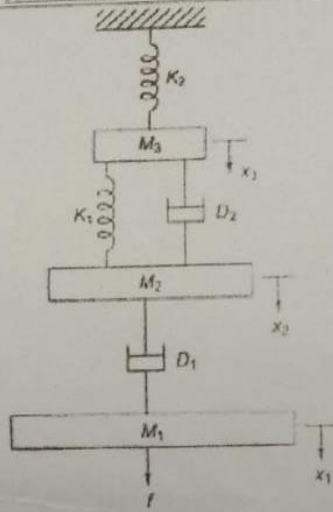
USN 1CR18ME061

Internal Assessment Test 2 – Dec 2022							
Sub:	Control Engineering				Sub Code:	18ME71	Bran
Date:	01/12/22	Duration:	90 min's	Max Marks: 50	Sem / Sec:		7^{th}/A & B

Answer all FIVE Questions

1



Draw equivalent mechanical system of the given system. Hence write the set o *equilibrium equations for it and obtain* FV analogous circuit.

2 *A unity feedback system has* $G(s) = \frac{K}{s(s+2)(s^2+2s+5)}$

i) For a unit ramp input, it is desired that $e_{ss} \leq 0.2$. Find K.

ii) Determine e_{ss} if $r(t) = 2 + 4t + \frac{t^2}{2}$

3 Derive unit step response for a second order underdamped system.

4 Derive rise time and peak time for an underdamped second order system.

5 A unity feedback system is characterized by $G(s) = \frac{10}{s^2+2s+6}$

Determine :

i) Undamped natural frequency ii) Damping ratio iii) Peak Time iv) Settling T

v) Peak Overshoot

CI

CCI

1)

Sol. : M_2 is under $y_2(t)$. K_2 is between $y_2(t)$ and $y_1(t)$. And M_1, K_1 and f are under $y_1(t)$. The equivalent mechanical system is shown in the Fig. 4.8.7 (a).

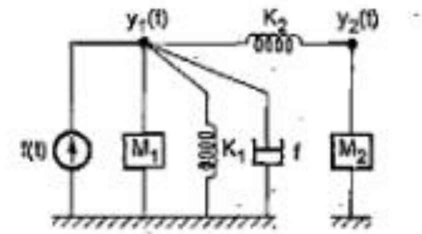


Fig. 4.8.7 (a)

The equilibrium equations are,

$$f(t) = M_1 \frac{d^2y_1}{dt^2} + K_1 y_1 + f\frac{dy_1}{dt} + K_2(y_1 - y_2) \quad ...(1)$$

$$0 = K_2(y_2 - y_1) + M_2 \frac{d^2y_2}{dt^2} \quad ...(2)$$

F-V analogy : $M \rightarrow L$, $f \rightarrow R$, $K \rightarrow \frac{1}{C}$, $y \rightarrow q$, $\frac{dy}{dt} \rightarrow \frac{dq}{dt} \rightarrow i$, $y \rightarrow \int i\,dt$, $\frac{d^2y}{dt^2} \rightarrow \frac{di}{dt}$

The equations are,

$$v(t) = L_1 \frac{di_1}{dt} + \frac{1}{C_1}\int i_1 dt + Ri_1 + \frac{1}{C_2}\int (i_1 - i_2)dt \quad ...(3)$$

$$0 = \frac{1}{C_2}\int (i_2 - i_1)dt + L_2 \frac{di_2}{dt} \quad ...(4)$$

Simulating on loop basis the analogous network is as shown in the Fig. 4.8.7 (b).

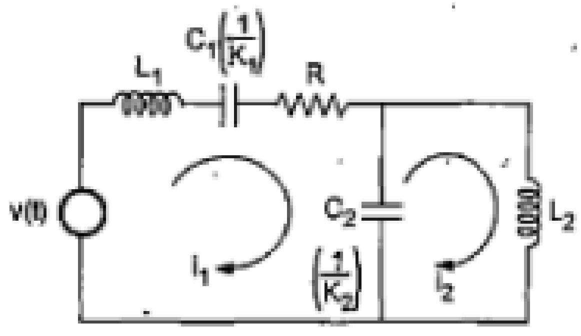


Fig. 4.8.7 (b)

2)

Sol. : For a given system,

$$G(s)H(s) = \frac{K}{s(s+2)\left(s^2+2s+5\right)}$$

i) For unit ramp input, K_v is required.

$$\therefore \quad K_v = \lim_{s\to 0} sG(s)H(s)$$

$$= \lim_{s\to 0} s\frac{K}{s(s+2)\left(s^2+2s+5\right)} = \frac{K}{10}$$

and $\quad A$ = Magnitude of ramp = 1

$$\therefore \quad e_{ss} = \frac{A}{K_v} = \frac{10}{K}$$

But $\quad e_{ss} \leq 0.2 \quad$ i.e. $\quad \frac{10}{K} \leq 0.2 \quad$ i.e. $\quad K \geq 50$

Thus for given condition **$50 \leq K < \infty$**.

ii) r(t) is $A_1 = 2$ step, $A_2 = 4$ ramp, $A_3 = 1$ parabolic

And $\quad K_p = \lim_{s\to 0} sG(s)H(s) = \infty$,

$$K_a = \lim_{s\to 0} s^2G(s)H(s) = 0$$

$$\therefore \quad e_{ss} = e_{ss1} + e_{ss2} + e_{ss3} = \frac{A_1}{1+K_p} + \frac{A_2}{K_v} + \frac{A_3}{K_a}$$

$$= \frac{2}{1+\infty} + \frac{4}{\left(\frac{K}{10}\right)} + \frac{1}{0} = \infty$$

3)

For underdamped systems, $\xi < 1$.

