Solution 21MAT31 –jan feb 2023

CBCS SCHEME

USN | | | | | | | | | | | **21MAT31**

Third Semester B.E. Degree Examination, Jan./Feb. 2023
Transform Calculus, Fourier Series and Numerical Techniques

Time: 3 hrs. Max. Marks: 100

Note: ***Answer any FIVE full questions, choosing ONE full question from each module.***

Module-1

1 a. Find the Laplace transform of
$te^{2t} - \frac{2\sin 3t}{t}$. **(06 Marks)**

b. Given that $f(t) = \begin{cases} E, & 0 < t < a/2 \\ -E, & a/2 < t < a \end{cases}$

where $f(t+a) = f(t)$ show that $L\{f(t)\} = \frac{E}{S}\tan h\left(\frac{as}{4}\right)$. **(07 Marks)**

c. Using convolution theorem obtain the inverse. Laplace transform of the following function :
$\frac{1}{(s-1)(s^2+1)}$. **(07 Marks)**

OR

2 a. Find the inverse Laplace transform of :
$\frac{s+5}{s^2-6s+13}$. **(06 Marks)**

b. Express the following function interms of unit step function and hence find their Laplace transform.

$f(t) = \begin{cases} 1, & 0 < t < 1 \\ t, & 1 < t \le 2 \\ t^2 & t > 2. \end{cases}$ **(07 Marks)**

c. Solve the following intial value problem by using Laplace transform :
$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = e^{-t}, y(0) = 0, \ y'(0) = 0.$ **(07 Marks)**

Module-2

3 a. Obtain Fourier series of $f(x) = \frac{\pi - x}{2}$ in $0 < x < 2\pi$. Hence deduce that

$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{------} = \frac{\pi}{4}$. **(06 Marks)**

b. Find a cosine Fourier series for $f(x) = (x-1)^2, 0 \le x \le 1$. **(07 Marks)**

c. Obtain the Fourier series of y upto the First harmonic for the following values.

x°	45	90	135	180	225	270	315	360
y	4.0	3.8	2.4	2.0	–1.5	0	2.8	3.4

(07 Marks)

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OR

4 a. Obtain Fourier series for

$$f(x)=\begin{cases}\pi x & \text{in } 0\le x\le 1\\ \pi(2-x) & \text{in } 1\le x\le 2\end{cases}$$ **(06 Marks)**

b. Obtain the sine half range series for the function:

$f(x)=1-\left(\frac{x}{\pi}\right)$ in $0\le x\le \pi$. **(07 Marks)**

c. The following values of y and x are given. Find Fourier series of upto first harmonics.

x	0	2	4	6	8	10	12
y	9.0	18.2	24.4	27.8	27.5	22.0	9.0

(07 Marks)

Module-3

5 a. If $f(x)=\begin{cases}1-x^2, & |x|<1\\ 0, & |x|\ge 1\end{cases}$. Find Fourier transform of f(x) and hence find the value of

$\int_0^\infty \frac{x\cos x-\sin x}{x^3}\,dx$. **(06 Marks)**

b. Find the Fourier sine transform of $f(x)=e^{-|x|}$ and hence evaluate

$\int_0^\infty \frac{x\sin mx}{1+x^2}\,dx,\ m>0$. **(07 Marks)**

c. Solve by using Z-Transforms $U_{n+2}+2U_{n+1}+U_n=n$ with $U_0=0=U_1$. **(07 Marks)**

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OR

6 a. Obtain the Fourier cosine transform of the function :

$$f(x)=\begin{cases}4x, & 0<x<1\\ 4-x, & 1<x\le 4\\ 0, & x>4.\end{cases}$$

(06 Marks)

b. Obtain the Z-transform of $\cos n\theta$ and $\sin n\theta$ **(07 Marks)**

c. Compute the inverse Z-transform of $\dfrac{3z^2+2z}{(5z-1)(5z+2)}$. **(07 Marks)**

Module-4

7 a. Classify the following partial differential equations :

i) $x^2u_{xx}+(1-y^2)u_{yy}=0, -\infty<x<\infty, -1<y<1$

ii) $(1+x^2)u_{xx}+(5+2x^2)u_{xt}+(4+x^2)u_{tt}=0$

iii) $(x+1)u_{xx}-2(x+2)u_{xy}+(x+3)u_{yy}=0$. **(10 Marks)**

b. Solve $u_t=u_{xx}$ subject to the conditions $u(0, t)=0=u(1, t)$ and $u(x, 0)=\sin(\pi x)$ by taking $h=0.2$ for 5 levels. Further write down the following values from the table

i) u(0.2, 0.04)

ii) u(0.4, 0.08)

iii) u(0.6m 0.06). **(10 Marks)**

OR

8 a. Solve the elliptic equation $u_{xx}+u_{yy}=0$ for the following square Mesh with boundary values as shown. Find the iterative values of u_i(1 to 9) to the nearest integer.

Fig.Q8(a) **(10 Marks)**

b. Solve $25u_{xx}=u_{tt}$ at the pivotal points given $u(0, t)=0=u(5, t)$, $u_t(x, 0)=0$ and

$$u(x,0)=\begin{cases}20x, & 0\le x\le 1\\ 5(5-x), & 1\le x\le 5\end{cases}$$

by taking $h=1$ compute $u(x, t)$ for $0\le t\le 1$. **(10 Marks)**

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Module-5

9 a. Given $y'' - xy' - y = 0$ with the initial conditions $y(0) = 1$, $y'(0) = 0$ compute $y(0.2)$ using fourth order Runge – Kutta method. **(06 Marks)**

b. Derive the Euler's equation. **(07 Marks)**

c. Find the extremal of the functional.

$$\int_{x_1}^{x_2} (y^2 + y'^2 + 2ye^x)dx .$$ **(07 Marks)**

OR

10 a. Obtain the solution of the equation $2\frac{d^2y}{dx^2} = 4x + \frac{dy}{dx}$ by computing the value of $y(1.4)$ by applying Milne's method using following data :

x	1	1.1	1.2	1.3
y	2	2.2156	2.4649	2.7514
y'	2	2.3178	2.6725	3.0657

(06 Marks)

b. Find the curve on which the functional $\int_0^1 \left[[y']^2 + 12xy\right]dx$ with $y(0) = 0$ and $y(1) = 1$ can be determined. **(07 Marks)**

c. Prove that the shortest distance between two points in a plane is straight line. **(07 Marks)**

1.a

Find the Laplace transforms of $t\,e^{2t} - \dfrac{2\sin 3t}{t}$

Let, $f(t) = t\,e^{2t} - \dfrac{2\sin 3t}{t} = f_1(t) - f_2(t)$ (say)

$L[f(t)] = L[f_1(t)] - L[f_2(t)]$

Now, $L[f_1(t)] = L(te^{2t}) = \{L(t)\}_{s\to s-2} = \left\{\dfrac{1}{s^2}\right\}_{s\to s-2}$

$\therefore \quad L[f_1(t)] = \dfrac{1}{(s-2)^2}$

Next, $L[f_2(t)] = 2L\left[\dfrac{\sin 3t}{t}\right]$

$$= 2\int_s^\infty L(\sin 3t)\,ds = 2\int_s^\infty \frac{3}{s^2+3^2}\,ds$$

$L[f_2(t)] = 2[\tan^{-1}(s/3)]_s^\infty = 2\{\pi/2 - \tan^{-1}(s/3)\} = 2\cot^{-1}(s/3)$

$\therefore \quad L[f_2(t)] = 2\cot^{-1}(s/3)$

Thus the required, $\boxed{L[f(t)] = \dfrac{1}{(s-2)^2} - 2\cot^{-1}(s/3)}$

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1 b)

[44] *Given* $f(t) = \begin{cases} E, & 0<t<a/2 \\ -E, & a/2<t<a \end{cases}$ *where* $f(t+a) = f(t)$,

show that $L[f(t)] = E/s \cdot \tanh(as/4)$.

☞ The given function is periodic with period $T = a$.

