6^{th} Sem July 2023

18EE61 - Control Systems – Solution

Question 1

a. Control system is a system in which the output quantity is controlled by varying the input quantity.

Comparison

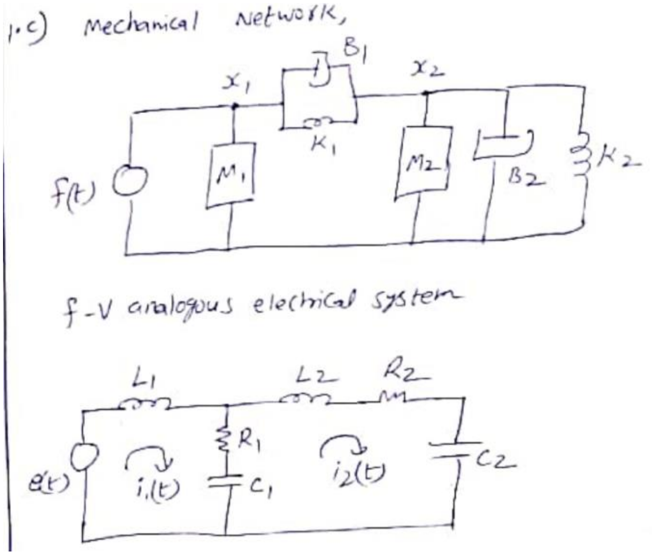
Open-Loop	Closed-Loop
Simple and economical	Complex and costlier
Consume less power	Consume more power
Easier to construct because of less number of components required	Not easy to construct because of more number of components required
Generally stable system	More care is needed to design a stable system
Inaccurate and unreliable	Accurate and more reliable
Changes in the output not corrected automatically	Changes in the output corrected automatically
	Feedback reduces the overall gain of the system.

b. Variables are angular acceleration, angular velocity and angular displacement. Newton's Law states that algebraic sum of torques about a fixed axis is equal to the product of inertia and its angular acceleration about the axis.

$$\sum Torques = J\alpha(t) = J\frac{d\omega(t)}{dt} = J\frac{d^2\theta(t)}{dt}$$

Component	Torque-angular velocity	Torque-angular displacement
Spring $T(t)$ $\theta(t)$ K	$T(t)=K\int_0^t \omega(\tau)d\tau$	$T(t)=K\theta(t)$
Viscous damper $T(t)$ $\theta(t)$ D	$T(t)=D\omega(t)$	$T(t)=D\frac{d\theta(t)}{dt}$
Inertia $T(t)$ $\theta(t)$ J	$T(t)=J\frac{d\omega(t)}{dt}$	$T(t)=J\frac{d^2\theta(t)}{dt^2}$

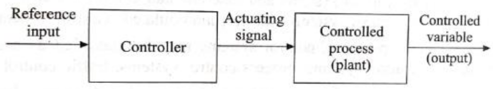
c.



Question 2

a.

1. Open-Loop Control Systems:

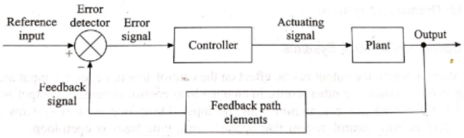


Any physical system which does not automatically correct the variation in its output.

- It is not a feedback system
- It operates on a time basis

Example: Washing machine, Electric Toaster, Traffic control.

2. Closed-Loop Control Systems:

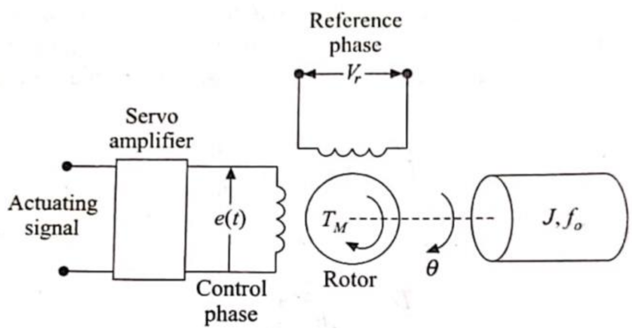


- Feedback control system.
- A system that maintains a prescribed relationship between the output and the reference input by comparing them and using the difference as a means of control.

