USN

Internal Assessment Test I – July 2023

Sub:	Mathematics-II for Electrical & Electronics Engineering Stream				Sub Code:	BMATE201	
Date:	04/07/2023	Duration:	90 mins	Max Marks: 50	Sem / Sec:	M,N,O,P (PHY CYCLE)	OBE

Answer any FIVE questions.

		MARKS	CO	RBT
1.	Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$	[10]	CO1	L3
2.	If $\varphi = \frac{xz}{x^2+y^2}$, find the directional derivative of φ at $(1, -1, 1)$ in the direction of $\hat{i} - 2\hat{j} + \hat{k}$	[10]	CO1	L3
3.	Find the divergence and curl of the vector $\vec{F}$ if $\vec{F} = grad\ (x^3 + y^3 + z^3 - 3xyz)$	[10]	CO1	L3

4. Show that $\vec{F} = (6xy + z^3)\hat{\imath} + (3x^2 - z)\hat{\jmath} + (3xz^2 - y)\hat{k}$ is irrotational and find φ such that $\vec{F} = \nabla\varphi$ [10] CO1 L3

5. Find the work done in moving a particle in the force field $\vec{F} = 3x^2\hat{\imath} + (2xz - y)\hat{\jmath} + z\hat{k}$ along the straight line from $(0,0,0)$ to $(2,1,3)$ [10] CO1 L3

6. Verify Green's theorem for [10] CO1 L3

$$\oint_C (xy + y^2)dx + x^2dy$$

where C is bounded by $y = x$ and $y = x^2$

7. Find $y(38)$ and $y(85)$ by using suitable interpolation formulae from the table given below: [10] CO4 L3

x	40	50	60	70	80	90
y	184	204	226	250	276	304

IAT-I July -2023

1) Find the angle between the surfaces $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at the point $(2,-1,2)$.

Sol^n:- The equation of the two surfaces are given by

$x^2+y^2+z^2=9$ and $z=x^2+y^2-3$.

Let $\phi_1 = x^2+y^2+z^2$ and $\phi_2 = x^2+y^2-z$.

We know that $\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$.

$$\nabla\phi_1 = \frac{\partial\phi_1}{\partial x} i + \frac{\partial\phi_1}{\partial y} j + \frac{\partial\phi_1}{\partial z} k$$

$$\nabla\phi_1 = 2x i + 2y j + 2z k$$

$$[\nabla\phi_1]_{(2,-1,2)} = 4i - 2j + 4k$$

$$\nabla\phi_2 = \frac{\partial\phi_2}{\partial x} i + \frac{\partial\phi_2}{\partial y} j + \frac{\partial\phi_2}{\partial z} k$$

$$\nabla\phi_2 = 2x i + 2y j - k$$

$$\therefore [\nabla\phi_2]_{(2,-1,2)} = 4i - 2j - k$$

If θ is the angle between the normals to the surface then we have

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1|\,|\nabla\phi_2|}$$

$$= \frac{(4i-2j+4k)\cdot(4i-2j-k)}{\sqrt{16+4+16}\,\sqrt{16+4+1}}$$

$$= \frac{16+4-4}{\sqrt{36}\,\sqrt{21}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\Rightarrow \boxed{\theta = \cos^{-1}\left(8/3\sqrt{21}\right)}$$

2) If $\phi = \dfrac{xz}{x^2+y^2}$, find the directional derivative of ϕ at $(1,-1,1)$ in the direction of $i-2j+k$.

Solⁿ:- The given scalar function is

$$\phi = \frac{xz}{x^2+y^2}.$$

We first find $\nabla\phi$.

$$\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$$

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LAB INTERNAL TEST

MARKS SCORED					EXAMINERS SIGNATURE	
Q.No	Procedure	Conducting The Practical	Viva-Voce	Total	INTERNAL	EXTERNAL
1						
2						
			Total			

$$\nabla\phi = \frac{\partial}{\partial x}\left(\frac{xz}{x^2+y^2}\right) i + \frac{\partial}{\partial y}\left(\frac{xz}{x^2+y^2}\right) j + \frac{\partial}{\partial z}\left(\frac{xz}{x^2+y^2}\right) k$$

$$= z\left[\frac{(x^2+y^2)\cdot 1 - x(2x+0)}{(x^2+y^2)^2}\right] i + xz \frac{-1}{(x^2+y^2)^2} \times 2y\, j$$

$$+ \frac{x}{x^2+y^2} k$$

$$\Rightarrow \nabla\phi = \frac{z(y^2-x^2)}{(x^2+y^2)^2} i - \frac{2xyz}{(x^2+y^2)^2} j + \frac{x}{x^2+y^2} k$$

$$[\nabla\phi]_{(1,-1,1)} = \frac{1\cdot(1-1)}{(1+1)^2} i - \frac{2\cdot 1\cdot(-1)(1)}{(1+1)^2} j + \frac{1}{1^2+1^2} k$$

$$= 0 + \frac{1}{2} j + \frac{1}{2} k$$

$$\therefore [\nabla\phi]_{(1,-1,1)} = \frac{1}{2} j + \frac{1}{2} k = \frac{1}{2}(j+k).$$

