-1}\left[\frac{-3z}{(z+3)^2}\right]$$

ie., $$y_n = \frac{1}{25}(2)^n - \frac{1}{25}(-3)^n + \frac{1}{15}(-3)^n \cdot n$$

Thus $\boxed{y_n = \frac{1}{5}\left\{\frac{1}{5}(2)^n - \frac{1}{5}(-3)^n + \frac{1}{3}(-3)^n \cdot n\right\}}$ is the required solution.

[39] *Find the Z - transform of* $\cos(n\pi/2 + \pi/4)$

☞ Let $u_n = \cos(n\pi/2 + \pi/4)$

$$= \cos(n\pi/2)\cos(\pi/4) - \sin(n\pi/2)\sin(\pi/4)$$

ie., $$u_n = \frac{1}{\sqrt{2}}[\cos(n\pi/2) - \sin(n\pi/2)]$$

$$\therefore \quad Z_T(u_n) = \frac{1}{\sqrt{2}}\left[Z_T\cos(n\pi/2) - Z_T\sin(n\pi/2)\right]$$

Consider, $e^{i(n\pi/2)} = (e^{i\pi/2})^n = k^n\,(say)$ where $k = e^{i\pi/2}$.

We know that, $Z_T(k^n) = \dfrac{z}{z-k}$ and hence we have,

$$Z_T(e^{in\pi/2}) = \frac{z}{z - e^{i\pi/2}} = \frac{z}{z - \cos(\pi/2) - i\sin(\pi/2)} = \frac{z}{z-i}$$

ie., $$Z_T(e^{in\pi/2}) = \frac{z(z+i)}{(z-i)(z+i)} = \frac{z^2+iz}{z^2+1}$$

ie., $$Z_T[\cos(n\pi/2) + i\sin(n\pi/2)] = \frac{z^2}{z^2+1} + i\frac{z}{z^2+1}$$

$$\Rightarrow \quad Z_T[\cos(n\pi/2)] = \frac{z^2}{z^2+1} \text{ and } Z_T[\sin(n\pi/2)] = \frac{z}{z^2+1}$$

We substitute these results in (1).

Thus, $$\boxed{Z_T(u_n) = \frac{1}{\sqrt{2}}\left[\frac{z^2}{z^2+1} - \frac{z}{z^2+1}\right] = \frac{z(z-1)}{\sqrt{2}(z^2+1)}}$$

where $u_n = \cos(n\pi/2 + n\pi/4)$

[38] *Show that* $Z_T\left[\frac{1}{n!}\right] \doteq e^{1/z}$. *Hence find* $Z_T\left[\frac{1}{(n+1)!}\right]$ *and* $Z_T\left[\frac{1}{(n+2)!}\right]$

☞ By the definition,

$$Z_T\left[\frac{1}{n!}\right] = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} = 1 + \frac{z^{-1}}{1!} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \cdots$$

But, $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ and here we have $x = z^{-1}$

Thus, $\boxed{Z_T\left[\frac{1}{n!}\right] = e^{z^{-1}} = e^{1/z}}$

We have the properties,

$$Z_T(u_{n+1}) = z\left[\bar{u}(z) - u_0\right] \quad \ldots (1)$$

$$Z_T(u_{n+2}) = z\left[\bar{u}(z) - u_0 - u_1 z^{-1}\right] \quad \ldots (2)$$

Let, $u_n = \frac{1}{n!}$ $\therefore$ $Z_T(u_n) = \bar{u}(z) = e^{1/z}$.

Also, $u_0 = \frac{1}{0!} = 1$ and $u_1 = \frac{1}{1!} = 1$

Thus by using these results in (1) and (2) we obtain,

$$\boxed{\begin{aligned} Z_T\left[\frac{1}{(n+1)!}\right] &= z\left[e^{1/z} - 1\right] \\ Z_T\left[\frac{1}{(n+2)!}\right] &= z\left[e^{1/z} - 1 - z^{-1}\right]. \end{aligned}}$$

- **Geodesics**

Given two arbitrary points P and Q on a surface S, there exists infinite number of curves on the surface having P and Q as their extremities. Of these curves that curve whose length is the least is called the ***geodesic*** between the points P and Q on the given surface.