Example: Traffic control, Room heating system.

b. A.C Servomotor

- An ac servomotor is basically a two-phase induction motor except for certain special design features.



Mechanical Load,

$$J\frac{d^2\theta}{dt^2} + f\frac{d\theta}{dt} = T_M \quad \text{— (1)}$$

Motor Torque: $T_M = K_1 e_c - K_2\frac{d\theta}{dt}$. — (2)

(2) in (1),

$$J\frac{d^2\theta}{dt^2} + f\frac{d\theta}{dt} = K_1 e_c - K_2\frac{d\theta}{dt} \quad \text{— (3)}$$

L.T (3),

$$J s^2\theta(s) + f s\,\theta(s) = K_1 E_c(s) - K_2 s\,\theta(s)$$

$$J s^2\theta(s) + f s\,\theta(s) + K_2 s\,\theta(s) = K_1 E_c(s)$$

$$\theta(s)\left[Js^2 + s\left[f + K_2\right]\right] = K_1 E_c(s)$$

$$\therefore \frac{\theta(s)}{E_c(s)} = \frac{K_1}{s\left[Js + (f + K_2)\right]} = \frac{K_1}{s(f+K_2)\left(\frac{Js}{f+K_2} + 1\right)}$$

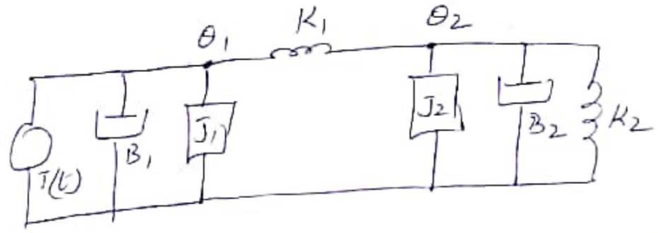
$$= \frac{K_1/(f+K_2)}{s\left[\frac{Js}{f+K_2} + 1\right]} = \frac{K_m}{s\left[\tau_m s + 1\right]}$$

where $K_m = \frac{K_1}{f+K_2}$, $\tau_m = \frac{J}{f+K_2}$

↳ Motor Gain ↳ Motor Time

c.

2· c) Mechanical network,



$$J_1\frac{d^2\theta_1}{dt^2} + B_1\frac{d\theta_1}{dt} + K_1\left[\theta_1 - \theta_2\right] = T(t) \quad \text{— (1)}$$

L.T (1),

$$J_1 s^2 \theta_1(s) + B_1 s \theta_1(s) + K_1 \theta_1(s) - K_1 \theta_2(s) = T(s)$$

$$\theta_1(s)\left[J_1 s^2 + B_1 s + K_1\right] - K_1 \theta_2(s) = T(s) \quad \text{--- (2)}$$

$$J_2 \frac{d^2\theta_2}{dt^2} + B_2 \frac{d\theta_2}{dt} + K_2 \theta_2 + K_1(\theta_2 - \theta_1) = 0 \quad \text{--- (3)}$$

L.T (3),

$$J_2 s^2 \theta_2(s) + B_2 s \theta_2(s) + K_2 \theta_2(s) + K_1 \theta_2(s) - K_1 \theta_1(s) = 0$$

$$-K_1 \theta_1(s) + \theta_2(s)\left[J_2 s^2 + B_2 s + (K_1 + K_2)\right] = 0 \quad \text{--- (4)}$$

By Krammer's rule in (2), (4).

$$\theta_2(s) = \frac{\begin{vmatrix} J_1 s^2 + B_1 s + K_1 & T(s) \\ -K_1 & 0 \end{vmatrix}}{\begin{vmatrix} \left[J_1 s^2 + B_1 s + K_1\right] & -K_1 \\ -K_1 & \left[J_2 s^2 + B_2 s + (K_1 + K_2)\right] \end{vmatrix}}$$

$$\theta_2(s) = \frac{T(s)\, K_1}{\left[J_1 s^2 + B_1 s + K_1\right]\left[J_2 s^2 + B_2 s + (K_1 + K_2)\right] - K_1^2}$$

$$\therefore \frac{\theta_2(s)}{T(s)} = \frac{K_1}{\left[J_1 s^2 + B_1 s + K_1\right]\left[J_2 s^2 + B_2 s + (K_1 + K_2)\right] - K_1^2}$$