$\therefore \quad s^2 + 2\xi\omega_n s + \omega_n^2 = 0$ has two roots,

$$s_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$

Now let $\quad \xi\omega_n = \alpha$

and $\quad \omega_n\sqrt{1-\xi^2} = \omega_d$ (as discussed earlier)

$\therefore \quad s_{1,2} = -\alpha \pm j\omega_d$

For unit step input R(s) = 1/s and

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Substituting R(s), $C(s) = \dfrac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)}$

The partial fraction can be calculated for the Laplace inverse as below,

$$C(s) = \frac{a_1}{s} + \frac{a_2 s + a_3}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$\therefore \quad \frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)} = \frac{a_1(s^2 + 2\xi\omega_n s + \omega_n^2) + s(a_2 s + a_3)}{s(s^2 + 2\xi\omega_n s + \omega_n^2)}$$

equating numerators on both sides, $\quad \omega_n^2 = s^2(a_1 + a_2) + s(a_1 2\xi\omega_n + a_3) + a_1\omega_n^2$

$\therefore \quad a_1\omega_n^2 = \omega_n^2 \quad$ equating constant

$a_1 + a_2 = 0 \quad$ equating coefficients of s^2

$a_1 2\xi\omega_n + a_3 = 0 \quad$ equating coefficients of s

$\therefore \quad a_1 = 1, \; a_2 = -1, \; a_3 = -2\xi\omega_n$

As $\quad \xi\omega_n = \alpha$ assumed earlier for ease of computations.

$\therefore \quad a_1 = 1, \; a_2 = -1, \; a_3 = -2\alpha$

$$\therefore \quad C(s) = \frac{1}{s} + \frac{-s - 2\alpha}{s^2 + 2\alpha s + \omega_n^2}$$

$$\therefore \quad C(s) = \frac{1}{s} - \left\{\frac{s + 2\alpha}{s^2 + 2\alpha s + \omega_n^2}\right\}$$ .... (Taking negative sign outside)

So adjusting denominator as, $s^2 + 2\alpha s + \alpha^2 + \omega_n^2 - \alpha^2 = (s+\alpha)^2 + \omega_n^2 - \alpha^2$

but $\quad \alpha = \xi\omega_n \quad \therefore \; \alpha^2 = \xi^2\omega_n^2$

Substituting in above we get, $(s+\alpha)^2 + \omega_n^2 - \xi^2\omega_n^2 = (s+\alpha)^2 + \omega_n^2(1-\xi^2)$

Now $\quad \omega_d = \omega_n\sqrt{1-\xi^2}$ i.e. $\omega_d^2 = \omega_n^2(1-\xi^2)$

Substituting this in the expression of C(s) we get,

$$\therefore \quad C(s) = \frac{1}{s} - \left\{\frac{s + 2\alpha}{(s+\alpha)^2 + \omega_d^2}\right\}$$

$$L^{-1}\left\{\frac{(s+a)}{(s+a)^2 + \omega^2}\right\} = e^{-at}\cos\omega t \quad \text{and} \quad L^{-1}\left\{\frac{\omega}{(s+a)^2 + \omega^2}\right\} = e^{-at}\sin\omega t$$

Adjusting C(s) as,

$$C(s) = \frac{1}{s} - \left\{\frac{s+\alpha}{(s+\alpha)^2 + \omega_d^2} + \frac{\alpha}{(s+\alpha)^2 + \omega_d^2}\right\}$$

Multiplying and dividing by ω_d to the last term,

$$C(s) = \frac{1}{s} - \left\{\frac{s+\alpha}{(s+\alpha)^2 + \omega_d^2} + \frac{\alpha}{\omega_d} \cdot \frac{\omega_d}{(s+\alpha)^2 + \omega_d^2}\right\}$$

Taking Laplace inverse,

$$c(t) = 1 - e^{-\alpha t}\cos\omega_d t - \frac{\alpha}{\omega_d} e^{-\alpha t}\sin\omega_d t$$

Using $\alpha = \xi\omega_n$, $\omega_d = \omega_n\sqrt{1-\xi^2}$

$$c(t) = 1 - e^{-\xi\omega_n t}\left[\cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t\right]$$

$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\sqrt{1-\xi^2}\cos\omega_d t + \xi\sin\omega_d t\right]$$

• Now, $\sin(\omega_d t + \theta) = \sin(\omega_d t)\cos\theta + \cos(\omega_d t)\sin\theta$

Comparing this with the expression in bracket we can write $\sin\theta = \sqrt{1-\xi^2}$ and $\cos\theta = \xi$.