We have, $L[f(t)] = \dfrac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$

$$= \frac{1}{1-e^{-sa}} \int_0^a e^{-st} f(t)\,dt$$

$$L[f(t)] = \frac{1}{1-e^{-as}} \left\{ \int_0^{a/2} e^{-st} E\,dt + \int_{a/2}^{a} e^{-st}(-E)\,dt \right\}$$

$$= \frac{E}{1-e^{-as}} \left\{ \left[\frac{e^{-st}}{-s}\right]_0^{a/2} + \left[\frac{e^{-st}}{s}\right]_{a/2}^{a} \right\}$$

$$= \frac{E}{s(1-e^{-as})} \left\{ -[e^{-st}]_0^{a/2} + [e^{-st}]_{a/2}^{a} \right\}$$

$$= \frac{E}{s(1-e^{-as})} \{ -e^{-as/2} + 1 + e^{-as} - e^{-as/2} \}$$

$$= \frac{E}{s(1-e^{-as})} (1 - 2e^{-as/2} + e^{-as}) = \frac{E(1-e^{-as/2})^2}{s(1-e^{-as})}$$

$$L[f(t)] = \frac{E(1-e^{-as/2})^2}{s(1-e^{-as/2})(1+e^{-as/2})} = \frac{E(1-e^{-as/2})}{s(1+e^{-as/2})}$$

Multiplying both the numerator and denominator by $e^{as/4}$ we get,

$$L[f(t)] = \frac{E(e^{as/4} - e^{-as/4})}{s(e^{as/4} + e^{-as/4})} = \frac{E \cdot 2\sinh(as/4)}{s \cdot 2\cosh(as/4)}$$

Thus, $$\boxed{L[f(t)] = \frac{E}{s}\tanh\left(\frac{as}{4}\right)}$$

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1c)

Let, $\bar{f}(s) = \frac{1}{s-1}$; $\bar{g}(s) = \frac{1}{s^2+1}$

$\Rightarrow \quad f(t) = L^{-1}[\bar{f}(s)] = e^t$; $g(t) = L^{-1}[\bar{g}(s)] = \sin t$

Now by applying convolution theorem we have,

$$L^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right] = \int_{u=0}^{t} e^u \cdot \sin(t-u)\,du$$

But, $\int e^{ax}\sin(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin(bx+c) - b\cos(bx+c)]$

$$L^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right] = \left[\frac{e^u}{1+1}\{\sin(t-u)+\cos(t-u)\}\right]_0^t$$

$$= \frac{1}{2}\{e^t(0+1) - 1(\sin t + \cos t)\}$$

Thus, $$\boxed{L^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right] = \frac{1}{2}(e^t - \sin t - \cos t)}$$

2a

$$L^{-1}\left[\frac{s+5}{s^2-6s+13}\right] = L^{-1}\left[\frac{s+5}{(s-3)^2+4}\right]$$

$$= L^{-1}\left[\frac{\overline{s-3}+3+5}{(s-3)^2+2^2}\right] = L^{-1}\left[\frac{(s-3)+8}{(s-3)^2+2^2}\right]$$

Here, $a = 3$ and $(s-3)$ changes to s

ie., $$= e^{3t}L^{-1}\left[\frac{s+8}{s^2+2^2}\right]$$

$$= e^{3t}\left\{L^{-1}\left(\frac{s}{s^2+2^2}\right) + 8L^{-1}\left(\frac{1}{s^2+2^2}\right)\right\}$$

Thus, $$\boxed{L^{-1}\left[\frac{s+5}{s^2-6s+13}\right] = e^{3t}(\cos 2t + 4\sin 2t)}$$

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2b)

$$f(t) = \begin{cases} 1, & 0<t\le 1 \\ t, & 1<t\le 2 \\ t^2, & t>2 \end{cases}$$

$f(t) = 1+(t-1)u(t-1)+(t^2-t)u(t-2)$ by a property.

$L[f(t)] = L(1)+L[(t-1)u(t-1)]+L[(t^2-t)u(t-2)]$ . . . (1)

Let, $F(t-1) = (t-1)$; $G(t-2) = t^2-t$

$\Rightarrow$ $F(t) = t$; $G(t) = (t+2)^2-(t+2) = t^2+3t+2$

$\therefore$ $\bar{F}(s) = \frac{1}{s^2}$; $\bar{G}(s) = \frac{2}{s^3}+\frac{3}{s^2}+\frac{2}{s}$

$L[F(t-1)u(t-1)] = e^{-s}\bar{F}(s)$ and $L[G(t-2)u(t-2)] = e^{-2s}\bar{G}(s)$

ie., $L[(t-1)u(t-1)] = \frac{e^{-s}}{s^2}$ and

$$L[(t^2-t)u(t-2)] = e^{-2s}\left(\frac{2}{s^3}+\frac{3}{s^2}+\frac{2}{s}\right)$$

We shall use these results in (1).

Thus, $$\boxed{L[f(t)] = \frac{1}{s}+\frac{e^{-s}}{s^2}+e^{-2s}\left(\frac{2}{s^3}+\frac{3}{s^2}+\frac{2}{s}\right)}$$

2c)

Solve the following initial value problem by using Laplace transforms :

The given equation is $y''(t)+4y'(t)+4y(t) = e^{-t}$

Taking Laplace transform on both sides we have,

$$L[y''(t)]+4L[y'(t)]+4L[y(t)] = L(e^{-t})$$

ie., $\{s^2L[y(t)]-sy(0)-y'(0)\}+4\{sL[y(t)]-y(0)\}+4L[y(t)] = \frac{1}{s+1}$

Using the given initial conditions we obtain,

$$L[y(t)]\{s^2+4s+4\} = \frac{1}{s+1} \quad \text{or} \quad L[y(t)] = \frac{1}{(s+1)(s+2)^2}$$

$$\therefore \quad y(t) = L^{-1}\left[\frac{1}{(s+1)(s+2)^2}\right]$$

Let, $$\frac{1}{(s+1)(s+2)^2} = \frac{A}{s+1}+\frac{B}{s+2}+\frac{C}{(s+2)^2}$$

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Multiplying with $(s+1)(s+2)^2$ we obtain

$$1 = A(s+2)^2 + B(s+1)(s+2) + C(s+1)$$

Putting $s = -1$ we get $A = 1$

Putting $s = -2$ we get $C = -1$

Putting $s = 0$ we have $1 = 1(4) + B(2) - 1(1) \therefore B = -1$

Hence, $\dfrac{1}{(s+1)(s+2)^2} = \dfrac{1}{s+1} + \dfrac{-1}{s+2} + \dfrac{-1}{(s+2)^2}$

$$\therefore \quad L^{-1}\left[\frac{1}{(s+1)(s+2)^2}\right] = L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{1}{s+2}\right] - L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

ie., $y(t) = e^{-t} - e^{-2t} - e^{-2t} L^{-1}\left(\dfrac{1}{s^2}\right)$

Thus, $\boxed{y(t) = e^{-t} - e^{-2t} - e^{-2t}\, t = e^{-t} - (1+t)e^{-2t}}$

3a)

☞ The Fourier series of $f(x)$ having period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \ldots (1)$$

where, $a_0 = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(x)\,dx,\ a_n = \dfrac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx,\ b_n = \dfrac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$

Now, $a_0 = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} \frac{\pi - x}{2}\,dx = \frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{2\pi} = \frac{1}{2\pi}\{(2\pi^2 - 2\pi^2) - 0\} = 0$

$a_0 = 0$

$a_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} \frac{\pi - x}{2}\cos nx\,dx.$ Applying Bernoulli's rule,

$$a_n = \frac{1}{2\pi}\left[(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{-1}{2\pi n^2}\left[\cos nx\right]_0^{2\pi}, \quad \text{since, } \sin 2n\pi = 0 = \sin 0$$

$$= \frac{-1}{2\pi n^2}\left[\cos 2n\pi - \cos 0\right] = 0, \quad \text{since, } \cos 2n\pi = 1 = \cos 0$$

