CMR INSTITUTE OF TECHNOLOGY		USN							CMRIT
Internal Assessment Test – I December 2022									
Sub:	Transform Calculus, Fourier Series and Numerical Techniques						Code:	21MAT31	
Date:	01/12/2022	Duration:	90 mins	Max Marks:	50	Sem:	III	Branch:	**CSE/ISE/ECE only**

Question 1 is compulsory and Answer any 6 from the remaining questions.

		Marks	OBE CO	OBE RBT
1	Solve by using Laplace transform techniques: $y'' - 3y' + 2y = e^{3t}, y(0) = 1, y'(0) = -1$.	[8]	CO1	L3
2	Find the Laplace transform of the triangular wave of period 2a given by $f(t) = \begin{cases} t, & 0 < t < a \\ 2a - t, & a < t < 2a \end{cases}$	[7]	CO1	L3
3	Express the following function in terms of the unit step function and hence find its Laplace transform $f(t) = \begin{cases} 1, & 0 < t < 1 \\ 2t, & 1 < t < 2 \\ 3t, & 2 < t < 3 \end{cases}$	[7]	CO1	L3
4	Using the convolution theorem find the inverse Laplace transform of $\frac{1}{(s^2+1)(s^2+9)}$.	[7]	CO1	L3

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1. $y''(t) - 3y'(t) + 2y = e^{3t}$, $y(0) = 1$,
$y'(0) = -1$

$$L[y''(t)] - 3L[y'(t)] + 2L[y(t)] = L(e^{3t})$$

$$\{s^2 Y(s) - s y(0) - y'(0)\} - 3\{s Y(s) - y(0)\} + 2Y(s) = \frac{1}{s-3}$$

$$(s^2 - 3s + 2) Y(s) - s + 1 + 3 = \frac{1}{s-3}$$

$$(s^2 - 3s + 2) Y(s) - s + 4 = \frac{1}{s-3}$$

$$(s^2 - 3s + 2) Y(s) = \frac{1}{s-3} + s - 4$$

$$Y(s) = \frac{1}{(s-3)(s-1)(s-2)} + \frac{s-4}{(s-2)(s-1)} \quad \text{(A)}$$

$$\frac{1}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3} \quad \text{(I)}$$

$$1 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2)$$

$s=1$: $1 = A(-1)(-2) \Rightarrow A = \frac{1}{2}$

$s=2$: $1 = B(1)(-1)$ $B = -1$

$s=3$: $1 = C(2)$ $C = 1/2$

$$\frac{s-4}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}$$

$$s-4 = A(s-2) + B(s-1)$$

$s=1$ $\quad -3 = -A$

$s=2$ $\quad -2 = B$

$A = 3$ $\quad B = -2$

$$\frac{s-4}{(s-1)(s-2)} = \frac{3}{s-1} - \frac{2}{s-2} \quad \text{— (II)}$$

(I) is $Y(s) = \left\{\frac{1}{2}\frac{1}{s-1} - \frac{1}{s-2} + \frac{1}{2}\frac{1}{s-3}\right\}$

$$+ \left\{\frac{3}{s-1} - \frac{2}{s-2}\right\}$$

$$Y(s) = \frac{7}{2}\frac{1}{s-1} - \frac{3}{s-2} + \frac{1}{2}\frac{1}{s-3}$$

Take inverse $y(t) = \frac{7}{2}e^{t} - 3e^{2t} + \frac{1}{2}e^{3t}$

Q2 $\quad L[f(t)] = \frac{1}{1-e^{-2as}} \int_{t=0}^{2a} f(t)e^{-st}\,dt$

$$F(s) = \frac{1}{1-e^{-2as}}\left[\int_{t=0}^{a} t e^{-st}\,dt + \int_{t=a}^{2a} (2a-t)e^{-st}\,dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left\{t\frac{e^{-st}}{-s} - 1.\frac{e^{-st}}{s^2}\right\}_{t=0}^{a} + \left\{(2a-t)\frac{e^{-st}}{-s} - (-1)\frac{e^{-st}}{s^2}\right\}_{t=a}^{2a}\right]$$

$$F(s) = \frac{1}{1-e^{-2as}} \left[-\frac{1}{s}\left(ae^{-as} - 0\right) - \frac{1}{s^2}\left(e^{-as} - 1\right) - \frac{1}{s}\left(0 - ae^{-as}\right) + \frac{1}{s^2}\left(e^{-2as} - e^{-as}\right) \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\frac{1}{s^2}\left(1 - 2e^{-as} + e^{-2as}\right) \right]$$

