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Internal Assessment Test III – Sept 2023

Sub:	Mathematics – II for CSE Stream				Sub Code:	BMATS201	
Date:	04/09/2023	Duration:	90 mins	Max Marks: 50	Sem / Sec:	II / A-G	OBE

Answer any FIVE questions.

		MARKS	CO	RBT
1.	Find the directional derivative of $\varphi = x^2yz + 4xz^2$ at (1, -2,-1) in the direction of the vector $2\hat{\imath} - \hat{\jmath} - 2\hat{k}$.	[10]	CO2	L3
2.	If $\vec{F} = \nabla(xy^3z^2)$, find div $\vec{F}$ and curl $\vec{F}$ at $(1,-1,1)$. Is $\vec{F}$ solenoidal or irrotational?	[10]	CO2	L3
3.	Find the angle between the surfaces $x^2 + y^2 - z^2 = 4$ and $x^2 + y^2 - 13 = z$ at the point (2,1,2).	[10]	CO2	L3
4.	Show that the spherical coordinate system is orthogonal.	[10]	CO2	L2

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4.	Show that the spherical coordinate system is orthogonal.	[10]	CO2	L2

5.	Prove that the subset $W = \{(x, y, z)/2x + 3y + z = 0\}$ of the vector space R^3 is a subspace of R^3.	[10]	CO3	L3
6.	Show that the function $f(x) = 3x - 2 \; and \; g(x) = x \; are \; orthogonal \; in \; P_n \; with \; inner \; product$ $\langle f.g\rangle = \int_0^1 f(x).g(x)\,dx.$	[10]	CO3	L3
7.	Prove that the transformation $T: R^2 \rightarrow R^2$ defined by $T(x, y) = (3x, x + y)$is linear. Find the images of the vectors $(1,3) \; and \; (-1,2)$ under this transformation.	[10]	CO3	L3

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Internal Assessment - III
Test

1. DD of $\phi = x^2yz + 4xz^2$ in dir^n of $\vec{d} = 2\hat{i} - \hat{j} - 2\hat{k}$ at $(1,-2,-1)$

→ $\text{grad}(\phi) = \hat{i}\left(\frac{\partial \phi}{\partial x}\right) + \hat{j}\left(\frac{\partial \phi}{\partial y}\right) + \hat{k}\left(\frac{\partial \phi}{\partial z}\right)$

$= \hat{i}(2xyz + 4z^2) + \hat{j}(x^2z + 0) + \hat{k}(x^2y + 8xz)$

$\therefore \text{grad}(\phi) = (2xyz + 4z^2)\hat{i} + (x^2z)\hat{j} + (x^2y + 8xz)\hat{k}$

WKT, DD of ϕ along $\vec{d} = \text{grad}(\phi) \cdot \frac{\vec{d}}{|\vec{d}|}$

$\text{grad}(\phi)_{(1,-2,-1)} = (2(2) + 4)\hat{i} + (-1)\hat{j} + (-2 + (-8))\hat{k}$

$= 8\hat{i} - \hat{j} - 10\hat{k}$

Now, DD of ϕ along $\vec{d} = \frac{(8\hat{i} - \hat{j} - 10\hat{k}) \cdot (2\hat{i} - \hat{j} - 2\hat{k})}{\sqrt{2^2 + (-1)^2 + (-2)^2}}$

$= \frac{16 + 1 + 20}{\sqrt{4 + 1 + 4}}$

$= \frac{37}{3}$

Hence, DD of $\phi_{(1,-2,-1)}$ along $\vec{d} = \frac{37}{3}$

2. Given,

$$\vec{F} = \nabla(xy^3z^2)$$

$$= \frac{\partial}{\partial x}(xy^3z^2)\hat{i} + \frac{\partial}{\partial y}(xy^3z^2)\hat{j} + \frac{\partial}{\partial z}(xy^3z^2)\hat{k}$$

$$\vec{F} = y^3z^2\hat{i} + 3xy^2z^2\hat{j} + 2xy^3z\hat{k}$$

$$\operatorname{div}(\vec{F}) = \frac{\partial}{\partial x}(y^3z^2) + \frac{\partial}{\partial y}(3xy^2z^2) + \frac{\partial}{\partial z}(2xy^3z)$$

$$\operatorname{div}(\vec{F}) = 0 + 6xyz^2 + 2xy^3$$

$$\operatorname{div}(\vec{F})_{(1,-1,1)} = 6(-1) + 2(-1)$$

$$= -6 - 2 = -8$$

$$\therefore \operatorname{div}(\vec{F})_{(1,-1,1)} = -8$$

$$\operatorname{curl}(\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3z^2 & 3xy^2z^2 & 2xy^3z \end{vmatrix}$$

$$= \hat{i}(6xy^2z - 6xy^2z) - \hat{j}(2y^3z - 2y^3z) + \hat{k}(3y^2z^2 - 3y^2z^2)$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$\therefore \operatorname{curl}(\vec{F})_{(1,-1,1)} = \vec{0}$$

Since, the $\operatorname{curl}(\vec{F})_{(1,-1,1)}$ is $\vec{0}$, $\vec{F}$ is irrotational.

