CMR INSTITUTE OF TECHNOLOGY	USN										

Internal Assessment Test – II December 2022

Sub:	Transform Calculus, Fourier Series and Numerical Techniques							Code:	21MAT31
Date:	26/12/2022	Duration:	90 mins	Max Marks:	50	Sem:	III	Branch:	

Question 1 is compulsory and Answer any 6 from the remaining questions.

		Marks	OBE CO	OBE RBT
1	Find the Fourier transform of $f(x) = f(x) = \begin{cases} 1-x^2, if \quad x<0 \\ 0, \quad if \quad x>0 \end{cases}$ and hence evaluate $\int_0^{\infty} \frac{x\cos x - \sin x}{x^3} \cos\frac{x}{2}\, dx$	[8]	CO2	L3
2	Obtain the Fourier series expansion for the function $f(x) = e^{-x}$ in $[0,2\pi]$	[7]	CO2	L3
3	Find the Fourier series of $f(x) = x\sin x$ in $[-\pi,\pi]$	[7]	CO2	L3
4	Find the sine series of $f(x)=\cos x$, in $(0,\pi)$ Find $f(x)$ at $x=\frac{\pi}{2}$.	[7]	CO2	L3

<table>
<tr><td>5</td><td>Find the cosine series of $f(x) = \begin{cases} kx, & 0 \le x \le \frac{l}{2} \\ k(l-x) & \frac{l}{2} < x \le l \end{cases}$ where k is a non- integer positive constant</td><td>[7]</td><td>CO2</td><td>L3</td></tr>
<tr><td>6</td><td>Find the Fourier transform of $f(x) = xe^{-|x|}$</td><td>[7]</td><td>CO2</td><td>L3</td></tr>
<tr><td>7</td><td>Find the Fourier expansion of $f(x) = \begin{cases} 2-x, & 0 < x < 4 \\ x-6, & 4 \le x < 8 \end{cases}$</td><td>[7]</td><td>CO2</td><td>L3</td></tr>
<tr><td>8</td><td>Find constant term and first harmonic by expanding y as a Fourier series if the values of y are given by
<table>
<tr><td>x</td><td>0</td><td>$\frac{\pi}{6}$</td><td>$\frac{\pi}{3}$</td><td>$\frac{\pi}{2}$</td><td>$\frac{2\pi}{3}$</td><td>$\frac{5\pi}{6}$</td></tr>
<tr><td>y</td><td>1.98</td><td>1.30</td><td>1.05</td><td>1.30</td><td>-0.88</td><td>-0.25</td></tr>
</table></td><td>[7]</td><td>CO2</td><td>L3</td></tr>
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IAT-II Solutions

1. $$F(u) = \int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx$$

$$= \int_{-\infty}^{0} (1-x^2)\, e^{iux}\, dx$$

$$= \left[(1-x^2)\frac{e^{iux}}{iu} - (-2x)\frac{e^{iux}}{i^2u^2} + (-2)\frac{e^{iux}}{i^3u^3} \right]_{x=-\infty}^{0}$$

$$= \left[\left\{ \frac{1}{iu} - \frac{2}{(-i)u^3} \right\} - \left\{ 0 \right\} \right]$$

$$F(u) = \frac{1}{iu}\left(1+\frac{2}{u^2}\right) = \frac{-i}{u}\left(\cancel{1}+\frac{2}{u^2}\right)$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-i}{u}\left(1+\frac{2}{u^2}\right) e^{-iux}\, du$$

$(\cos ux - i \sin ux)$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-i}{u}\left(1+\frac{2}{u^2}\right)\left(\cos ux - i\sin ux\right) du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \underset{o}{\frac{-i}{u}}\underset{e}{\left(1+\frac{2}{u^2}\right)} \underset{e}{\cos(ux)}\, du$$

$$- \frac{1}{2\pi}\int_{-\infty}^{\infty} \cancel{0}\,\underbrace{\underset{o}{\frac{1}{u}}\underset{e}{\left(1+\frac{2}{u^2}\right)} \underset{o}{\sin(ux)}\, du}$$

$$= -\frac{1}{\pi}\int_{0}^{\infty} \underbrace{\frac{1}{u}\left(1+\frac{2}{u^2}\right) \sin(ux)\, du}$$

Evaluation cannot be done for the given limits.