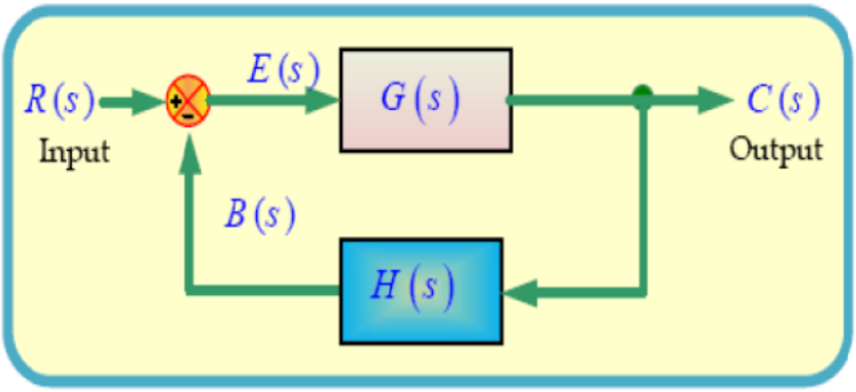
Question 3

a. Block Diagram: A *block diagram* is a pictorial representation of the cause and effect relationship between the input and output of a physical system.

Properties:

- Block diagram reduction refers to simplification of block diagrams of complex systems through certain rearrangements
- Simplification enables easy calculation of the overall transfer function of the system
- Simplification is done using certain rules called the ‘rules of block diagram algebra’
- All these rules are derived by simply algebraic manipulations of the equations representing the blocks

b.



T.F

$$C(s) = G(s)E(s) \quad \text{--- (1)}$$

$$B(s) = H(s)C(s) \quad \text{--- (2)}$$

$$E(s) = R(s) - B(s) \quad \text{--- (3)}$$

(2), (3) in (1).