3. Given surfaces

$$\phi_1 = x^2+y^2-z^2-4 \qquad \phi_2 = x^2+y^2-z-13$$

$$\text{grad}(\phi_1) = \frac{\partial(\phi_1)}{\partial x}\hat{i} + \frac{\partial(\phi_1)}{\partial y}\hat{j} + \frac{\partial(\phi_1)}{\partial z}\hat{k}$$

$$= 2x\,\hat{i} + 2y\,\hat{j} - 2z\,\hat{k}$$

$$\text{grad}(\phi_1)_{(2,1,2)} = 4\hat{i} + 2\hat{j} - 4\hat{k}$$

$$\text{grad}(\phi_2) = 2x\,\hat{i} + 2y\,\hat{j} - \hat{k}$$

$$\text{grad}(\phi_2)_{(2,1,2)} = \cancel{4\hat{i}+4\hat{j}-2\hat{k}}\quad 4\hat{i} + 2\hat{j} - \hat{k}$$

If θ is the angle b/w ϕ_1 and ϕ_2, then

$$\cos\theta = \frac{\cancel{\phi_1 \cdot \phi_2}}{\cancel{|\phi_1| \times |\phi_2|}} = \frac{\text{grad}(\phi_1)\cdot\text{grad}(\phi_2)}{|\text{grad}(\phi_1)|\,|\text{grad}(\phi_2)|}$$

$$\cos\theta = \frac{16+4+4}{\sqrt{16+4+16}\,\sqrt{16+4+1}}$$

$$\cos\theta = \frac{24}{\sqrt{36}\,\sqrt{21}} = \frac{4}{\sqrt{21}}$$

$\therefore$ Angle b/w the surfaces at $(2,1,2) \Rightarrow \theta = \cos^{-1}\left(\frac{4}{\sqrt{21}}\right)$

4. <u>Spherical Coordinate System</u> :-

If P is a point in spherical coordinate system, it can be denoted as $P(r, \theta, \phi)$.

Comparing to cartesian system,

$$x = r\sin\theta\cos\phi,\quad y = r\sin\theta\sin\phi,\quad z = r\cos\theta$$

Let's denote $\vec{r}$

If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

In spherical system, $\vec{r} = r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\,\hat{k}$

$$\hat{e}_\theta = \frac{\partial \vec{r}}{\partial \theta} = \frac{r\cos\theta\cos\phi\,\hat{i} + r\cos\theta\sin\phi\,\hat{j} - r\sin\theta\,\hat{k}}{\sqrt{r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta}}$$

$$= \frac{r\cos\theta\cos\phi\,\hat{i} + r\cos\theta\sin\phi\,\hat{j} - r\sin\theta\,\hat{k}}{\sqrt{r^2\cos^2\theta(\cos^2\phi + \sin^2\phi) + r^2\sin^2\theta}}$$

$$= \frac{r\cos\theta\cos\phi\,\hat{i} + r\cos\theta\sin\phi\,\hat{j} - r\sin\theta\,\hat{k}}{\sqrt{r^2(\sin^2\theta + \cos^2\theta)}}$$

$$= \frac{r\cos\theta\cos\phi}{r}\,\hat{i} + \frac{r\cos\theta\sin\phi}{r}\,\hat{j} - \frac{r\sin\theta}{r}\,\hat{k}$$

$$\therefore \hat{e}_\theta = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$$

$$\hat{e}_\phi = \frac{d\vec{r}}{d\phi} = \frac{-r\sin\theta\sin\phi\,\hat{i} + r\sin\theta\cos\phi\,\hat{j} + 0\,\hat{k}}{\sqrt{r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi}}$$

$$= \frac{-r\sin\theta\sin\phi\,\hat{i} + r\sin\theta\cos\phi\,\hat{j} + 0\hat{k}}{\sqrt{r^2\sin^2\theta(\sin^2\phi+\cos^2\phi)}}$$

$$= \frac{-r\sin\theta\sin\phi}{r\sin\theta}\,\hat{i} + \frac{r\sin\theta\cos\phi}{r\sin\theta}\,\hat{j} + 0\,\hat{k}$$