$a_n = 0$

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$$b_n = \frac{1}{\pi}\int_0^{2\pi} \frac{\pi - x}{2} \sin nx\, dx.$$ Again by Bernoulli's rule,

$$b_n = \frac{1}{2\pi}\left[(\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{-1}{2\pi n}\left[(\pi - x)\cos nx\right]_0^{2\pi} + 0$$

$$= \frac{-1}{2\pi n}(-\pi\cos 2n\pi - \pi\cos 0) = \frac{-1}{2\pi n}(-\pi - \pi) = \frac{1}{n}$$

$b_n = 1/n$

Thus by substituting the values of a_0, a_n, b_n in (1), the Fourier series is given by,

$$\boxed{f(x) = \frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{1}{n}\sin nx}$$

To deduce the required series we put $x = \pi/2$ in the Fourier series of $f(x)$.

$$\therefore \quad f(\pi/2) = \sum_{n=1}^{\infty} \frac{1}{n}\sin\left(\frac{n\pi}{2}\right)$$

ie., $$\frac{\pi - (\pi/2)}{2} = \frac{\sin(\pi/2)}{1} + \frac{\sin\pi}{2} + \frac{\sin(3\pi/2)}{3} + \frac{\sin 2\pi}{4} + \frac{\sin(5\pi/2)}{5} + \cdots$$

Thus, $$\boxed{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots}$$

3b)

Find a cosine series for $f(x) = (x-1)^2, 0 \le x \le 1$

☞ Comparing the given interval [0, 1] with half range [0, l] we have $l = 1$. The corresponding cosine half range Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x \text{ where,}$$

$$a_0 = \frac{2}{1}\int_0^1 f(x)\,dx;\; a_n = \frac{2}{1}\int_0^1 f(x)\cos n\pi x\, dx$$

$$a_0 = 2\int_0^1 (x-1)^2\, dx = 2\left[\frac{(x-1)^3}{3}\right]_0^1 = \frac{2}{3}\{0 - (-1)^3\} = \frac{2}{3}$$

$\therefore \quad a_0/2 = 1/3$

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$$a_n = 2\int_0^1 (x-1)^2 \cos n\pi x \, dx$$

$$= 2\left[(x-1)^2 \cdot \frac{\sin n\pi x}{n\pi} - 2(x-1)\cdot -\frac{\cos n\pi x}{n^2\pi^2} + 2\cdot -\frac{\sin n\pi x}{n^3\pi^3}\right]_0^1$$

$$= \frac{4}{n^2\pi^2}\left[(x-1)\cos n\pi x \right]_0^1 = \frac{4}{n^2\pi^2}\{0-(-1)\} = \frac{4}{n^2\pi^2}$$

$$a_n = 4/n^2\pi^2$$

Thus the required cosine half range Fourier series is given by

$$f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x$$

3c) obtain up to first harmonics

x^0	45	90	135	180	225	270	315	360
y	4.0	3.8	2.4	2.0	–1.5	0	2.8	3.4

The interval of x is $0 < x \le 2\pi$ and period of $y = f(x)$ is 2π.

$$a_0 = \frac{2}{N}\sum y, \; a_1 = \frac{2}{N}\sum y\cos x, \; b_1 = \frac{2}{N}\sum y \sin x$$

x^0	y	$\cos x$	$y \cos x$	$\sin x$	$y \sin x$
45	4.0	0.7071	2.8284	0.7071	2.8284
90	3.8	0	0	1	3.8
135	2.4	−0.7071	−1.69704	0.7071	1.69704
180	2.0	−1	−2.0	0	0
225	−1.5	−0.7071	1.06065	−0.7071	1.06065
270	0	0	0	−1	0
315	2.8	0.7071	1.97988	−07071	−1.97988
360	3.4	1	3.4	0	0
Totals	**16.9**		**5.57189**		**7.40621**

From the table,

$$\sum y = 16.9, \; \sum y\cos x = 5.57189, \; \sum y \sin x = 7.40621,$$

$a_0 = 1/4 \cdot (16.9) = 4.225$, $\quad a_0/2 = 2.1125$

$a_1 = 1/4 \cdot (5.57189) = 1.393$. $\quad b_1 = 1/4 \cdot (7.40621) = 1.8516$

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4a) find fourier series of

$$f(x) = \begin{cases} \pi x & in \quad 0 \le x \le 1 \\ \pi(2-x) & in \quad 1 \le x \le 2 \end{cases}$$

Here $f(x)$ is defined in $[0, 2]$ and period of $f(x) = 2 - 0 = 2$.

$\therefore \quad 2l = 2$ or $l = 1$.

The Fourier series of $f(x)$ having period 2 is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

$$a_0 = \frac{1}{1}\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$$

$$a_0 = \int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx$$

$$a_0 = \pi\left\{\left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2\right\}$$

$$= \pi\left\{\left(\frac{1}{2} - 0\right) + (4-2) - \left(2 - \frac{1}{2}\right)\right\} = \pi$$

$$\therefore \quad a_0/2 = \pi/2$$

$$a_n = \frac{1}{1}\int_0^2 f(x)\cos(n\pi x)\,dx$$

$$= \int_0^1 f(x)\cos(n\pi x)\,dx + \int_1^2 f(x)\cos(n\pi x)\,dx$$

$$= \int_0^1 \pi x\cos(n\pi x)\,dx + \int_1^2 \pi(2-x)\cos(n\pi x)\,dx$$

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$$= \pi\left\{\int_0^1 x\cos(n\pi x)\,dx + \int_1^2 (2-x)\cos(n\pi x)\,dx\right\}$$

Applying Bernoulli's rule to both the integrals,

$$a_n = \pi\left\{\left[x\cdot\frac{\sin(n\pi x)}{n\pi} - 1\cdot - \frac{\cos(n\pi x)}{n^2\pi^2}\right]_0^1 + \left[(2-x)\frac{\sin(n\pi x)}{n\pi} - (-1)\cdot - \frac{\cos(n\pi x)}{n^2\pi^2}\right]_1^2\right\}$$

$$= \frac{\pi}{n^2\pi^2}\left\{[\cos n\pi x]_0^1 - [\cos n\pi x]_1^2\right\},$$ since, $\sin n\pi = 0 = \sin 0$.

$$= \frac{1}{n^2\pi}(\cos n\pi - \cos 0 - \cos 2n\pi + \cos n\pi).$$ But, $\cos 2n\pi = 1 = \cos 0$.

$$a_n = \frac{1}{n^2\pi}(-2 + 2\cos n\pi)$$

$$\therefore \quad a_n = \frac{-2}{\pi n^2}\{1-(-1)^n\}$$

$$b_n = \frac{1}{1}\int_0^2 f(x)\sin(n\pi x)\,dx$$

$$= \int_0^1 f(x)\sin(n\pi x)\,dx + \int_1^2 f(x)\sin(n\pi x)\,dx$$

$$= \int_0^1 \pi x\sin(n\pi x)\,dx + \int_1^2 \pi(2-x)\sin(n\pi x)\,dx$$

$$= \pi\left\{\left[x\cdot\frac{-\cos(n\pi x)}{n\pi} - (1)\cdot\frac{-\sin(n\pi x)}{n^2\pi^2}\right]_0^1 + \left[(2-x)\cdot\frac{-\cos(n\pi x)}{n\pi} - (-1)\cdot\frac{-\sin(n\pi x)}{n^2\pi^2}\right]_1^2\right.$$

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$$= \frac{-\pi}{n\pi}\left\{\left[x\cos(n\pi x)\right]_0^1 + \left[(2-x)\cos(n\pi x)\right]_1^2\right\}$$

$$= \frac{-1}{n}\left\{(\cos n\pi - 0) + (0 - \cos n\pi)\right\} = 0$$

$\therefore \quad b_n = 0$

The required Fourier series is given by

$$f(x) = \frac{\pi}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{1-(-1)^n}{n^2}\right\}\cos(n\pi x)$$

But, $1-(-1)^n = \begin{cases} 1-(+1) = 0 & \text{if } n \text{ is even} \\ 1-(-1) = 2 & \text{if } n \text{ is odd} \end{cases}$

Hence, $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1,3,5,\cdots}^{\infty}\frac{\cos(n\pi x)}{n^2}$

Thus, $\boxed{f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \cdots\right)}$

4b)

4b) $1 - x/\pi$ in $0, \pi$ Sine half Range.