In other words, **a geodesic on a surface is a curve along which the distance between any two points of the surface is a minimum.**

Finding the geodesic on a surface is a variational problem involving the condition for the extremum of the associatied functional.

5.24 Standard variational problems.

[31] *Prove that the shortest distance between two points in a plane is along the straight line joining them or porve that the geodesics on a plane are straight lines.*

[June 2017, Dec 16, 18]

☞ Let $y = y\,(\,x\,)$ be a curve joining two points $P(x_1\,,y_1)$ and $Q(x_2\,,y_2)$ in the XOY plane.

We know that the arc length between P and Q is given by

$$s = \int_{x_1}^{x_2} \frac{ds}{dx}\,dx = \int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

ie., $$s = I = \int_{x_1}^{x_2} \sqrt{1+y'^2}\,dx$$

We need to find the curve $y\,(\,x\,)$ such that I is minimum.

Let, $f\,(x, y, y') = \sqrt{1+y'^2}$

Euler's equation, $\dfrac{\partial f}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'}\right) = 0$ becomes,

$$0-\frac{d}{dx}\left[\frac{2y'}{2\sqrt{1+y'^2}}\right]=0$$

or $$\frac{d}{dx}\left[\frac{y'}{\sqrt{1+y'^2}}\right]=0$$

ie., $$y''\sqrt{1+y'^2}-y'\frac{2y'y''}{2\sqrt{1+y'^2}}=0,$$ by quotient rule and cross multiplying.

ie., $$y''(1+y'^2)-y''y'^2=0 \text{ or } y''=0.$$

ie., $$\frac{d^2y}{dx^2}=0$$

Let us integrate twice w.r.t x

Thus $\boxed{y = c_1x + c_2}$ which is a straight line.

[15] *Find the extremal of the functional* $\int_{x_1}^{x_2} (y^2 + y'^2 + 2ye^x)\, dx$

☞ Let, $f(x, y, y') = y^2 + y'^2 + 2ye^x$

Euler's equation, $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ becomes,

$$(2y + 2e^x) - \frac{d}{dx}(2y') = 0 \quad \text{or} \quad y + e^x - y'' = 0$$

ie., $y'' - y = e^x$ or $(D^2 - 1)y = e^x$ where $D = \frac{d}{dx}$

AE is $m^2 - 1 = 0 \quad \therefore \; m = \pm 1$

Hence, $CF = y_c = c_1 e^x + c_2 e^{-x}$

$$PI = y_p = \frac{e^x}{D^2 - 1} = \frac{e^x}{0},$$ on replacing D by 1.

$$y_p = x\frac{e^x}{2D} = \frac{x\,e^x}{2}$$

We have, $y = y_c + y_p$

Thus, $\boxed{y = c_1 e^x + c_2 e^{-x} + x e^x/2}$

[3] *Compute* $y\ (\ 0.1\ )$ *given* $\frac{d^2y}{dx^2} = y^3$ *and* $y = 10,\ \frac{dy}{dx} = 5$ *at* $x\ = 0$ *by Runge-Kutta method of fourth order.*

☞ Putting $\frac{dy}{dx} = z$ and differentiating w.r.t x we obtain $\frac{d^2y}{dx^2} = \frac{dz}{dx}$ so that the given equation assumes the form $\frac{dz}{dx} = y^3$. Hence we have a system of equations:

$$\frac{dy}{dx} = z\ ;\ \frac{dz}{dx} = y^3 \text{ where } y = 10, z = 5,\ x = 0.$$