$$\therefore \hat{e}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j} + 0\,\hat{k}$$

$$\hat{e}_r = \frac{d\vec{r}}{dr} = \frac{\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}}{\sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta}}$$

$$= \frac{\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}}{\sqrt{\sin^2\theta(\cos^2\phi+\sin^2\phi)+\cos^2\theta}}$$

$$= \frac{\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}}{\sqrt{\sin^2\theta+\cos^2\theta}}$$

$$\therefore \hat{e}_r = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$

Now,

$$\hat{e}_\theta \cdot \hat{e}_\phi = -\cos\theta\cos\phi\sin\phi + \cos\theta\sin\phi\cos\phi + 0$$

$$= \underline{0}$$

$$\hat{e}_\phi \cdot \hat{e}_r = -\sin\phi\sin\theta\cos\phi + \cos\phi\sin\theta\sin\phi + 0$$

$$= \underline{0}$$

$$\hat{e}_r \cdot \hat{e}_\theta = \sin\theta\cos\phi\cos\theta\cos\phi + \cos\theta\sin\phi\sin\theta\sin\phi - \sin\theta\cos\theta$$

$$= \sin\theta\cos\theta(\cos^2\phi+\sin^2\phi) - \sin\theta\cos\theta$$

$$= \sin\theta\cos\theta - \sin\theta\cos\theta = \underline{0}$$

Since,

$$\hat{i}_r \cdot \hat{i}_\theta = \hat{e}_\theta \cdot \hat{i}_\phi = \hat{e}_\phi \cdot \hat{i}_\theta = \underline{0} \quad \{\text{angle b/w. them is } 90^\circ\}$$

we can conclude that spherical coordinate system is orthogonal

5. $W = \{(x, y, z \mid 2x + 3y + z = 0)\}$

W.K.T $\quad 2(0) + 3(0) + 0 = 0$

$(0,0,0) \in W$

$\therefore$ W is a non-empty set

Now,

$u, v \in W$ & $a, b \in F$

$u = (x_1, y_1, z_1)$, $\quad V = (x_2, y_2, z_2)$

$2x_1 + 3y_1 + z_1 = 0$, $\quad 2x_2 + 3y_2 + z_2 = 0$

$au + bv = 0$,

$a(x_1, y_1, z_1) + b(x_2, y_2 + z_2)$

$(ax_1, ay_1, az_1) + (bx_2, by_2 + bz_2)$

Now to check whether it belongs to W,

$2(3ax_1 + 3bx_2) + 3(ay_1 + by_2) + (az_1 + bz_2)$

$= (2ax_1 + 2bx_2) + (3ay_1 + 3by_2) + (az_1 + bz_2)$

$= (2ax_1 + 3ay_1 + az_1) + (2bx_2 + 3by_2 + bz_2)$

$= a(2x_1 + 3y_1 + z_1) + b(2x_2 + 3y_2 + z_2)$

$= a(0) + b(0)$

$= 0$

$\therefore$ The subset W belongs ✓

6) Consider $\langle f.g \rangle = \int_0^1 f(x) \cdot g(x)\, dx$

$= \int_0^1$ given $f(x) = 3x - 2$

$g(x) = x$

$= \int_0^1 (3x-2)(x)\, dx$

$= \int_0^1 3x^2 - 2x \quad dx$

$= \left[\frac{3x^3}{3} - \frac{2x^2}{2}\right]_0^1$

$= [1 - 1] = 0$

$\therefore$ They are orthogonal

7) i) consider $u = (x_1, y_1)$

$v = (x_2, y_2)$

$T(u+v) = (x_1+x_2, y_1+y_2)$

$= (3(x_1+x_2), x_1+x_2+y_1+y_2)$

$= (3x_1+3x_2, x_1+x_2+y_1+y_2)$

$= (3x_1, x_1+y_1) + (3x_2, x_2+y_2)$

$= T(u) + T(v)$

ii) Now,

$T(au) = (ax_1, ay_1)$

$= (3ax_1, ax_1+ay_1) \longleftarrow (1)$

R.H.S $= aT(u) = a(x_1, y_1)$

$= a(3x_1, x_1+y_1)$

$= (3ax_1, ax_1+ay_1) \longrightarrow (2)$

From (1) & (2)

R.H.S = L.H.S

$\therefore$ It is a linear transformation.

$T(1,3) = (3(1), 1+3)$ $(3,4)$

$T(-1,2) = (3(-1), 1+2) = (-3,1)$

$\therefore$ The images of vectors $(1,3)$ and $(-1,2)$ are $(3,4)$ and $(3,1)$