$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^\pi (1 - x/\pi)\sin nx\,dx$

$= \frac{2}{\pi}\int_0^\pi \left(\sin nx - \frac{1}{\pi}x\sin nx\right)dx$

$= \frac{2}{\pi}\left[-\frac{\cos nx}{n} - \frac{1}{\pi}\left(x\left(-\frac{\cos nx}{n}\right)_0^\pi - \left(1\left(-\frac{\sin nx}{n^2}\right)\right)\right)\right]_0^\pi$

$= \frac{2}{\pi}\left[-\frac{\cos n\pi}{n} + \frac{1}{n} - \frac{1}{\pi}\left(-\frac{\pi\cos n\pi}{n}\right)\right]$

$b_n = \frac{2}{\pi}\left[\frac{1}{n}\right]$

$f(x) = \sum_{n=1}^{\infty} b_n \sin nx = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin nx.$

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4c)

x	0	2	4	6	8	10	12
y	9.0	18.2	24.4	27.8	27.5	22.0	9.0

☞ The values of y at $x = 0$ and $x = 12$ are same. Hence the interval of x is $(0, 12)$. That is $0 \le x \le 12$ and we shall omit the value of y for $x = 12$ in the process of calculation.

The Fourier series of period $2l$ is given by

$$y = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

Putting $l = 6$, the Fourier series upto the second harmonics is given by

$$y = f(x) = \frac{a_0}{2} + \left(a_1 \cos\frac{\pi x}{6} + b_1 \sin\frac{\pi x}{6} \right) + \left(a_2 \cos\frac{2\pi x}{6} + b_2 \sin\frac{2\pi x}{6} \right)$$

Putting $\theta = \pi x/6$ we have,

$$y = a_0/2 + (a_1 \cos\theta + b_1 \sin\theta) + (a_2 \cos 2\theta + b_2 \sin 2\theta)$$

The relevant table is as follows.

x	y	$\theta^0 = \pi x/6$	$\cos\theta$	$y\cos\theta$	$\sin\theta$	$y\sin\theta$
0	9.0	0	1	9	0	0
2	18.2	60	0.5	9.1	0.866	15.7612
4	24.4	120	−0.5	−12.2	0.866	21.1304
6	27.8	180	−1	−27.8	0	0
8	27.5	240	−0.5	−13.75	−0.866	−23.815
10	22.0	300	0.5	11.0	−0.866	−19.052
Total	**128.9**			**−24.65**		**−5.9754**

$$a_0 = \frac{2}{N}\sum y \qquad = \frac{2}{6}(128.9) \approx 42.967 \quad \therefore \quad \frac{a_0}{2} \approx 21.4835$$

$$a_1 = \frac{2}{N}\sum y\cos\theta \qquad = \frac{2}{6}(-24.65) \approx -8.217$$

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$$b_1 = \frac{2}{N}\sum y\sin\theta \quad = \frac{2}{6}(-5.9754) \approx -1.9918$$

5a)

[3] *If* $f(x) = \begin{cases} 1 - x^2, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$

find the Fourier transform of $f(x)$

☞ $$F(u) = \int_{x=-\infty}^{\infty} f(x)\,e^{iux}\,dx = \int_{-1}^{1}(1-x^2)\,e^{iux}\,dx,$$

$\because$ $f(x) = 0$ for $|x| \geq 1$ and $1-x^2$ for $|x| < 1$.

$\therefore$ $$F(u) = \left[(1-x^2)\frac{e^{iux}}{iu} - (-2x)\frac{e^{iux}}{i^2u^2} + (-2)\frac{e^{iux}}{i^3u^3}\right]_{x=-1}^{1},$$ by Bernoulli's rule.

$$= \frac{-i}{u}\left[(1-x^2)\,e^{iux}\right]_{x=-1}^{1} - \frac{2}{u^2}\left[x\,e^{iux}\right]_{x=-1}^{1} - \frac{2i}{u^3}\left[e^{iux}\right]_{x=-1}^{1}$$

$$\left(i^2 = -1,\ \frac{1}{i} = -i,\ \frac{1}{i^3} = i\right)$$

$$F(u) = \frac{-i}{u}(0-0) - \frac{2}{u^2}\left\{1\cdot e^{iu} - (-1)\,e^{-iu}\right\} - \frac{2i}{u^3}(e^{iu} - e^{-iu})$$

$$= -\frac{2}{u^2}(e^{iu} + e^{-iu}) - \frac{2i}{u^3}(e^{iu} - e^{-iu})$$

But, $e^{iu} = \cos u + i\sin u,\ e^{-iu} = \cos u - i\sin u$

$\therefore$ $e^{iu} + e^{-iu} = 2\cos u,\ e^{iu} - e^{-iu} = 2i\sin u$

Hence, $$F(u) = \frac{-4\cos u}{u^2} + \frac{4\sin u}{u^3}$$

Let us evaluate $\int_{-\infty}^{\infty} \frac{x\cos x - \sin x}{x^3}\,dx$

By inverse Fourier transform, we have,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\,e^{-iux}\,du$$

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If $x = 0 : f(x) = 1 - 0^2 = 1$ at $x = 0$. By putting $x = 0$ in the integral and using the expression of $F(u)$ we get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} 4\left(\frac{\sin u - u\cos u}{u^3}\right)e^0\,du = f(0) = 1.$$

$$\int_{-\infty}^{\infty} \frac{\sin u - u\cos u}{u^3}\,du = \frac{2\pi}{4} = \frac{\pi}{2}$$

If u is changed to $-u$, the expression $\dfrac{\sin u - u\cos u}{u^3}$ becomes

$\dfrac{\sin(-u)-(-u)\cos(-u)}{(-u)^3} = \dfrac{\sin u - u\cos u}{u^3}$ itself. Therefore the function is even and hence the integral from $-\infty$ to ∞ is twice the integral from 0 to ∞.

$$\therefore \quad 2\int_0^{\infty} \frac{\sin u - u\cos u}{u^3}\,du = \frac{\pi}{2} \quad \text{or} \quad \int_0^{\infty} \frac{u\cos u - \sin u}{u^3}\,du = -\frac{\pi}{4}$$

Changing u to x we get $\boxed{\displaystyle\int_0^{\infty} \frac{x\cos x - \sin x}{x^3}\,dx = -\frac{\pi}{4}}$

5b_)

Find the Fourier sine transform of $f(x) = e^{-|x|}$ *and hence evaluate*

$$\int_0^{\infty} \frac{x\sin mx}{1+x^2}\,dx,\ m>0.$$

☞ Fourier sine transform is given by

$$F_s(u) = \int_0^{\infty} f(x)\sin ux\,dx$$

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$$F_s(u) = \int_0^\infty e^{-|x|} \sin ux \, dx = \int_0^\infty e^{-x} \sin ux \, dx, \text{ since } |x| = x, x > 0.$$

$$F_s(u) = \left[\frac{e^{-x}(-1 \sin ux - u \cos ux)}{(-1)^2 + u^2} \right]_0^\infty$$

But, $e^{-x} \to 0$ as $x \to \infty, e^0 = 1, \cos 0 = 1, \sin 0 = 0.$

Thus, $F_s(u) = \dfrac{u}{1+u^2}$

By inverse Fourier sine transform we have,

$$\frac{2}{\pi}\int_0^\infty F_s(u) \sin ux \, du = f(x)$$

ie., $\displaystyle\int_0^\infty \frac{u}{1+u^2} \sin ux \, du = \frac{\pi}{2} f(x)$

Putting $x = m$ where $m > 0$ we have $f(x) = e^{-|m|} = e^{-m}$

$$\therefore \quad \int_0^\infty \frac{u \sin mu}{1+u^2} du = \frac{\pi}{2} e^{-m}$$

Thus by changing the variable u to x, $\boxed{\displaystyle\int_0^\infty \frac{x \sin mx}{1+x^2} dx = \frac{\pi}{2} e^{-m}}$

5c)

Solve by using Z-transforms : $y_{n+2} + 2y_{n+1} + y_n = n$ *with* $y_0 = 0 = y_1$

Taking Z-transforms on both sides of the given equation we have,

$$z^2 [\bar{y}(z) - y_0 - y_1 z^{-1}] + 2z[\bar{y}(z) - y_0] + \bar{y}(z) = \frac{z}{(z-1)^2}$$

$[z^2 + 2z + 1]\, \bar{y}(z) = \dfrac{z}{(z-1)^2}$, by using the given values.