Hence $\tan\theta = \frac{\sqrt{1-\xi^2}}{\xi}$

$$\therefore \quad \theta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi} \text{ radians}$$

• Hence using this in the expression.

$$\therefore \quad c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \theta)$$ ... Required expression

where $\omega_d = \omega_n\sqrt{1-\xi^2}$

and $\theta = \tan^{-1}\left\{\frac{\sqrt{1-\xi^2}}{\xi}\right\}$ radians

4)

7.16.1 Derivation of Peak Time T_p

Transient response of second order underdamped system is given by,

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \theta) \quad \text{where} \quad \theta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$$

As at $t = T_p$, c(t) will achieve its maxima. According to Maxima theorem,

$$\left.\frac{dc(t)}{dt}\right|_{t=T_p} = 0$$

So differentiating c(t) w.r.t. 't' we can write,

$$\text{i.e.} \; -\frac{e^{-\xi\omega_n t}(-\xi\omega_n)\sin(\omega_d t + \theta)}{\sqrt{1-\xi^2}} - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\omega_d\cos(\omega_d t + \theta) = 0$$

Substituting $\omega_d = \omega_n\sqrt{1-\xi^2}$

$$\frac{\xi\omega_n e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \theta) - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\omega_n\sqrt{1-\xi^2}\cos(\omega_d t + \theta) = 0$$

$$\xi\sin(\omega_d t + \theta) - \sqrt{1-\xi^2}\cos(\omega_d t + \theta) = 0 \quad \text{i.e.} \quad \tan(\omega_d t + \theta) = \frac{\sqrt{1-\xi^2}}{\xi}$$

Now $\theta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$ i.e. $\frac{\sqrt{1-\xi^2}}{\xi} = \tan\theta$

$$\tan(\omega_d t + \theta) = \tan\theta$$

From trigonometric formula,

$$\tan(n\pi + \theta) = \tan\theta$$

$$\omega_d t = n\pi \quad \text{where } n = 1, 2, 3$$

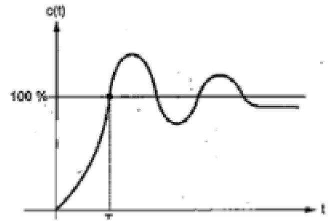
But T_p, time required for first peak overshoot. $\therefore$ $n = 1$

$$\omega_d T_p = \pi$$

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} \text{ sec}$$

7.16.3 Derivation of T_r

- Time required by output to achieve 100 % of its final value, starting from zero during the first attempt is the rise time.



$\{c(t)_{t = T_r}\} = 1$ for unit step input

$$\therefore \quad 1 = 1 - \frac{e^{-\xi \omega_n T_r}}{\sqrt{1-\xi^2}} \sin(\omega_d T_r + \theta)$$

$$\therefore \quad -\frac{e^{-\xi \omega_n T_r}}{\sqrt{1-\xi^2}} \sin(\omega_d T_r + \theta) = 0$$

Equation will get satisfied only if,

$$\sin(\omega_d T_r + \theta) = 0$$

Trigonometrically this is true only if,

$$\omega_d T_r + \theta = n\pi \quad \text{where } n = 1, 2 \ldots$$

As we are interested in first attempt use n = 1

$$\therefore \quad \omega_d T_r + \theta = \pi$$

$$\therefore \quad \boxed{T_r = \frac{\pi - \theta}{\omega_d} \text{ sec}}$$

5)

Fig. 7.16.4

Sol. : $$\frac{C(s)}{R(s)} = \frac{\frac{20}{(s+1)(s+4)}}{1 + \frac{20}{(s+1)(s+4)}} = \frac{20}{s^2 + 5s + 24}$$

> **Key Point** *Now though T.F. is not in standard form, denominator always reflect $2\xi\omega_n$ and ω_n^2 from middle term and the last term respectively.*

$\therefore$ Comparing, $s^2 + 5s + 24$ with $s^2 + 2\xi\omega_n s + \omega_n^2$

$$\therefore \quad \omega_n^2 = 24 \quad \therefore \omega_n = \mathbf{4.8989 \text{ rad/sec.}}$$

$$2\xi\omega_n = 5 \quad \therefore \xi = \mathbf{0.51031}$$

$$\omega_d = \omega_n \sqrt{1-\xi^2} = \mathbf{4.2129 \text{ rad/sec.}}$$

Now, for c(t) we can use standard expression for $\frac{C(s)}{R(s)}$ in standard form. So writing

$$\frac{C(s)}{R(s)} = \frac{20}{24} \cdot \left\{\frac{24}{s^2 + 5s + 24}\right\}$$

For the bracket term use standard expression, and then c(t) can be obtained by multiplying this expression by constant $\frac{20}{24}$.

$$\therefore \quad c(t) = \frac{20}{24}\left[1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)\right]$$

$$\theta = \tan^{-1} \frac{\sqrt{1-\xi^2}}{\xi} \text{ radians} = 1.03 \text{ radians}$$

$$\therefore \quad c(t) = \frac{20}{24}\left[1 - 1.1628\, e^{-2.5t} \sin(4.2129\, t + 1.03)\right]$$