$$= \frac{\left(1-e^{-as}\right)^2}{s^2\left(1-e^{-as}\right)\left(1+e^{-as}\right)} = \frac{1-e^{-as}}{s^2\left(1+e^{-as}\right)}$$

$$= \left(\frac{1-e^{-as}}{1+e^{-as}}\right) \frac{1}{s^2} \frac{e^{as/2}}{e^{as/2}} = \frac{1}{s^2} \frac{e^{as/2} - e^{-as/2}}{e^{as/2} + e^{-as/2}}$$

$$= \frac{1}{s^2} \tanh\left(\frac{as}{2}\right) \;//$$

Q3 $f(t) = f_1(t) + \{f_2(t) - f_1(t)\} u(t-a)$
$+ \{f_3(t) - f_2(t)\} u(t-b)$

$f(t) = 1 + (2t-1) u(t-1) + (3t-2t) u(t-2)$
$f(t) = 1 + (2t-2+1) u(t-1) + t\, u(t-2)$
$f(t) = 1 + 2(t-1) u(t-1) + u(t-1)$
$+ (t-2+2) u(t-2)$

$f(t) = 1 + 2(t-1) u(t-1) + u(t-1)$
$+ (t-2) u(t-2) + 2u(t-2)$

Take LT
$F(s) = \frac{1}{s} + 2e^{-s} \frac{1}{s^2} + \frac{e^{-s}}{s} + e^{-2s} \cdot \frac{1}{s^2}$
$+ \frac{2e^{-2s}}{s}$

Q4 $H(s) = \dfrac{1}{(s^2+1)(s^2+9)}$

$$\frac{1}{s^2+1} \cdot \frac{1}{s^2+9} \qquad G(s) = \frac{1}{s^2+9}$$

Let $F(s) = \dfrac{1}{s^2+1}$

$$f(t) = \sin t \qquad g(t) = \frac{1}{3}\sin 3t$$

$$L^{-1}\left[F(s)\,G(s)\right] = f(t) * g(t)$$

$$\sin t * \frac{1}{3}\sin 3t$$

$$= \frac{1}{3}\int_{u=0}^{t} \sin \overset{B}{u}\, \sin 3(\overset{A}{t} - u)\, du$$

$$= \frac{1}{3\times 2}\int_{u=0}^{t}\left\{\cos(3t-4u) - \cos(3t-2u)\right\} du$$

$$= \frac{1}{6}\left(\frac{\sin(3t-4u)}{-4}\right)_{u=0}^{t} - \frac{1}{6}\left(\frac{\sin(3t-2u)}{-2}\right)_{u=0}^{t}$$

$$= -\frac{1}{24}\left\{\sin(-t) - \sin 3t\right\} + \frac{1}{12}\left\{\sin t - \sin 3t\right\}$$

$$= \sin t\left(\frac{1}{12} + \frac{1}{24}\right) + \sin 3t\left(\frac{1}{24} - \frac{1}{12}\right)$$

$$= \frac{1}{8}\sin t - \frac{1}{24}\sin 3t = \frac{3\sin t - \sin 3t}{24}$$

Q5 i) $F(s) = \frac{(s^2-1)^2}{s^5} = \frac{s^4 - 2s^2 + 1}{s^5} = \frac{1}{s} - \frac{2}{s^3} + \frac{1}{s^5}$

$$L^{-1}[F(s)] = L^{-1}\left(\frac{1}{s}\right) - \cancel{2} L^{-1}\left(\frac{2}{s^3}\right) + L^{-1}\left(\frac{1}{s^5}\right)$$

$$= 1 - \cancel{t^2} + \frac{t^4}{4!} = 1 - t^2 + \frac{t^4}{24}$$ //

ii) $F(s) = \frac{s}{s^2+6s+13} = \frac{s}{(s+3)^2+2^2}$

$$L^{-1}[F(s+a)] = e^{-at} L^{-1}[F(s)]$$

$$L^{-1}\left[\frac{(s+3)-3}{(s+3)^2+2^2}\right] = e^{-3t} L^{-1}\left(\frac{s}{s^2+2^2}\right) - e^{-3t} \cdot \frac{3}{2} L^{-1}\left(\frac{2}{s^2+2^2}\right)$$