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or $\quad \bar{y}(z) = \dfrac{z}{(z-1)^2(z+1)^2}$

Let, $\dfrac{z}{(z-1)^2(z+1)^2} = A \cdot \dfrac{z}{z-1} + B \cdot \dfrac{z}{(z-1)^2} + C \cdot \dfrac{z}{(z+1)} + D \cdot \dfrac{z}{(z+1)^2}$ ...(1)

or $\quad 1 = A(z-1)(z+1)^2 + B(z+1)^2 + C(z-1)^2(z+1) + D(z-1)^2$

Put $\quad z = 1 \quad : \quad 1 = B(4) \qquad \therefore \quad B = 1/4$

Put $\quad z = -1 \quad : \quad 1 = D(4) \qquad \therefore \quad D = 1/4$

Equating the coefficient of z^3 on both sides we get,

$$A + C = 0 \quad \text{or} \quad C = -A$$

Put $\quad z = 0 \; : \; 1 = -A + B + C + D$

ie., $\quad 1 = C + 1/4 + C + 1/4 \quad$ or $\quad 1/2 = 2C \qquad \therefore \quad C = 1/4$. Also $A = -1/4$

Substituting, $A = -1/4$, $B = C = D = 1/4$ in (1) and taking the inverse Z-transform we have,

$$Z_T^{-1}[\bar{y}(z)] = -\frac{1}{4} Z_T^{-1}\left[\frac{z}{z-1}\right] + \frac{1}{4} Z_T^{-1}\left[\frac{z}{(z-1)^2}\right] + \frac{1}{4} Z_T^{-1}\left[\frac{z}{z+1}\right] + \frac{1}{4} Z_T^{-1}\left[\frac{z}{(z+1)^2}\right]$$

ie., $\quad y_n = -\dfrac{1}{4} \cdot 1 + \dfrac{1}{4} n + \dfrac{1}{4}(-1)^n + \dfrac{1}{4} \cdot (-1)(-1)^n n$

$$= \frac{1}{4}\{(n-1) - (-1)^n (n-1)\}$$

Thus, $\boxed{y_n = \dfrac{(n-1)}{4}[1 - (-1)^n]}$ is the required solution.

6a)

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| *Obtain the Fourier cosine transform of the function* **[Dec. 2**

$$f(x) = \begin{cases} 4x, & 0<x<1 \\ 4-x, & 1<x<4 \\ 0, & x>4 \end{cases}$$

Fourier cosine transform is given by

$$F_c(u) = \int_0^\infty f(x)\cos ux\,dx$$

$$= \int_0^1 f(x)\cos ux\,dx + \int_1^4 f(x)\cos ux\,dx + \int_4^\infty f(x)\cos ux\,dx$$

$$\therefore \quad F_c(u) = \int_0^1 4x\cos ux\,dx + \int_1^4 (4-x)\cos ux\,dx + \int_4^\infty 0\cdot\cos ux\,dx$$

Applying Bernoulli's rule to the integrals we have,

$$F_c(u) = \left[4x\cdot\frac{\sin ux}{u} - 4\frac{-\cos ux}{u^2}\right]_0^1 + \left[(4-x)\frac{\sin ux}{u} - (-1)\frac{-\cos ux}{u^2}\right]_1^4 + 0$$

$$= \frac{4}{u}[x\sin ux]_0^1 + \frac{4}{u^2}[\cos ux]_0^1 + \frac{1}{u}[(4-x)\sin ux]_1^4 - \frac{1}{u^2}[\cos ux]_1^4$$

$$= \frac{4}{u}(\sin u - 0) + \frac{4}{u^2}(\cos u - 1) + \frac{1}{u}(0 - 3\sin u) - \frac{1}{u^2}(\cos 4u - \cos u)$$

$$= \frac{4}{u}\sin u + \frac{4}{u^2}\cos u - \frac{4}{u^2} - \frac{3}{u}\sin u - \frac{1}{u^2}\cos 4u + \frac{1}{u^2}\cos u$$

Thus, $$\boxed{F_c(u) = \frac{1}{u}\sin u + \frac{5\cos u - 4}{u^2} - \frac{1}{u^2}\cos 4u}$$

6b)

☞ We know that, $e^{in\theta} = \cos n\theta + i\sin n\theta$

We can write, $e^{in\theta} = (e^{i\theta})^n = k^n$ where $k = e^{i\theta}$

We have, $Z_T(k^n) = \dfrac{z}{z-k}$, k being $e^{i\theta}$

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$$\therefore \quad Z_T(e^{in\theta}) = \frac{z}{z-e^{i\theta}} = \frac{z(z-e^{-i\theta})}{(z-e^{-i\theta})(z-e^{i\theta})}$$

$$= \frac{z[z-(\cos\theta - i\sin\theta)]}{z^2 - z(e^{i\theta}+e^{-i\theta})+1}$$

$$= \frac{z[(z-\cos\theta)+i\sin\theta]}{z^2-2z\cos\theta+1}$$

ie., $$Z_T(\cos n\theta + i\sin n\theta) = \frac{z[(z-\cos\theta)+i\sin\theta]}{z^2-2z\cos\theta+1}$$

or $$Z_T(\cos n\theta) + iZ_T(\sin n\theta) = \left[\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}\right] + i\left[\frac{z\sin\theta}{z^2-2z\cos\theta+1}\right]$$

Equating the real and imaginary parts we get

$$Z_T(\cos n\theta) = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$$

$$Z_T(\sin n\theta) = \frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

6c)

Compute the inverse z-transform of $\frac{3z^2+2z}{(5z-1)(5z+2)}$

Let, $$\bar{u}(z) = \frac{3z^2+2z}{(5z-1)(5z+2)}$$

$$\therefore \quad \frac{\bar{u}(z)}{z} = \frac{3z+2}{(5z-1)(5z+2)}$$

Let, $$\frac{3z+2}{(5z-1)(5z+2)} = \frac{A}{5z-1} + \frac{B}{5z+2}$$

or $3z+2 = A(5z+2)+B(5z-1)$

Put $z = 1/5 : 13/5 = A(3) \quad \therefore A = 13/15$

Put $z = -2/5 : 4/5 = B(-3) \quad \therefore B = -4/15$

Hence, $$\frac{\bar{u}(z)}{z} = \frac{13}{15}\frac{1}{5z-1} - \frac{4}{15}\frac{1}{5z+2}$$

or $$\bar{u}(z) = \frac{13}{75}\frac{z}{z-(1/5)} - \frac{4}{75}\frac{z}{z+(2/5)}$$

$$\Rightarrow \quad Z_T^{-1}[\bar{u}(z)] = \frac{13}{75}Z_T^{-1}\left[\frac{z}{z-(1/5)}\right] - \frac{4}{75}Z_T^{-1}\left[\frac{z}{z+(2/5)}\right]$$

Thus, $$\boxed{Z_T^{-1}[\bar{u}(z)] = u_n = \frac{1}{75}\left\{13(1/5)^n - 4(-2/5)^n\right\}}$$

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7a)

7 a) (1) $x^2 u_{xx} + (1-y^2) u_{yy} = 0$, $-\infty < x < \infty$, $-1 < y < 1$

Soln: General linear pde of IInd order

$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0.$

Check $B^2 - 4AC$.

