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Internal Assessment Test I – January 2023

Sub:	Engineering Mathematics-I				Sub Code:	22MATS11	
Date:	19/01/2023	Duration:	90 mins	Max Marks:	50	Sem / Sec:	I / A to G (PHY CYCLE)

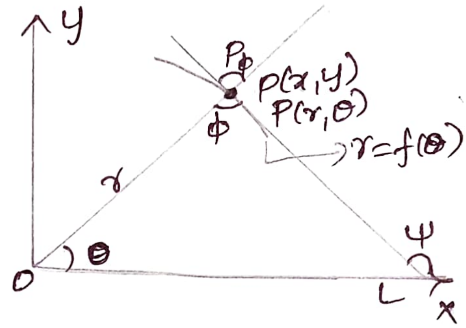
Question 1 is compulsory and answer any SIX questions from the rest.

		MARKS	CO	RBT
1.	With usual notations, prove that $\cot\varphi = \frac{1}{r}\frac{dr}{d\theta}$.	[08]	CO1	L2
2.	Find the rank of the matrix: $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$.	[07]	CO4	L3
3.	Investigate the values of λ and μ such that the following system may have (i) unique solution, (ii) infinitely many solutions and (iii) no solution. $$x + y + z = 6$$ $$x + 2y + 3z = 10$$ $$x + 2y + \lambda z = \mu$$	[07]	CO4	L3

4.	Solve the following system using Gauss Jordan method: $2x + y + z = 10, \quad 3x + 2y + 3z = 18, \quad x + 4y + 9z = 16.$	[07]	CO4	L3
5.	Find the dominant eigenvalue and the corresponding eigenvector of the matrix $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ by power method taking the initial vector as $[1,1,1]^T$ **(Perform only 4 iterations)**.	[07]	CO4	L3
6.	Solve using Gauss Seidel method **(perform only 3 iterations)**: $20x + y - 2z = 17$, $3x + 20y - z = -18$, $2x - 3y + 20z = 25.$	[07]	CO4	L3
7.	Show that the curves $r = a(1 + cos\theta)$ and $r = a(1 - cos\theta)$ intersect orthogonally.	[07]	CO1	L3
8.	Find the pedal equation of the curve $r^m = a^m(cosm\theta + sinm\theta)$.	[07]	CO1	L3

4m / 6m
V.V.V.I

Angle between radius vector and tangent (or) prove with usual notations $\tan\phi = r\frac{d\theta}{dr}$ (or) $\cot\phi = \frac{1}{r}\frac{dr}{d\theta}$ //



Let

$P(r,\theta)$ be any point on the curve $r=f(\theta)$

$\angle XOP = \theta$ and $OP = r$

Let

PL be the tangent to the curve at P subtending an angle ψ with the positive direction of the initial line [x-axis] and ϕ be the angle between radius vector OP and the tangent PL $\angle OPL = \phi$

From the figure we have an exterior angle is equal to the sum of interior angle

$$\psi = \phi + \theta$$

Take tan on both sides

$$\tan\psi = \tan(\phi+\theta)$$

We know that

$$\tan(A+B) = \frac{\tan A + \tan B}{1-\tan A\tan B}$$

$$\tan\psi = \frac{\tan\phi + \tan\theta}{1-\tan\phi\tan\theta} \rightarrow (1)$$

let x, y be the Cartesian Co-ordinates of P

$x = r\cos\theta$

$y = r\sin\theta$

Since, r is a function of θ

$\therefore \tan\psi = \frac{dy}{dx}$ = slope of the tangent

$$\tan\psi = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{\frac{d}{d\theta}[r\sin\theta]}{\frac{d}{d\theta}[r\cos\theta]}$$

$$= \frac{r\cos\theta + \sin\theta \cdot r'}{r(-\sin\theta) + \cos\theta\, r'}$$

$$\tan\psi = \frac{r\cos\theta + r'\sin\theta}{r'\cos\theta - r\sin\theta}$$

Dividing both the numerator and denominator by $r'\cos\theta$

$$\tan\psi = \frac{\frac{r\cos\theta}{r'\cos\theta} + \frac{r'\sin\theta}{r'\cos\theta}}{\frac{r'\cos\theta}{r'\cos\theta} - \frac{r\sin\theta}{r'\cos\theta}}$$

$$\tan\psi = \frac{\frac{r}{r'} + \tan\theta}{1 - \frac{r}{r'}\tan\theta} \rightarrow ②$$

Comparing eqn ① & ② we get

$$\tan\phi = \frac{r}{r'}$$

$$\tan\phi = \frac{r}{\frac{dr}{d\theta}}$$

$$\tan\phi = \frac{r\,d\theta}{d\dots}$$

$$\cot \phi = \frac{1}{r}\frac{dr}{d\theta}$$

Hence proved

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2. Find the rank of the matrix: $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

Solution

Given $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$R_3 \to R_3 - 3R_1, \quad R_4 \to R_4 - R_1$

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

$R_4 \to R_4 - R_3$

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho[A] = 2$ //

8. Investigate the Values of λ and μ Such that of equations
(or)
For what values of λ and μ the it satisfies the following condition.