2) $f(x) = e^{-x}$ in $[0, 2\pi]$

The required Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx = -\frac{1}{\pi}\left(e^{-x}\right)_0^{2\pi}$$

$$= \frac{1}{\pi}\left(1 - e^{-2\pi}\right)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$$

$$= \frac{1}{\pi}\left\{\frac{e^{-x}}{n^2+1}\left[-\cos nx + n\sin nx\right]_0^{2\pi}\right\}$$

$$= \frac{1 - e^{-2\pi}}{\pi(n^2+1)}$$

and

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{\frac{e^{-x}}{n^2+1}\left[-n\cos nx - \sin nx\right]_0^{2\pi}\right\}$$

$$= \frac{n(1 - e^{-2\pi})}{\pi(n^2+1)}$$

$\therefore f(x) = e^{-x} = \frac{1-e^{-2\pi}}{\pi}\left\{\frac{1}{2} + \sum_{n=1}^{\infty} \frac{\cos nx + n\sin nx}{n^2+1}\right\}$

3) $f(x) = x\sin x$ in $[-\pi, \pi]$

$f(-x) = (-x)\sin(-x) = x\sin x = f(x)$

$\Rightarrow f(x)$ is an even function.

$\therefore b_n = 0$.

Fourier series in $[-\pi, \pi]$ is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin x\,dx = \frac{2}{\pi}\Big[x(-\cos x) + \sin x\Big]_0^{\pi}$$

$$= \frac{2}{\pi} \times \pi = 2.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\,dx$$

This gives $a_1 = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos x\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - \left(-\frac{\sin 2x}{2^2}\right)\right]_0^{\pi} = \frac{1}{\pi}\left(-\frac{\pi}{2} + 0\right)$$

$$= -\frac{1}{2}$$

For $n \geq 2$, we obtain

$$a_n = \frac{1}{\pi}\int_0^{\pi} x\left\{\sin(n+1)x - \sin(n-1)x\right\}dx$$

$$= \frac{1}{\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - \left\{-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left\{-\frac{\pi(-1)^{n+1}}{n+1} + \frac{\pi(-1)^{n-1}}{n-1} + 0\right\}$$

$$= (-1)^{n-1}\left(\frac{1}{n-1} - \frac{1}{n+1}\right) = \frac{2(-1)^{n-1}}{n^2-1}$$

$$\therefore f(x) = 1 - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n^2-1}\cos nx.$$

4) $f(x) = \cos x$ in $(0, \pi)$.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx = \frac{2}{\pi}\int_0^{\pi} \cos x \sin nx\, dx$$

For $n=1$, this yields

$$b_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x\, dx = \frac{2}{\pi}\left(\frac{\sin^2 x}{2}\right)_0^{\pi} = 0.$$

For $n>1$,

$$b_n = \frac{1}{\pi}\int_0^{\pi}\left\{\sin(n+1)x + \sin(n-1)x\right\} dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= -\frac{1}{\pi}\left[\frac{1}{n+1}\left\{(-1)^{n+1}-1\right\} + \frac{1}{n-1}\left\{(-1)^{n-1}-1\right\}\right]$$

$$= -\frac{1}{\pi}\left\{(-1)^{n+1}-1\right\}\left\{\frac{1}{n+1}+\frac{1}{n-1}\right\}$$

$$= \frac{1}{\pi}\left\{(-1)^{n}+1\right\}\left(\frac{2n}{n^2-1}\right)$$

$$= \frac{2n\left[1+(-1)^n\right]}{\pi(n^2-1)}$$

$$\therefore f(x) = \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{n\left[1+(-1)^n\right]}{n^2-1}\sin nx.$$

5) $$f(x) = \begin{cases} kx, & 0 \le x \le l/2 \\ k(l-x), & \frac{l}{2} < x \le l \end{cases}$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\left\{\int_0^{l/2} kx\,dx + \int_{l/2}^{l} k(l-x)\,dx\right\}$$

$$= \frac{2k}{l}\left\{\left[\frac{x^2}{2}\right]_0^{l/2} - \left[\frac{(l-x)^2}{2}\right]_{l/2}^{l}\right\}$$

$$= \frac{2k}{l}\left(\frac{l^2}{8}+\frac{l^2}{8}\right) = \frac{1}{2}kl$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi}{l}\right)x\,dx$$

$$= \frac{2k}{l}\left\{\int_0^{l/2} x\cos\left(\frac{n\pi}{l}\right)x\,dx + \int_{l/2}^{l}(l-x)\cos\left(\frac{n\pi}{l}\right)x\,dx\right\}$$

$$= \frac{2k}{l}\left\{\left[x\left\{\frac{l}{n\pi}\sin\left(\frac{n\pi}{l}\right)x\right\} - \left\{-\frac{l^2}{n^2\pi^2}\cos\left(\frac{n\pi}{l}\right)x\right\}\right]_0^{l/2}\right.$$

$$\left. + \left[(l-x)\left\{\frac{l}{n\pi}\sin\left(\frac{n\pi}{l}\right)x\right\} + \left\{-\frac{l^2}{n^2\pi^2}\cos\left(\frac{n\pi}{l}\right)x\right\}\right]_{l/2}^{l}\right\}$$