Here $B = 0$, $A = x^2$, $C = (1-y^2)$

$B^2 - 4AC = -4x^2(1-y^2)$ (y is in (−1,1) range)

So $B^2 - 4AC < 0$

$\Rightarrow$ The equation is elliptic.

(2) $(1+x^2) u_{xx} + (5+2x^2) u_{xt} + (4+x^2) u_{tt} = 0$

Here $A = (1+x^2)$, $B = 5+2x^2$, $C = 4+x^2$

$B^2 - 4AC = (5+2x^2)^2 - 4(1+x^2)(4+x^2)$

$= 25 + 4x^4 + 20x^2 - 16 - 4x^4 - 20x^2$

$= 9 > 0$

$\Rightarrow$ gv. eq is ~~parabolic~~ hyperbolic

(3) $(x+1) u_{xx} - 2(x+2) u_{xy} + (x+3) u_{yy} = 0$

$A = (x+1)$, $B = -2(x+2)$, $C = (x+3)$

$B^2 - 4AC = 4(x+2)^2 - 4(x+1)(x+3)$

$= 4 > 0 \Rightarrow$ eqn is ~~parabolic~~ hyperbolic.

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7b)

☞ The values of x in $0 \le x \le 1$ with $h = 0.2$ are 0, 0.2, 0.4, 0.6, 0.8, 1.

Here, $u_t = u_{xx}$ in comparison with $u_t = c^2 u_{xx}$ gives $c^2 = 1$ and $k = h^2 / 2c^2 = 0.02$

The values of t are 0.02, 0.04, 0.6, 0.08 and 0.1 upto 5 levels.

Also by data $u(x,0) = \sin \pi x$

$\therefore \quad u_{1,0} = u(0.2,0) = \sin(\pi/5) = 0.59 \; ; \; u_{2,0} = u(0.4,0) = \sin(2\pi/5) = 0.95$

$u_{3,0} = u(0.6,0) = \sin(3\pi/5) = 0.95 \; ; \; u_{4,0} = u(0.8,0) = \sin(4\pi/5) = 0.59$

The basic table is formed along with values to be computed being $u_i\,(i = 1 \text{ to } 20)$.

t \ x	0	0.2	0.4	0.6	0.8	1
0	0	0.59	0.95	0.95	0.59	0
0.02	0	u_1	u_2	u_3	u_4	0
0.04	0	u_5	u_6	u_7	u_8	0
0.06	0	u_9	u_{10}	u_{11}	u_{12}	0
0.08	0	u_{13}	u_{14}	u_{15}	u_{16}	0
0.1	0	u_{17}	u_{18}	u_{19}	u_{20}	0

The unknowns $u_i\,(i = 1 \text{ to } 20)$ are computed by applying Schmidt scheme

$$u_1 = \frac{1}{2}(0 + 0.95) = \mathbf{0.475}, \; u_2 = \frac{1}{2}(0.59 + 0.95) = \mathbf{0.77}$$

$$u_3 = \frac{1}{2}(0.95 + 0.59) = \mathbf{0.77}, \; u_4 = \frac{1}{2}(0.95 + 0) = \mathbf{0.475}$$

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$u_5 = \frac{1}{2}(0+0.77) = \mathbf{0.385}, \quad u_6 = \frac{1}{2}(0.475+0.77) = \mathbf{0.6225}$

$u_7 = \frac{1}{2}(0.77+0.475) = \mathbf{0.6225}, \quad u_8 = \frac{1}{2}(0.77+0) = \mathbf{0.385}$

$u_9 = \frac{1}{2}(0+0.6225) = \mathbf{0.3113}, \quad u_{10} = \frac{1}{2}(0.385+0.6225) = \mathbf{0.504}$

$u_{11} = \frac{1}{2}(0.6225+0.385) = \mathbf{0.504}, \quad u_{12} = \frac{1}{2}(0.6225+0) = \mathbf{0.3113}$

$u_{13} = \frac{1}{2}(0+0.504) = \mathbf{0.252}, \quad u_{14} = \frac{1}{2}(0.3113+0.504) = \mathbf{0.408}$

$u_{15} = \frac{1}{2}(0.504+0.3113) = \mathbf{0.408}, \quad u_{16} = \frac{1}{2}(0.504+0) = \mathbf{0.252}$

$u_{17} = \frac{1}{2}(0+0.408) = \mathbf{0.204}, \quad u_{18} = \frac{1}{2}(0.252+0.408) = \mathbf{0.33}$

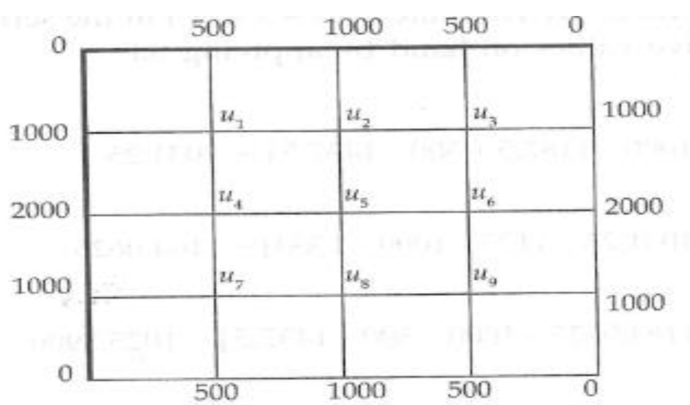
$u_{19} = \frac{1}{2}(0.408+0.252) = \mathbf{0.33}, \quad u_{20} = \frac{1}{2}(0.408+0) = \mathbf{0.204}$

Also by referring to the table, the desired values are written.

(i) $u(0,2,0.04) = u_5 = \mathbf{0.385}$ **(ii)** $u(0.4,0.008) = u_{14} = \mathbf{0.408}$

(iii) $u(0.6,0.06) = u_{11} = \mathbf{0.504}$

8a)



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☞ u_5 is located at the centre of the region and hence by the standard five point formula,

$$u_5 = \frac{1}{4}(2000 + 2000 + 1000 + 1000) = 1500$$

Next we shall compute u_1, u_3, u_7, u_9 by the diagonal five point formula.

$$u_1 = \frac{1}{4}(0 + 1500 + 2000 + 1000) = 1125$$

$$u_3 = \frac{1}{4}(1000 + 2000 + 1500 + 0) = 1125$$

Also $u_7 = 1125 = u_9$

Further we compute u_2, u_4, u_6, u_8 by S.F.

$$u_2 = \frac{1}{4}(1125 + 1125 + 1000 + 1500) = 1187.5$$

$$u_4 = \frac{1}{4}(2000 + 1500 + 1125 + 1125) = 1437.5$$

$$u_6 = \frac{1}{4}(1500 + 2000 + 1125 + 1125) = 1437.5$$

$$u_8 = \frac{1}{4}(1125 + 1125 + 1500 + 1000) = 1187.5$$

These values are regarded as the initial approximations to commence the Liebmann's iteration. We compute u_i ($i = 1$ *to* 9) in the serial order by using the latest iterative values on hand by applying S.F.