$x+y+z=6$
$x+2y+3z=10$
$x+2y+\lambda z=\mu$ may have

(i) unique solution
(ii) infinite solution
(iii) No solution (or) inconsistent.

Soln

$$[A ; B] = \begin{bmatrix} 1 & 1 & 1 & ; 6 \\ 1 & 2 & 3 & ; 10 \\ 1 & 2 & \lambda & ; \mu \end{bmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$= \begin{bmatrix} 1 & 1 & 1 & ; 6 \\ 0 & 1 & 2 & ; 4 \\ 0 & 1 & \lambda-1 & ; \mu-6 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$= \begin{bmatrix} 1 & 1 & 1 & ; 6 \\ 0 & 1 & 2 & ; 4 \\ 0 & 0 & \lambda-3 & ; \mu-10 \end{bmatrix}$$

(i) unique solution
$\rho[A ; B] = \rho[A] =$ No. of unknowns
$\therefore \mu \neq 10, \lambda \neq 3$

(ii) Infinite solution
$\rho[A ; B] = \rho[A] <$ No. of unknowns
$\therefore \mu = 10, \lambda = 3$

(iii) No solution
$\rho[A ; B] \neq \rho[A]$
$\therefore \lambda = 3, \mu \neq 10$

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4. Solve the following system using Gauss Jordan method
$2x+y+z=10, \quad 3x+2y+3z=18, \quad x+4y+9z=16$

Sol:

$$[A;B] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{array}\right]$$

$R_1 \leftrightarrow R_3$

$$[A;B] = \left[\begin{array}{ccc|c} 1 & 4 & 9 & 16 \\ 3 & 2 & 3 & 18 \\ 2 & 1 & 1 & 10 \end{array}\right]$$

$R_2 \to R_2 - 3R_1, \quad R_3 \to R_3 - 2R_1$

$$[A;B] = \left[\begin{array}{ccc|c} 1 & 4 & 9 & 16 \\ 0 & -10 & -24 & -30 \\ 0 & -7 & -17 & -22 \end{array}\right]$$

$2-12$ $\quad 1-8$
$3-27$ $\quad 1-18$
$18-48$ $\quad 10-32$

$R_2 \to \dfrac{R_2}{-2}$

$$[A;B] = \left[\begin{array}{ccc|c} 1 & 4 & 9 & 16 \\ 0 & 5 & 12 & 15 \\ 0 & -7 & -17 & -22 \end{array}\right]$$

$R_1 \to 5R_1 - 4R_2, \quad R_3 \to 5R_3 + 7R_2$

$$[A;B] = \left[\begin{array}{ccc|c} 5 & 0 & -3 & 20 \\ 0 & 5 & 12 & 15 \\ 0 & 0 & -1 & -5 \end{array}\right]$$

$\frac{16\times5}{80}$
$45-48$
$80-60$
$-17(5)+7(12)$
$-85+84$
$-22(5)+7(15)$
$-110+105$

$R_1 \to R_1 - 3R_3, \quad R_2 \to R_2 + 12R_3$

$$[A:B] = \left[\begin{array}{ccc:c} 5 & 0 & 0 & 35 \\ 0 & 5 & 0 & -45 \\ 0 & 0 & -1 & -5 \end{array}\right]$$

$20 - 3(-5)$

$20 + 15$

$12 + 12$

$15 + 12(-5)$

$15 - 60$

$$\begin{array}{r} 60 \\ 15 \\ \hline 45 \end{array}$$

$\therefore \quad 5x = 35, \quad 5y = -45, \quad -z = -5$

$\therefore \quad x = 7, \quad y = -9, \quad z = 5$

$(x, y, z) = (\cancel{8}\,7, -9, 5)$

Find the dominant Eigen value and the corresponding vector of the matrix by power method by taking initial eigen vector as $x_0 = [1\ 1\ 1]^T$