$$= e^{-3t}\cos(2t) - \frac{3}{2}e^{-3t}\cos(2t)$$

Q6 $f(t) = e^{-2t}\sin(5t)\sin(3t)$

$$= e^{-2t}\left[\frac{1}{2}(\cos 2t - \cos 8t)\right]$$

$$= \frac{1}{2}\left[e^{-2t}\cos(2t) - e^{-2t}\cos(8t)\right]$$

$$L[f(t)] = \frac{1}{2}\left\{L\left[e^{-2t}\cos(2t)\right] - L\left[e^{-2t}\cos(8t)\right]\right\}$$

$$= \frac{1}{2}\left\{\frac{s+2}{(s+2)^2+2^2} - \frac{s+2}{(s+2)^2+8^2}\right\}$$

$$= \frac{s+2}{2}\left[\frac{1}{(s+2)^2+2^2} - \frac{1}{(s+2)^2+8^2}\right]$$

$$= \frac{s+2}{2} \cdot \frac{60}{\{(s+2)^2+2^2\}\{(s+2)^2+8^2\}}$$

$$F(s) = \frac{30(s+2)}{\{(s+2)^2+2^2\}\{(s+2)^2+8^2\}}$$ //

~~(ii) $L\left[\frac{\cos 3t}{t}\right] = \int_s^\infty F(s)\,ds$~~

~~$= \int_s^\infty L(\cos 3t)\,ds$~~

~~$= \int_s^\infty \frac{s}{s^2+3^2}\,ds = \frac{1}{2}\log(s^2+9)_s^\infty$~~

~~$= \frac{1}{2}\left[\qquad \right]$~~

(ii) $L\left[\frac{1-\cos 3t}{t}\right] = \int_s^\infty F(s)\,ds$

$$= \int_s^\infty L[1-\cos 3t]\,ds$$

$$= \int_s^\infty \{L(1) - L(\cos 3t)\}\,ds$$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+9}\right)ds = \cancel{\log s - \frac{1}{2}\log\left(\frac{s}{s^2+9}\right)}$$

$$\left\{\log s - \frac{1}{2}\log(s^2+9)\right\}_s^\infty$$

$$= \left(\log\frac{s}{\sqrt{s^2+9}}\right)_s^\infty = \log\sqrt{\frac{s^2+9}{s}}$$

Q7 Consider $\int_0^{\infty} e^{-st}(t \sin 3t)\, dt$

$= L(t \sin 3t) = (-1) F'(s)$

$= (-1) \frac{d}{ds} F(s)$

$= (-1) \frac{d}{ds} L[f(t)] = -1 \frac{d}{ds} L(\sin 3t)$

$= (-1) \frac{d}{ds} \frac{3}{s^2+9} = (-1) \frac{3(-2s)}{(s^2+9)^2}$

$L(t \sin 3t) = \frac{6s}{(s^2+9)^2}$

$\Rightarrow \int_0^{\infty} e^{-2t}\, t \sin(3t)\, dt = \frac{6(2)}{(2^2+9)^2} = \frac{12}{169}$ //

Q8 $f(x) = x^2$ in $-\pi \le x \le \pi$

$f(-x) = (-x)^2 = x^2 = f(x)$

$\Rightarrow f(x)$ is even

F.S $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$

$a_0 = \frac{\cancel{1}}{\pi} \int_{x=0}^{\pi} f(x)\, dx = \frac{2}{\pi} \int_{x=0}^{\pi} x^2\, dx$

$= \frac{2}{\pi} \left(\frac{x^3}{3}\right)_0^{\pi} = \frac{2\pi^2}{3}$

$a_0 = \frac{2\pi^2}{3}$

$$a_n = \frac{2}{\pi}\int_{x=0}^{\pi} \underbrace{x^2}_{u}\,\underbrace{\cos(nx)}_{v}\,dx \qquad n \in Z^+$$

$$= \frac{2}{\pi}\left[\underbrace{x^2}_{u}\,\underbrace{\frac{\sin(nx)}{n}}_{v_1} - \underbrace{2x}_{u'}\,\underbrace{\frac{-\cos(nx)}{n^2}}_{v_2} + 2\,\frac{-\sin(nx)}{n^3}\right]_{x=0}^{\pi}$$

$$a_n = \frac{2}{\pi}\left[\frac{2}{n^2}\left(\pi\cos(n\pi)\right)\right] = 4\,\frac{(-1)^n}{n^2}$$

$$\boxed{a_n = 4\,\frac{(-1)^n}{n^2}} \qquad n \in Z^+$$

<u>FS</u> $\quad x^2 \sim \frac{\pi^2}{3} + 4\sum \frac{(-1)^n}{n^2}\cos(nx)$