First iteration :

$$u_1^{(1)} = \frac{1}{4}[1000 + 1187.5 + 500 + 1437.5] = 1031.25$$

$$u_2^{(1)} = \frac{1}{4}[1031.25 + 1125 + 1000 + 1500] = 1164.0625$$

$$u_3^{(1)} = \frac{1}{4}[1164.0625 + 1000 + 500 + 1437.5] = 1025.3906$$

$$u_4^{(1)} = \frac{1}{4}[2000 + 1500 + 1031.25 + 1125] = 1414.0625$$

$$u_5^{(1)} = \frac{1}{4}[1414.0625 + 1437.5 + 1164.0625 + 1187.5] = 1300.7813$$

$$u_6^{(1)} = \frac{1}{4}[1300.78 + 2000 + 1025.3906 + 1125] = 1362.793$$

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$$u_7^{(1)} = \frac{1}{4}[1000 + 1187.5 + 1414.0625 + 500] = 1025.3906$$

$$u_8^{(1)} = \frac{1}{4}[1025.4 + 1125 + 1300.793 + 1000] = 1112.8$$

$$u_9^{(1)} = \frac{1}{4}[1112.8 + 1000 + 1362.793 + 500] = 993.8975$$

Thus the required first iterative values to the nearest integer are as follows.

$$u_1 = 1031,\ u_2 = 1164,\ u_3 = 1025,\ u_4 = 1414,\ u_5 = 1301$$
$$u_6 = 1363,\ u_7 = 1025,\ u_8 = 1113,\ u_9 = 994$$

8b)

Solve $25u_{xx} = u_{tt}$ *at the pivotal points given* $u(0,t) = 0 = u(5,t)$;

$u_t(x,0) = 0$ *and* $u(x,0) = \begin{cases} 20x, & 0 \le x \le 1 \\ 5(5-x), & 1 \le x \le 5 \end{cases}$ *by taking* $h = 1$.

Compute $u(x, t)$ *for* $0 \le t \le 1$.

☞ Comparing the wave equation $c^2 u_{xx} = u_{tt}$ with the given equation $25u_{xx} = u_{tt}$, we have $c^2 = 25$ or $c = 5$. Also $k = h/c = 1/5 = 0.2$

Since $h = 1$, the values of x in $0 \le x \le 5$ are 0, 1, 2, 3, 4, 5 and the values of t are 0, 0.2, 0.4, 0.6, 0.8, 1.

We have the following initial table. The values in the first and last column are zero by the first two initial conditions.

t \\ x		x_0	x_1	x_2	x_3	x_4	x_5
		0	1	2	3	4	5
t_0	0	0	20	15	10	5	0
t_1	0.2	0	$u_1 = 7.5$	$u_2 = 15$	$u_3 = 10$	$u_4 = 5$	0
t_2	0.4	0	$u_5 = -5$	$u_6 = 2.5$	$u_7 = 10$	$u_8 = 5$	0
t_3	0.6	0	$u_9 = -5$	$u_{10} = -10$	$u_{11} = -2.5$	$u_{12} = 5$	0
t_4	0.8	0	$u_{13} = -5$	$u_{14} = -10$	$u_{15} = -15$	$u_{16} = -7.5$	0
t_5	1	0	$u_{1,5}$	$u_{2,5}$	$u_{3,5}$	$u_{4,5}$	0

We have, $u(x,0) = \begin{cases} 20x, & 0 \le x \le 1 \\ 5(5-x), & 1 \le x \le 5 \end{cases}$

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$\therefore \quad u_{1,0} = u(1,0) = 20\ ;\ u_{2,0} = u(2,0) = 5\times 3 = 15\ ;$

$u_{3,0} = u(3,0) = 5\times 2 = 10\ ;\ u_{4,0} = u(4,0) = 5$

Next consider, $u_{i,1} = \frac{1}{2}\left[u_{i-1,0} + u_{i+1,0}\right]$

(*Average of the appropriate values on the left and right sides of the first row*)

$u_1 = \frac{1}{2}(0+15) = 7.5 \quad ; \quad u_2 = \frac{1}{2}(20+10) = 15$

$u_3 = \frac{1}{2}(15+5) = 10 \quad ; \quad u_4 = \frac{1}{2}(10+0) = 5$

The rest of the values are computed easily by applying the explicit formula scheme and entered in the table.

$u_5 = 0+15-20 = -5 \quad ; \quad u_6 = 7.5+10-15 = 2.5$

$u_7 = 15+5-10 = 10 \quad ; \quad u_8 = 10+0-5 = 5$

Next, $u_9 = 0+2.5-7.5 = -5 \quad ; \quad u_{10} = -5+10-15 = -10$

$u_{11} = 2.5+5-10 = -2.5 \quad ; \quad u_{12} = 10+0-5 = 5$

Next, $u_{13} = 0-10+5 = -5 \quad ; \quad u_{14} = -5-2.5-2.5 = -10$

$u_{15} = -10+5-10 = -15 \quad ; \quad u_{16} = -2.5+0-5 = -7.5$

Next, $u_{17} = 0-10+5 = -5 \quad ; \quad u_{18} = -5-15+10 = -10$

$u_{19} = -10-7.5+2.5 = -15 \quad ; \quad u_{20} = -15+0-5 = -20$

Thus the required values of $u_{i,j}$ are computed and tabulated.

9a)

Given $y'' - xy' - y = 0$ *with the initial conditions* $y(0) = 1,\ y'(0) = 0,$

☞ **Putting $y' = z$, we obtain $y'' = \frac{dz}{dx}$. The given equation becomes**

$$\frac{dz}{dx} = xz + y\ ;\ y(0) = 1,\ z(0) = 0$$

Hence we have a system of equations,

$$\frac{dy}{dx} = z\ ;\ \frac{dz}{dx} = xz + y \text{ where } y = 1,\ z = 0,\ x = 0$$

Let, $f(x,y,z) = z,\ g(x,y,z) = xz + y,\ x_0 = 0,\ y_0 = 1,\ z_0 = 0$ and $h = 0.2$

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We shall first compute the following.

$$k_1 = h f(x_0, y_0, z_0) = (0.2) f(0,1,0) = (0.2)0 = 0$$

$$l_1 = (0.2)[0 \times 0 + 1] = 0.2$$

$$k_2 = h f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2},\ z_0 + \frac{l_1}{2} \right)$$

$$k_2 = (0.2) f(0.1,1,0.1) = (0.2)(0.1) = 0.02$$

$$l_2 = (0.2)[0.1 \times 0.1 + 1] = 0.202$$

$$k_3 = h f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2},\ z_0 + \frac{l_2}{2} \right)$$

$$k_3 = (0.2) f(0.1,\ 1.01, 0.101) = (0.2)(0.101) = 0.0202$$

$$l_3 = (0.2)[0.1 \times 0.101 + 1.01] = 0.204$$

$$k_4 = h f(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$k_4 = (0.2) f(0.2,\ 1.0202, 0.204) = (0.2)(0.204) = 0.0408$$

$$l_4 = (0.2)[0.2 \times 0.204 + 1.0202] = 0.2122$$

We have, $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$z(x_0 + h) = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

Substituting the appropriate values we obtain $y(0.2) = 1.0202$ and $z(0.2) = 0.204$

Thus, $\boxed{y(0.2) = 1.0202 \ \text{ and } \ y'(0.2) = 0.204}$

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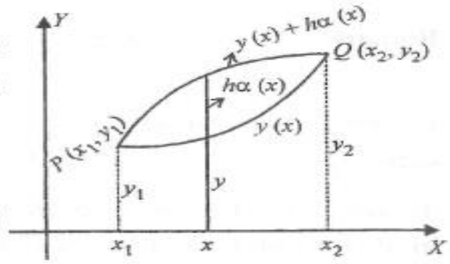
9b)

Euler's Equation

A necessary condition for the integral $I = \int_{x_1}^{x_2} f(x, y, y')\,dx$ *where* $y(x_1) = y_1$ *and* $y(x_2) = y_2$ *to be an extremum is that*

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \quad [\textit{Euler's equation}]$$

Proof :



Let I be an extremum along some curve $y = y(x)$ passing through $P(x_1, y_1)$ and $Q(x_2, y_2)$.

Also, let $y = y(x) + h\alpha(x)$ . . . (1)

be the neighbouring curve (*where h is small*) joining these points so that we must have

$$\alpha(x_1) = 0 \text{ at } P \text{ and } \alpha(x_2) = 0 \text{ at } Q. \quad \ldots (2)$$

When $h = 0$ these two curves coincide thus making I an extremum. That is to say that,

$$I = \int_{x_1}^{x_2} f\big(x, y(x) + h\alpha(x), y'(x) + h\alpha'(x)\big)\,dx$$

is an extremum when $h = 0$.