Let, $f(x, y, z) = z,\ g(x, y, z) = y^3,\ x_0 = 0,\ y_0 = 10,\ z_0 = 5$ and $h = 0.$

We shall first compute the following :

$$k_1 = h f(x_0, y_0, z_0) = (0.1) f(0, 10, 5) = (0.1)5 = 0.5$$

$$l_1 = (0.1)[10^3] = 100$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$k_2 = (0.1)\,f(0.05,\ 10.25,\ 55) = (0.1)(55) = 5.5$$

$$l_2 = (0.1)[(10.25)^3] = 107.7$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2},\ z_0 + \frac{l_2}{2}\right)$$

$$k_3 = (0.1)\,f(0.05,\ 12.75,\ 58.85) = (0.1)(58.85) = 5.885$$

$$l_3 = (0.1)(12.75)^3 = 207.27$$

$$k_4 = h\,f(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$k_4 = (0.1)\,f(0.1,\ 15.885,\ 212.27) = (0.1)(212.27) = 21.227$$

$$l_4 = (0.1)(15.885)^3 = 400.83$$

We have, $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$\therefore$ $y(0.1) = 10 + \frac{1}{6}[\,0.5 + 2(5.5) + 2(5.885) + 21.227]$

Thus, $\boxed{y(0.1) = 17.4162}$

[9] *Given the ODE $y'' + xy' + y = 0$ and the following table of initial values, compute $y\ (0.4)$ by applying Milne's method.*

x	0	0.1	0.2	0.3
y	1	0.995	0.9801	0.956
y'	0	–0.0995	–0.196	–0.2867

☞ Putting $y' = z$, we get $y'' = z'$.

Also we have, $z' = -(xz + y)$ from the given equation.

Further, $z'\ (0) = -[0+1] = -1$

$$z'\ (0.1) = -[(0.1)(-0.0995) + 0.995] = -0.985$$

$$z'(0.2) = -[(0.2)(-0.196) + 0.9801] = -0.941$$

$$z'\ (0.3) = -[(0.3)(-0.2867) + 0.956] = -0.87$$

We also have the following table.

x	$x_0 = 0$	$x_1 = 0.1$	$x_2 = 0.2$	$x_3 = 0.3$
y	$y_0 = 1$	$y_1 = 0.995$	$y_2 = 0.9801$	$y_3 = 0.956$
$y' = z$	$z_0 = 0$	$z_1 = -0.0995$	$z_2 = -0.196$	$z_3 = -0.2867$
$y'' = z'$	$z'_0 = -1$	$z'_1 = -0.985$	$z'_2 = -0.941$	$z'_3 = -0.87$

We first consider Milne's predictor formulae,

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3)$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}(2z'_1 - z'_2 + 2z'_3)$$

On substituting the appropriate values from the table we obtain

$$y_4^{(P)} = 0.9231 \text{ and } z_4^{(P)} = -0.3692$$

Next we consider Milne's corrector formulae,

$$y_4^{(C)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$$

$$z_4^{(C)} = z_2 + \frac{h}{3}(z_2' + 4z_3' + z_4')$$

We have, $z_4' = -(x_4 z_4^{(P)} + y_4^{(P)}) = -[(0.4)(-0.3692) + 0.9231] = -0.7754$

Hence by substituting the appropriate values in the corrector formulae we obtain

$$y_4^{(C)} = 0.9230 \text{ and } z_4^{(C)} = -0.3692$$

Thus the required, $\boxed{y(0.4) = 0.923}$

14. *Solve* $\nabla^2 u = 0$ *in the square region bounded by the co ordinate axes and the lines* $x = 4,\ y = 4$ *with the boundary conditions given by the analytical expressions,*

(i) $u(0, y) = 0$ *for* $0 \le y \le 4$

(ii) $u(4, y) = 12 + y$ *for* $0 \le y \le 4$

(iii) $u(x, 0) = 3x$ *for* $0 \le x \le 4$

(iv) $u(x, 4) = x^2$ *for* $0 \le x \le 4$

Also employ Liebmann's iteration process to compute the second iterative values of $u(x, y)$ *correct to two decimal places.*

>> We have $\nabla^2 u = 0$ represented by $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ in two dimensions.