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Soln

$$Ax_0 = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6-2+2 \\ -2+3-1 \\ 2-1+3 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 0 \\ 4 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 0 \\ 0.66 \end{bmatrix} = \lambda^{(1)} x^{(1)}$$

$$Ax^{(1)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.66 \end{bmatrix} = \begin{bmatrix} 6-0+1.32 \\ -2+0-0.66 \\ 2+0+1.98 \end{bmatrix}$$

$$= \begin{bmatrix} 7.32 \\ -2.66 \\ 3.98 \end{bmatrix} = 7.32 \begin{bmatrix} 1 \\ -0.36 \\ 0.54 \end{bmatrix}$$

$$= \lambda^{(2)} x^{(2)}$$

$$Ax^{(2)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -0.36 \\ 0.54 \end{bmatrix} = \begin{bmatrix} 6+0.72+1.08 \\ -2-1.08-0.54 \\ 2+0.36+1.62 \end{bmatrix}$$

$$= \begin{bmatrix} 7.8 \\ -3.62 \\ 3.98 \end{bmatrix} = 7.8 \begin{bmatrix} 1 \\ -0.46 \\ 0.51 \end{bmatrix} = \lambda^{(3)} x^{(3)}$$

$$Ax^{(3)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -0.46 \\ 0.51 \end{bmatrix} = \begin{bmatrix} 6+0.92+1.02 \\ -2-1.38-0.51 \\ 2+0.46+1.53 \end{bmatrix}$$

$$= \begin{bmatrix} 7.94 \\ -3.89 \\ 3.99 \end{bmatrix} = 7.94 \begin{bmatrix} 1 \\ -0.48 \\ 0.50 \end{bmatrix} = \lambda^{(4)} x^{(4)}$$

$$Ax^{(4)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -0.48 \\ 0.50 \end{bmatrix} = \begin{bmatrix} 6+0.96+1 \\ -2-1.44-0.50 \\ 2+0.48+1.5 \end{bmatrix}$$

$$= \begin{bmatrix} 7.96 \\ -3.94 \\ 3.98 \end{bmatrix} = 7.96 \begin{bmatrix} 1 \\ -0.49 \\ 0.5 \end{bmatrix} = \lambda^{(5)} x^{(5)}$$

$\therefore$

Largest Eigen value $= 7.98$

Largest Eigen vector $= \begin{bmatrix} -0.49 \\ 0.5 \end{bmatrix}$

Solve the following system of equations by Gauss Seidal method

$20x + y - 2z = 17$

$3x + 20y - z = -18$ carry out 3 iterations.

$2x - 3y + 20z = 25$

Soln

$$x = \frac{1}{20}[17 - y + 2z]$$

$$y = \frac{1}{20}[-18 - 3x + z]$$

$$z = \frac{1}{20}[25 - 2x + 3y]$$

I^{st} iteration

$x = 0, \ y = 0, \ z = 0$

$x^{(1)} = \frac{1}{20}[17 - 0 + 2(0)] = 0.85$

$y^{(1)} = \frac{1}{20}[-18 - 3(0.85) + 0] = -1.027$

$z^{(1)} = \frac{1}{20}[25 - 2(0.85) + 3(-1.027)] = 1.010.$

II^{nd} iteration

$x^{(1)} = 0.85, \ y^{(1)} = -1.027, \ z^{(1)} = 1.010$

$x^{(2)} = \frac{1}{20}[17 - (-1.027) + 2(1.010)] = 1.0023$

$y^{(2)} = \frac{1}{20}[-18 - 3(1.0023) + 1.010] = -0.9998$

$z^{(2)} = \frac{1}{20}[25 - 2(1.0023) + 3(-0.9998)] = 0.9998$

III^{rd} iteration

$x^{(2)} = 1.0023, \ y^{(2)} = -0.9998, \ z^{(2)} = 0.998$

$x^{(3)} = \frac{1}{20}[17 - (-0.9998) + 2(0.998)] = 0.999 \approx 1$

$y^{(3)} = \frac{1}{20}[-18 - 3(0.999) + 0.998] = -0.999 \approx -1$

$z^{(3)} = \frac{1}{20}[25 - 2(0.999) + 3(0.999)] = 1.000$

$\therefore$

$x = 1$
$y = -1$
$z = 1.$

Problem

Show that the following pair of curves intersect each other orthogonally

$r = a(1+\cos\theta)$, $r = a(1-\cos\theta)$.