This requires $\frac{dI}{dh} = 0$ when $h = 0$, treating I to be a function of h.

$$\therefore \quad \frac{dI}{dh} = \int_{x_1}^{x_2} \frac{\partial}{\partial h} f\big(x, y(x) + h\alpha(x), y'(x) + h\alpha'(x)\big)\,dx$$

Applying chain rule for the partial derivative in RHS, we have,

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial x}\frac{\partial x}{\partial h} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial h} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial h}\right] dx \quad \ldots (3)$$

But h is independent of x and hence $\frac{\partial x}{\partial h} = 0$.

Let us consider (1) and differentiate w.r.t. x.

$$\therefore \quad y' = y'(x) + h\alpha'(x) \quad \ldots (4)$$

Also, we have from (1), $\frac{\partial y}{\partial h} = \alpha(x)$ and from (4) $\frac{\partial y'}{\partial h} = \alpha'(x)$

Using these results in (3) we have,

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$$\frac{dI}{dh} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} \alpha(x) + \frac{\partial f}{\partial y'} \alpha'(x) \right] dx \quad \ldots (5)$$

Keeping the first term in the RHS of (5) as it is and integrating the second term by parts we have,

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \frac{\partial f}{\partial y} \alpha(x)\, dx + \left\{ \left[\frac{\partial f}{\partial y'} \alpha(x) \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \alpha(x) \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) dx \right\}$$

$$= \int_{x_1}^{x_2} \frac{\partial f}{\partial y} \alpha(x)\, dx + \left\{ \frac{\partial f}{\partial y'} \alpha(x_2) - \frac{\partial f}{\partial y'} \alpha(x_1) \right\} - \int_{x_1}^{x_2} \alpha(x) \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) dx$$

But $\alpha(x_1) = 0 = \alpha(x_2)$ from (2) and we have by combining the two integrals,

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) \right] \alpha(x)\, dx$$

But we have already stated that $\frac{dI}{dh}$ must be zero when $h = 0$ for I to be an extremum. Hence integrand in the RHS must be zero.

Since $\alpha(x)$ is arbitrary we must have

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) = 0$$

This is the required **Euler's equation** being the necessary condition for the extremum of the functional $I = \int_{x_1}^{x_2} f(x, y, y')\, dx$.

9c)

☞ Let, $f(x, y, y') = y^2 + y'^2 + 2ye^x$

Euler's equation, $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) = 0$ becomes,

$$(2y + 2e^x) - \frac{d}{dx}(2y') = 0 \quad \text{or} \quad y + e^x - y'' = 0$$

ie., $\quad y'' - y = e^x$ or $(D^2 - 1)y = e^x$ where $D = \frac{d}{dx}$

AE is $m^2 - 1 = 0 \quad \therefore \ m = \pm 1$

Hence, $CF = y_c = c_1 e^x + c_2 e^{-x}$

$$PI = y_p = \frac{e^x}{D^2 - 1} = \frac{e^x}{0}, \text{ on replacing } D \text{ by } 1.$$

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$$y_p = x\frac{e^x}{2D} = \frac{x e^x}{2}$$

We have, $y = y_c + y_p$

Thus, $\boxed{y = c_1 e^x + c_2 e^{-x} + x e^x/2}$

10a)

Obtain the solution of the equation $2\frac{d^2y}{dx^2} = 4x + \frac{dy}{dx}$ *by computing the value*

0f y(1.4) by Milne method

x	1	1.1	1.2	1.3
y	2	2.2156	2.4649	2.7514
y'	2	2.3178	2.6725	3.0657

☞ Dividing the given equation by 2 we have,

$$\frac{d^2y}{dx^2} = 2x + \frac{1}{2}\frac{dy}{dx} \quad \text{or} \quad y'' = 2x + \frac{y'}{2}$$

Putting, $y' = z$ we obtain $y'' = z'$ and the given equation becomes

$$z' = 2x + \frac{z}{2}$$

Now, $z_0' = 2(1) + \frac{2}{2} = 3$

$$z_1' = 2(1.1) + \frac{2.3178}{2} = 3.3589$$

$$z_2' = 2(1.2) + \frac{2.6725}{2} = 3.73625$$

$$z_3' = 2(1.3) + \frac{3.0657}{2} = 4.13285$$

We have the following table.

x	$x_0 = 1$	$x_1 = 1.1$	$x_2 = 1.2$	$x_3 = 1.3$
y	$y_0 = 2$	$y_1 = 2.2156$	$y_2 = 2.4649$	$y_3 = 2.7514$
$y' = z$	$z_0 = 2$	$z_1 = 2.3178$	$z_2 = 2.6725$	$z_3 = 3.0657$
$y'' = z'$	$z_0' = 3$	$z_1' = 3.3589$	$z_2' = 3.73625$	$z_3' = 4.13285$

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We first consider Milne's predictor formulae,

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3)$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}(2z'_1 - z'_2 + 2z'_3)$$

On substituting the appropriate values from the table we obtain,

$$y_4^{(P)} = 3.0793 \text{ and } z_4^{(P)} = 3.4996$$

Next we consider Milne's corrector formulae,

$$y_4^{(C)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$$

$$z_4^{(C)} = z_2 + \frac{h}{3}(z'_2 + 4z'_3 + z'_4)$$

We have, $z'_4 = 2x_4 + \frac{z_4^{(P)}}{2} = 2(1.4) + \frac{3.4996}{2} = 4.5498$

Hence by substituting the appropriate values in the corrector formulae we obtain

$$y_4^{(C)} = 3.0794 \text{ and } z_4^{(C)} = 3.4997$$

Thus the required, $\boxed{y(1.4) = 3.0794}$

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10b)

Find the curve on which the functional $\int_0^1 \left[(y')^2 + 12xy \right] dx$ *with*

$y(0) = 0$ and $y(1) = 1$ can be determined.

☞ Let, $I = \int_0^1 \left[(y')^2 + 12xy \right] dx$

Let $\quad f(x, y, y') = (y')^2 + 12xy$

Euler's equation $\dfrac{\partial f}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'}\right) = 0$ becomes,

$$12x - \frac{d}{dx}(2y') = 0. \text{ That is, } 12x - 2y'' = 0 \text{ or } y'' = 6x$$

ie., $\dfrac{d^2y}{dx^2} = 6x$ and integrating w.r.t. x we get, $\dfrac{dy}{dx} = 3x^2 + c_1$

Again integrating w.r.t. x we get,

$$y = x^3 + c_1 x + c_2$$

Using the condition $y = 0$ at $x = 0$ and $y = 1$ at $x = 1$,

we obtain $c_1 = 0$ and $c_2 = 0$

Thus, $\boxed{y = x^3}$ is the required curve.

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10c)

☞ Let $y = y(x)$ be a curve joining two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the XOY plane.

We know that the arc length between P and Q is given by

$$s = \int_{x_1}^{x_2} \frac{ds}{dx}\, dx = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

ie., $$s = I = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx$$

We need to find the curve $y(x)$ such that I is minimum.

Let, $f(x, y, y') = \sqrt{1 + y'^2}$

Euler's equation, $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ becomes,

$$0 - \frac{d}{dx}\left[\frac{2y'}{2\sqrt{1 + y'^2}}\right] = 0$$

or $$\frac{d}{dx}\left[\frac{y'}{\sqrt{1 + y'^2}}\right] = 0$$

ie., $y''\sqrt{1 + y'^2} - y'\frac{2y'y''}{2\sqrt{1 + y'^2}} = 0$, by quotient rule and cross multiplying.

ie., $y''(1 + y'^2) - y''y'^2 = 0$ or $y'' = 0$.

ie., $$\frac{d^2y}{dx^2} = 0$$

Let us integrate twice w.r.t x

Thus $\boxed{y = c_1 x + c_2}$ which is a straight line.