We shall divide the square region into 16 squares of side one unit.

We shall derive the values of $u(x, y)$ on the boundary from the given expressions.

(i) $u(0, y) = 0 \Rightarrow u(0, 1) = 0 = u(0, 2) = u(0, 3) = u(0, 4)$

(ii) $u(4, y) = 12 + y \Rightarrow u(4, 0) = 12\ ;\ u(4, 1) = 13\ ;$

$u(4, 2) = 14\ ;\ u(4, 3) = 15\ ;\ u(4, 4) = 16$

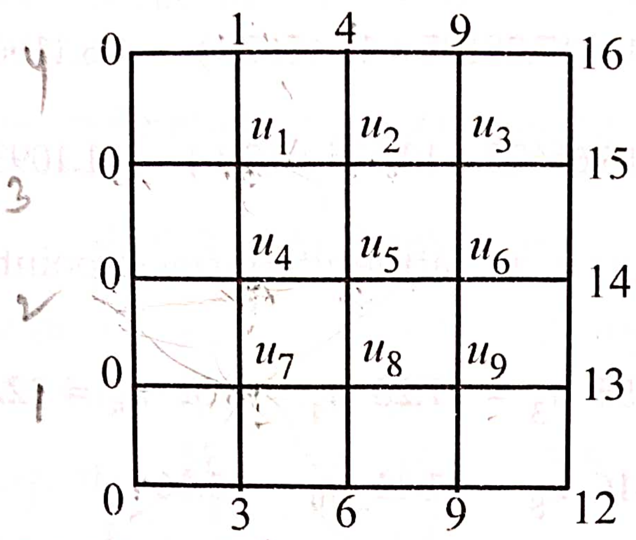
(iii) $u(x, 0) = 3x \Rightarrow u(0, 0) = 0\ ;\ u(1, 0) = 3\ ;$

$u(2, 0) = 6\ ;\ u(3, 0) = 9\ ;\ u(4, 0) = 12$

(iv) $u(x, 4) = x^2 \Rightarrow u(0, 4) = 0\ ;\ u(1, 4) = 1\ ;$

$u(2, 4) = 4\ ;\ u(3, 4) = 9\ ;\ u(4, 4) = 16.$

We shall represent these values on the square region and let $u_1, u_2, \cdots u_9$ be the interior mesh points of the region.



u_5 is located at the centre of the region.

$$\therefore \quad u_5^{(1)} = \frac{1}{4}(0+14+4+6) = 6 \quad \text{by applying S.F}$$

Next we apply D.F to compute u_7, u_9, u_1, u_3

$$u_7^{(1)} = \frac{1}{4}(0+6+0+6) = 3 \quad ; \quad u_9^{(0)} = \frac{1}{4}(6+14+6+12) = 9.5$$

$$u_1^{(0)} = \frac{1}{4}(0+4+0+6) = 2.5 \quad ; \quad u_3^{(0)} = \frac{1}{4}(6+16+4+14) = 10$$

Now we shall compute u_2, u_4, u_6, u_8 by S.F

$$u_2^{(0)} = \frac{1}{4}(2.5+10+4+6) = 5.625$$

$$u_4^{(0)} = \frac{1}{4}(0+6+2.5+3) = 2.875$$

$$u_6^{(0)} = \frac{1}{4}(6+14+10+9.5) = 9.875$$

$$u_8 = \frac{1}{4}(3+9.5+6+6) = 6.125$$

These values are regarded as the initial approximations to commence the Liebmann's iterative process for greater accuracy. We compute $u_i\ (i = 1 \text{ to } 9)$ in the serial order by using the latest iterative value on hand by applying the standard five point formula only.