Soln

$r = a(1+\cos\theta)$

Take log on both sides

$\log r = \log a + \log(1+\cos\theta)$

$\frac{1}{r}\frac{dr}{dt} = 0 + \frac{1}{(1+\cos\theta)}(0-\sin\theta)$

$\cot\phi_1 = \frac{-\sin\theta}{1+\cos\theta}$

$\cot\phi_1 = \frac{-2\sin\theta/2\,\cos\frac{\theta}{2}}{2\cos^2\theta/2}$

$\cot\phi_1 = -\tan\theta/2$

$\cot\phi_1 = \cot\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$

$\phi_1 = \frac{\pi}{2}+\frac{\theta}{2}$

$r = b(1-\cos\theta)$

Take log on both sides

$\log r = \log b + \log(1-\cos\theta)$

$\frac{1}{r}\frac{dr}{dt} = 0 + \frac{1}{(1-\cos\theta)}(0+\sin\theta)$

$\cot\phi_2 = \frac{\sin\theta}{(1-\cos\theta)}$

$\cot\phi_2 = \frac{2\sin\theta/2\,\cos\theta/2}{2\sin^2\theta/2}$

$\cot\phi_2 = \cot\frac{\theta}{2}$

$$\phi_2 = \frac{\theta}{2}$$

$$\therefore |\phi_2 - \phi_1| = \frac{\pi}{2}$$

$$\left|\frac{\theta}{2} - \frac{\pi}{2} - \theta/2\right| = \pi/2$$

$$\frac{\pi}{2} = \frac{\pi}{2}$$

The curves intersect orthogonally

Method - 2

$$\cot\phi_1 = \frac{-\sin\theta}{1+\cos\theta}$$

$$\frac{1}{\cot\phi_1} = \frac{(1+\cos\theta)}{-\sin\theta}$$

$$\tan\phi_1 = \frac{(1+\cos\theta)}{-\sin\theta}$$

$$\cot\phi_2 = \frac{\sin\theta}{(1-\cos\theta)}$$

$$\frac{1}{\cot\phi_2} = \frac{(1-\cos\theta)}{\sin\theta}$$

$$\tan\phi_2 = \frac{(1-\cos\theta)}{\sin\theta}$$

$$\tan\phi_1 \cdot \tan\phi_2 = -1$$

$$\frac{(1+\cos\theta)}{-\sin\theta} \cdot \frac{(1-\cos\theta)}{\sin\theta} = -1$$

$$\frac{(1^2 - \cos^2\theta)}{-\sin^2\theta} = -1$$

$$\frac{\sin^2\theta}{-\sin^2\theta} = -1$$

$$-1 = -1$$

$\therefore$ The curves intersect orthogonally.

$r^m = a^m (\cos m\theta + \sin m\theta)$

Soln

$\log r^m = \log a^m + \log(\cos m\theta + \sin m\theta)$

$$m \frac{1}{r} \frac{dr}{d\theta} = 0 + \frac{1}{(\cos m\theta + \sin m\theta)} m(-\sin m\theta + \cos m\theta)$$

$$\cot\phi = \frac{\cos m\theta - \sin m\theta}{\cos m\theta + \sin m\theta}$$

$$\cot\phi = \frac{\cos m\theta(1 - \tan m\theta)}{\cos m\theta(1 + \tan m\theta)}$$

$$\cot\phi = \cot\left(\frac{\pi}{4} + m\theta\right)$$

$$\phi = \frac{\pi}{4} + m\theta$$

we know that

$$p = r\sin\phi$$

$$\frac{p}{r} = \sin\left(\frac{\pi}{4} + m\theta\right)$$

$\therefore \sin(A+B) = \sin A \cos B + \sin B \cos A.$

$$\frac{p}{r} = \left(\sin\frac{\pi}{4}\cos m\theta + \cos\frac{\pi}{4}\sin m\theta\right)$$

$$\frac{p}{r} = \left(\frac{1}{\sqrt{2}}\cos m\theta + \frac{1}{\sqrt{2}}\sin m\theta\right)$$

$$\frac{p}{r} = \frac{1}{\sqrt{2}}(\cos m\theta + \sin m\theta)$$

$$p = \frac{r}{\sqrt{2}}(\cos m\theta + \sin m\theta) \rightarrow ①$$

$$\frac{r^m}{a^m} = \cos m\theta + \sin m\theta \rightarrow ② \text{ in } ①$$

$$p = \frac{r}{\sqrt{2}}\left(\frac{r^m}{a^m}\right)$$

$$p = \frac{r^{m+1}}{a^m\sqrt{2}}.$$