$$C(s) = G(s)\{R(s) - H(s)C(s)\}$$

$$C(s) = G(s)R(s) - G(s)H(s)C(s)$$

$$C(s) + G(s)H(s)C(s) = G(s)R(s)$$

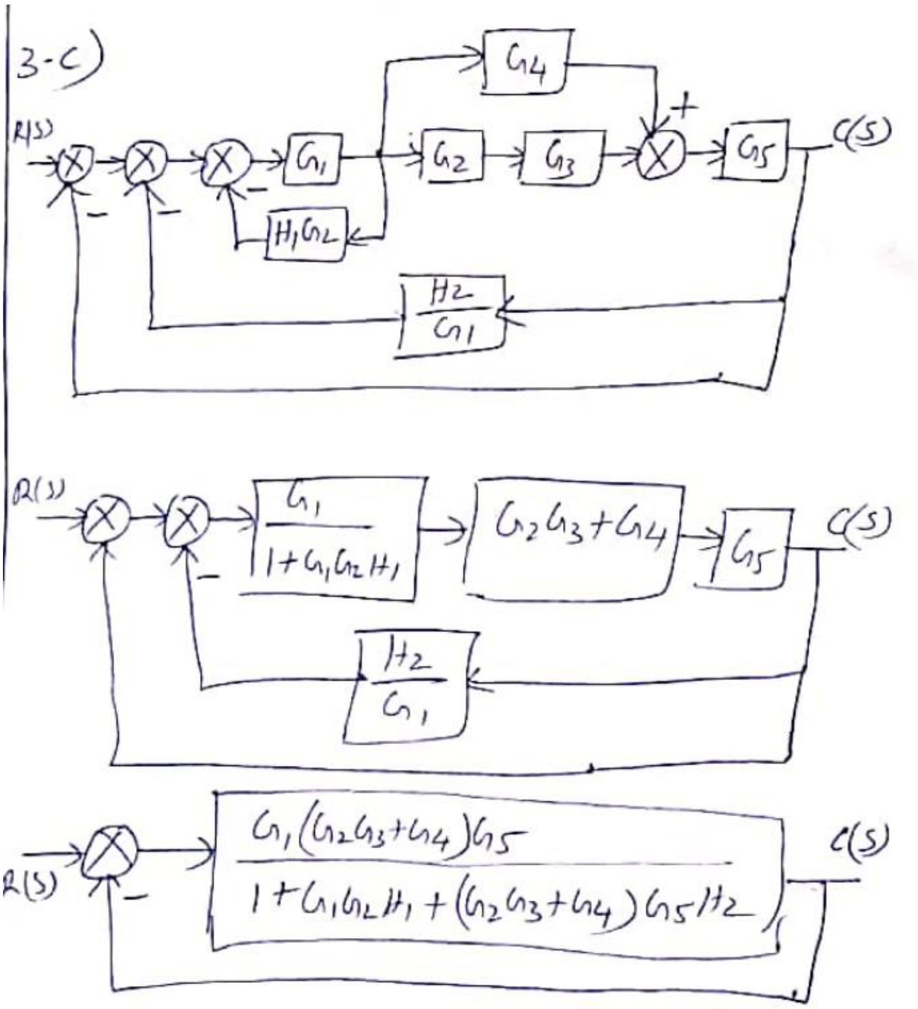
$$C(s)[1 + G(s)H(s)] = G(s)R(s)$$

$$\therefore \boxed{\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}}$$

Closed Loop T.F $\Rightarrow \dfrac{G(s)}{1 \pm G(s)H(s)}$

Open Loop T.F $\Rightarrow G(s)H(s)$

c.



$$\frac{C(s)}{R(s)} = \frac{G_1(G_2G_3+G_4)G_5}{1+G_1G_2H_1+(G_2G_3+G_4)G_5H_2+G_1(G_2G_3+G_4)G_5}$$

Question 4

a. i) **Input node or Source:** It is a node with only outgoing branches.
   Output node or Sink: It is a node with only incoming branches.

   ii) **Loop:** It is a path which originates and terminates at the same node and along which no node is traversed more than once.
   Loop gain: The product of branch gains encountered in traversing the loop

   iii) **Forward path:** It is a path that starts at an input node and ends at an output node, and along which no node is traversed more than once.

b. Mason Gain Formula:

$$M = \frac{y_{out}}{y_{in}} = \sum_k \frac{M_k \Delta_k}{\Delta}$$

where

- y_{in} = input-node variable
- y_{out} = output-node variable
- M = Overall gain of the system
- k = total number of forward paths between y_{in} and y_{out}
- M_k = path gain of the k^{th} forward path

Δ = 1 – (sum of the gains of all individual loops) + (sum of products of gains of all possible combinations of two nontouching loops) – (sum of products of gains of all possible combinations of three nontouching loops) + …

Or $\Delta = 1 - \sum_i L_{i1} + \sum_j L_{j2} - \sum_k L_{k3} + \ldots$

Δ_k = the Δ for that part of *SFG* that is non-touching with the k^{th} forward path.

c.

4·c) Forward Path Gain

$M_1 = G_1 G_2 G_3 G_4$, $\Delta_1 = 1$.

$M_2 = G_1 G_2 G_4 G_5$, $\Delta_2 = 1$

Individual Loops

$L_{11} = -G_2 H_1$

$L_{12} = -G_4 H_2$

$L_{13} = -G_1 G_2 G_3 G_4 H_3$

$L_{14} = -G_1 G_2 G_5 G_4 H_3$

Two nontouching loops

$L_{21} = G_2 G_4 H_1 H_2$

$\Delta = 1 - [L_{11} + L_{12} + L_{13} + L_{14}] + L_{21}$

$$\Delta = 1 + G_2H_1 + G_4H_2 + G_1G_2G_3G_4H_3 + G_1G_2G_5G_4H_3 + G_2G_4H_1H_2 .$$

$$\therefore \frac{X_6}{X_1} = \frac{M_1\Delta_1 + M_2\Delta_2}{\Delta}$$

$$\frac{X_6}{X_1} = \frac{G_1G_2G_3G_4 + G_1G_2G_4G_5}{1 + G_2H_1 + G_4H_2 + G_1G_2G_3G_4H_3 + G_1G_2G_5G_4H_3 + G_2G_4H_1H_2}$$

Question 5

a.