$$= \frac{2k}{l}\left\{\frac{l}{2}\cdot\frac{l}{n\pi}\sin\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)\right.$$

$$\left. - \frac{l}{2}\cdot\frac{l}{n\pi}\sin\frac{n\pi}{2} - \frac{l^2}{n^2\pi^2}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)\right\}$$

$$= \frac{2kl}{n^2\pi^2}\left\{2\cos\frac{n\pi}{2} - 1 - (-1)^n\right\}$$

Fourier cosine series is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{l} x$$

$$= \frac{kl}{4} + \frac{2kl}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left\{ 2\cos\frac{n\pi}{2} - 1 - (-1)^n \right\} \cos\frac{n\pi}{l}$$

6) $f(x) = xe^{-|x|} = \begin{cases} xe^{x} & \text{for } x<0 \\ xe^{-x} & \text{for } x>0 \end{cases}$

$$F(u) = \int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx$$

$$= \int_{-\infty}^{0} x e^{x} e^{iux}\, dx + \int_{0}^{\infty} x e^{-x} e^{iux}\, dx$$

$$= \int_{-\infty}^{0} x e^{(1+iu)x}\, dx + \int_{0}^{\infty} x e^{-(1-iu)x}\, dx$$

$$= \left[x\frac{e^{(1+iu)x}}{1+iu} - \frac{e^{(1+iu)x}}{(1+iu)^2} \right]_{-\infty}^{0}$$

$$+ \left[x\frac{e^{-(1-iu)x}}{-(1-iu)} - \frac{e^{-(1-iu)x}}{(1-iu)^2} \right]_{0}^{\infty}$$

$$F(u) = \frac{1}{(1-iu)^2} - \frac{1}{(1+iu)^2} = \frac{4iu}{(1+u^2)^2}.$$

7) $(0, 2l) = (0, 8)$

$\Rightarrow l = 4.$

$\phi(x) = 2 - x, \quad \psi(x) = x - 6$

$\phi(8 - x) = 2 - 8 + x$

$= x - 6 = \psi(x)$

$\Rightarrow f(x)$ is an even function.

$\therefore b_n = 0.$

$$a_0 = \frac{1}{2}\int_0^4 f(x)\,dx = \frac{1}{2}\int_0^4 (2-x)\,dx = 0.$$

$$a_n = \frac{1}{2}\int_0^4 (2-x)\cos\frac{n\pi x}{4}\,.\,dx$$

$$= \frac{1}{2}\left[(2-x).\frac{\sin\frac{n\pi x}{4}}{n\pi/4} - (-1)\frac{-\cos\frac{n\pi x}{4}}{(n\pi/4)^2}\right]_0^4$$

$$= -\frac{8}{n^2\pi^2}\left\{\cos n\pi - 1\right\}$$

$$a_n = +\frac{8}{n^2\pi^2}\left\{1 - (-1)^n\right\}$$

$\therefore$ Fourier series is,

$$f(x) = \sum_{n=1}^{\infty} \frac{8}{n^2\pi^2}\left\{1 - (-1)^n\right\}\cos\left(\frac{n\pi x}{4}\right).$$

8)

x	y	$\pi x/l = 2x$	$y\cos 2x$	$y\sin 2x$
0	1.98	0	1.98	0
$\pi/6$	1.3	$\pi/3$	0.65	1.126
$\pi/3$	1.05	$2\pi/3$	-0.525	0.9093
$\pi/2$	1.3	π	-1.3	0
$2\pi/3$	-0.88	$4\pi/3$	0.44	0.7621
$5\pi/6$	-0.25	$5\pi/3$	-0.125	0.2165
	$\sum y = 4.5$	$\sum =$	1.12	3.0139

Interval = $(0, \pi) = (0, 2l)$

$\Rightarrow l = \pi/2 \ , \ N = 6$

$$a_0 = 2/N \sum y = \frac{2}{6}(4.5) = 1.5$$

$$a_1 = \frac{2}{N}\sum y \cos\left(\frac{\pi x}{l}\right) = \frac{2}{6}(1.12) = 0.3733$$

$$b_1 = \frac{2}{N}\sum y \sin\left(\frac{\pi x}{l}\right) = \frac{2}{6}(3.0139) = 1.0046$$

Fourier series is $f(x) = \frac{a_0}{2} + (a_1 \cos 2x + b_1 \sin 2x)$

$$= \underbrace{0.75}_{\text{constant term}} + \underbrace{(0.3733 \cos 2x + 1.0046 \sin 2x)}_{\text{first harmonic}}$$