The unit vector normal in the direction $\vec{d} = i-2j+k$ is

$$\hat{n} = \frac{\vec{d}}{|\vec{d}|} = \frac{i-2j+k}{\sqrt{1+4+1}} = \frac{i-2j+k}{\sqrt{6}}$$

Thus the required directional derivative is,

$$\nabla\phi \cdot \hat{n} = \frac{1}{2}(j+k) \cdot \frac{(i-2j+k)}{\sqrt{6}}$$

$$= \frac{1}{2\sqrt{6}}(0-2+1) = \boxed{\frac{-1}{2\sqrt{6}}}.$$

3) Find the divergence and curl of the vector $\vec{F}$ if $\vec{F} = \text{grad}(x^3+y^3+z^3-3xyz)$.

Sol:- Given that,

$$\vec{F} = \text{grad}(x^3+y^3+z^3-3xyz)$$

$$= \nabla(x^3+y^3+z^3-3xyz)$$

$$= \frac{\partial}{\partial x}(x^3+y^3+z^3-3xyz)\,i + \frac{\partial}{\partial y}(x^3+y^3+z^3-3xyz)\,j + \frac{\partial}{\partial z}(x^3+y^3+z^3-3xyz)\,k$$

$$= (3x^2-3yz)i + (3y^2-3xz)j + (3z^2-3xy)k$$

$$\vec{F} = 3\left[(x^2-yz)i + (y^2-xz)j + (z^2-xy)k\right]$$

Divergence:-

$$\text{div}\,\vec{F} = \nabla\cdot\vec{F} = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)\cdot 3\left[(x^2-yz)i + (y^2-xz)j + (z^2-xy)k\right]$$

$$= 3\left[\frac{\partial}{\partial x}(x^2-yz) + \frac{\partial}{\partial y}(y^2-xz) + \frac{\partial}{\partial z}(z^2-xy)\right]$$

$$\boxed{\text{div}\,\vec{F} = 3[2x+2y+2z] = 6(x+y+z).}$$

d):-

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \times 3\left[(x^2-yz) i + (y^2-xz) j + (z^2-xy) k\right]$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3(x^2-yz) & 3(y^2-xz) & 3(z^2-xy) \end{vmatrix}$$

$$= i\left[-3x+3x\right] - j\left[-3y+3y\right] + k\left[-3z+3z\right] = 0.$$

$$\Rightarrow \boxed{\text{curl } \vec{F} = 0}$$

4) Show that $\vec{F} = (6xy + z^3) i + (3x^2 - z) j + (3xz^2 - y) k$ is irrotational and find ϕ such that $\vec{F} = \nabla \phi$.

Sol^n:- To prove $\vec{F}$ is irrotational.

we need to show that curl $\vec{F} = \nabla \times \vec{F} = 0$.

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \times \left[(6xy+z^3) i + (3x^2-z) j + (3xz^2-y) k\right]$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy+z^3 & 3x^2-z & 3xz^2-y \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(3xz^2-y) - \frac{\partial}{\partial z}(3x^2-z)\right] - j\left[\frac{\partial}{\partial x}(3xz^2-y) - \frac{\partial}{\partial z}(6xy+z^3)\right]$$
$$+ k\left[\frac{\partial}{\partial x}(3x^2-z) - \frac{\partial}{\partial y}(6xy+z^3)\right]$$

$\text{curl } \vec{F} = (-1+1)i - j(3z^2 - 3z^2) + k(6x - 6x) = 0$

$\Rightarrow \text{curl } \vec{F} = 0$

$\Rightarrow \vec{F}$ is irrotational.

<u>To find ϕ (scalar potential).</u>

We have,

$$\vec{F} = \nabla \phi$$

$$\Rightarrow (6xy + z^3)i + (3x^2 - z)j + (3xz^2 - y)k = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$$

Equating the coefficients of i, j, k, we obtain

$$\frac{\partial \phi}{\partial x} = 6xy + z^3 \Rightarrow \partial \phi = (6xy + z^3)\partial x$$

$$\Rightarrow \phi = 3x^2y + z^3x + f_1(y, z)$$

$$\frac{\partial \phi}{\partial y} = 3x^2 - z \Rightarrow \partial \phi = (3x^2 - z)\,dy$$

$$\Rightarrow \phi = 3x^2y - yz + f_2(x, z)$$

$$\frac{\partial \phi}{\partial z} = 3xz^2 - y \Rightarrow \partial \phi = (3xz^2 - y)\,dz$$

$$\Rightarrow \phi = xz^3 - yz + f_3(x, y).$$

Let us choose $f_1(y,z) = -yz$, $f_2(x,z) = xz^3$, $f_3(x,y) = 3x^2y$

Therefore the required scalar potential is

$$\boxed{\phi = 3x^2y - yz + xz^3}$$

(7)

Find the work done in moving particle in the force field $\vec{F} = 3x^2 i + (2xz - y) j + zk$ along the straight line from $(0,0,0)$ to $(2,1,3)$.

Sol^n:- Eq^n of straight line from $(0,0,0)$ to $(2,1,3)$,

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0} = t \text{ (say)}$$

$$\boxed{\begin{array}{l} \Rightarrow x = 2t, \; y = t, \; z = 3t. \\ \therefore dx = 2dt, \; dy = dt \text{ and } dz = 3dt. \end{array}}$$

$$\vec{F} \cdot d\vec{r} = \left(3x^2 i + (2xz - y) j + zk\right) \cdot (dx\, i + dy\, j + dz\, k)$$

$$= 3x^2 dx + (2xz - y) dy + z\, dz$$

$$\Rightarrow \vec{F} \cdot d\vec{r} = 3(2t)^2 (2dt) + (2 \cdot 2t \cdot 3t - t)\, dt + 3t\,(3dt)$$

$$\vec{F} \cdot d\vec{r} = 24t^2 dt + (12t^2 - t) dt + 9t\, dt$$

Range of t : $0 - 1$.

$$\therefore \text{Work done} = \int_0^1 \vec{F} \cdot d\vec{r} = \int_0^1 \left(24t^2 + 12t^2 - t + 9t\right) dt$$

$$= \int_0^1 (36t^2 + 8t)\, dt = \left(\frac{36}{3} t^3 + \frac{8t^2}{2}\right)_0^1$$

$\Rightarrow$ W.D $= \left[12t^3 + 4t^2\right]_0^1$

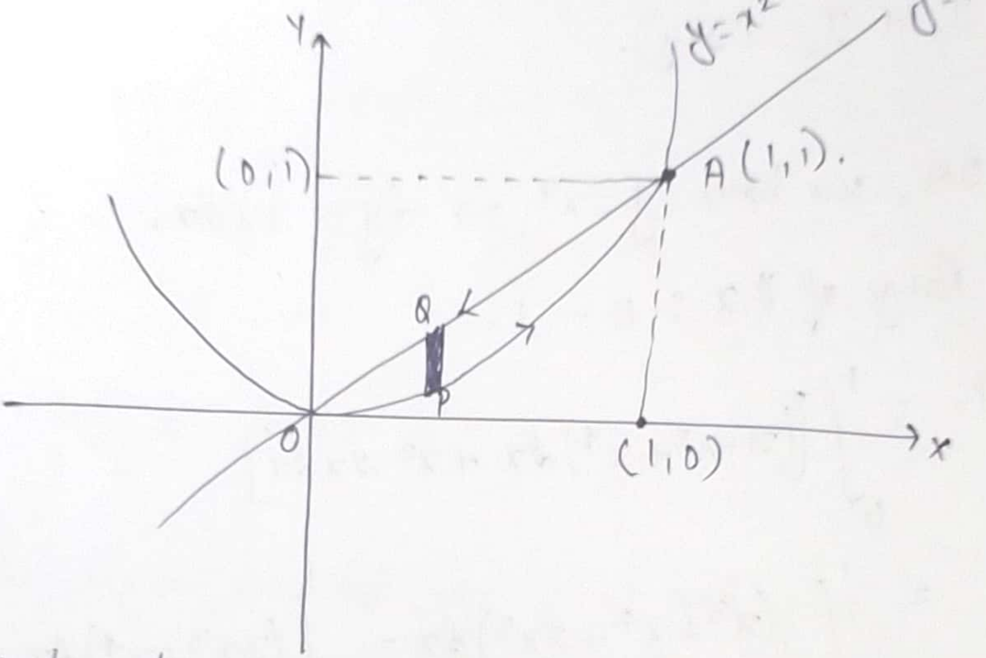
$= 12 + 4 = 16$

$\boxed{\text{Work done} = 16} = \int_C \vec{F} \cdot \vec{dr}.$

Verify Green's theorem for $\oint_C (xy+x^2)\,dx + x^2\,dy$, where C is boundary by $y=x$ and $y=x^2$.

Sol^n:- The given eq^{ns} of curves are $y=x$ & $y=x^2$.

We first draw the these curves.



Point of intersection of $y=x$ and $y=x^2$.

$$y=x^2 \Rightarrow x=x^2 \Rightarrow x^2-x=0 \Rightarrow x(x-1)=0$$

$$\Rightarrow x=0,1.$$

$$\Rightarrow y=0,1.$$

Therefore the points of intersection are $(0,0)$ and $(1,1)$.

By Green's theorem we have

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

L.H.S :- $\oint_C M\,dx + N\,dy = \int_C (xy+y^2)\,dx + x^2\,dy$

$= \int_{OA} (xy+y^2)\,dx + x^2\,dy + \int_{AO} (xy+y^2)\,dx + x^2\,dy.$

$= I_1 + I_2$ (say).

Along OA, we have $y = x^2 \Rightarrow dy = 2x\,dx$.

Range of x : 0 – 1.

$\therefore I_1 = \int_0^1 \left[(x \cdot x^2 + x^4)\,dx + x^2\, 2x\,dx\right]$

$= \int_0^1 (x^3 + x^4 + 2x^3)\,dx = \int_0^1 (3x^3 + x^4)\,dx$

$= \left[\frac{3}{4}x^4 + \frac{x^5}{5}\right]_0^1 = \frac{3}{4} + \frac{1}{5} = \frac{15+4}{20} = \frac{19}{20}.$

Along AO, we have $y = x \Rightarrow dy = dx$, and x : 1 – 0.

$I_2 = \int_1^0 \left[(x \cdot x + x^2)\,dx + x^2\,dx\right] = \int_1^0 3x^2\,dx$

$= \left[3 \cdot \frac{x^3}{3}\right]_1^0 = -1$

$\therefore$ L.H.S $= I_1 + I_2 = \frac{19}{20} - 1 = \frac{-1}{20}$.

For R.H.S :-

Compare the given integral $\oint_C (xy+y^2)\,dx + x^2 dy$ with $\oint_C M dx + N dy$, we get

$$M = xy + y^2 \quad \text{and} \quad N = x^2$$

$$\frac{\partial M}{\partial y} = x + 2y \quad \& \quad \frac{\partial N}{\partial x} = 2x.$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - x - 2y = x - 2y.$$

R is the region bounded by $y = x^2$ and $y = x$.

Choosing the vertical strip PQ in the figure, we obtain the limits for x & y as,

$$y : x^2 - x$$

$$\text{and} \quad x : 0 - 1$$

$$\text{RHS} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_{x=0}^{1} \left[\int_{y=x^2}^{x} (x-2y)\,dy\right] dx$$

$$= \int_{x=0}^{1} \left[xy - y^2\right]_{x^2}^{x} dx = \int_{x=0}^{1} \left[x^2 - x^2 - (x^3 - x^4)\right] dx$$

$$= \int_0^1 (x^4 - x^3)\,dx = \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1 = \frac{1}{5} - \frac{1}{4} = \frac{-1}{20}.$$

$\therefore$ L.H.S = R.H.S = $\frac{-1}{20}$.

Thus the theorem is verified.

7) Find y(38) and y(85) by using suitable interpolation formulae from the table given below:

x	40	50	60	70	80	90
y	184	204	226	250	276	304.

Sol^n:- For finding f(38), we use Newton's forward interpolation formula and to calculate f(85), we will use Newton's backward interpolation formula.

We first construct the difference table

x	$y = f(x)$	1st difference	2nd difference	3rd difference
40	184			
		20		
50	204		2	
				0
60	226	22		
			2	
70	250	24		0
80	276	26	2	
				0
90	304	28	2	

find $f(38)$:-

From the table we have.

$$x_0 = 40, \; y_0 = 184, \; \Delta y_0 = 20, \; \Delta^2 y_0 = 2, \; \Delta^3 y_0 = 0.$$

By Newton's forward interpolation formula,

$$y_r = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots$$

where $r = \frac{x - x_0}{h} = \frac{38-40}{10}$ (h is interval difference or step size)

$$\boxed{r = -0.2}$$

$$f(38) = 184 + (-0.2)(20) + \frac{(-0.2)(-0.2-1)}{2!}(2) + 0$$

$$\boxed{f(38) = 180.24}$$

To find $f(85)$:-

From the table, we have

$$x_n = 90, \; y_n = 304, \; \nabla y_n = 28, \; \nabla^2 y_n = 2, \; \nabla^3 y_n = 0.$$

By Newton's backward interpolation formula,

$$y_r = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \cdots$$

where $r = \frac{x - x_n}{h} = \frac{85-90}{10} = -0.5$

$$\boxed{r = -0.5}$$

(17)

$$y(85) = f(85) = 304 + (-0.5)(28) + \frac{(-0.5)(-0.5+1)}{2!}(2) + 0$$

$$\Rightarrow \boxed{f(85) = 289.75}$$