Underdamped, $0 < \delta < 1$

Unit step, $R(S) = \frac{1}{S}$.

Second order system, $\frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2 + 2\delta\omega_n S + \omega_n^2}$

$$C(S) = R(S)\,\frac{\omega_n^2}{S^2 + 2\delta\omega_n S + \omega_n^2}$$

$$C(S) = \frac{\omega_n^2}{S(S^2 + 2\delta\omega_n S + \omega_n^2)}$$

By PFE, $C(S) = \frac{A}{S} + \frac{BS + C}{S^2 + 2\delta\omega_n S + \omega_n^2}$

$A = 1,\ B = -1,\ C = -2\delta\omega_n$

$\therefore C(S) = \frac{1}{S} - \frac{S + 2\delta\omega_n}{S^2 + 2\delta\omega_n S + \omega_n^2}$

$\mathcal{L}^{-1}\frac{b}{(S+a)^2 + b^2} = e^{-at}\sin bt$

$\mathcal{L}^{-1}\frac{S+a}{(S+a)^2 + b^2} = e^{-at}\cos bt$

Add and subtract by $\delta^2\omega_n^2$ to denominator

$$C(S) = \frac{1}{S} - \frac{S + 2\delta\omega_n}{S^2 + 2\delta\omega_n S + \omega_n^2 + \delta^2\omega_n^2 - \delta^2\omega_n^2}$$

$$= \frac{1}{S} - \frac{S + 2\delta\omega_n}{(S^2 + 2\delta\omega_n S + \delta^2\omega_n^2) + \omega_n^2(1 - \delta^2)}$$

$\because \omega_d = \omega_n\sqrt{1-\delta^2}$

$$= \frac{1}{S} - \frac{S + 2\delta\omega_n}{(S + \delta\omega_n)^2 + \omega_d^2}$$

$\rightarrow a = \delta\omega_n$, $b = \omega_d$.

$$C(s) = \frac{1}{s} - \frac{s+\delta\omega_n}{(s+\delta\omega_n)^2+\omega_d^2} - \frac{\delta\omega_n \times \omega_d}{\left((s+\delta\omega_n)^2+\omega_d^2\right)\omega_d}$$

$$= \frac{1}{s} - \frac{s+\delta\omega_n}{(s+\delta\omega_n)^2+\omega_d^2} - \frac{\delta\omega_n}{\omega_d}\,\frac{\omega_d}{(s+\delta\omega_n)^2+\omega_d^2}$$

$$c(t) = L^{-1}\left[C(s)\right]$$

$$= 1 - e^{-\delta\omega_n t}\cos\omega_d t - \frac{\delta\omega_n}{\omega_d}e^{-\delta\omega_n t}\sin\omega_d t$$

$$= 1 - e^{-\delta\omega_n t}\left[\cos\omega_d t + \frac{\delta\omega_n}{\omega_n\sqrt{1-\delta^2}}\sin\omega_d t\right]$$

$$c(t) = 1 - e^{-\delta\omega_n t}\left[\cos\omega_d t + \frac{\delta}{\sqrt{1-\delta^2}}\sin\omega_d t\right]$$

$$= 1 - \frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\left[\sqrt{1-\delta^2}\cos\omega_d t + \delta\sin\omega_d t\right]$$

$$= 1 - \frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\left[\sin\omega_d t\cdot\delta + \cos\omega_d t\cdot\sqrt{1-\delta^2}\right]$$

$$\cos\theta = \delta, \quad \sin\theta = \sqrt{1-\delta^2}$$

$$= 1 - \frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\left[\sin\omega_d t\cdot\cos\theta + \cos\omega_d t\cdot\sin\theta\right]$$

$$c(t) = 1 - \frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\sin(\omega_d t + \theta)$$

where $\theta = \tan^{-1}\frac{\sqrt{1-\delta^2}}{\delta}$

b.

5.b) i) $K_p = \lim_{s \to 0} G(s)$

$$= \lim_{s \to 0} \frac{20(s+1)}{s^2(s^2+6s+8)} = \infty.$$

$$K_v = \lim_{s \to 0} s\,G(s) = \infty.$$

$$K_a = \lim_{s \to 0} s^2 G(s) = \frac{20}{8} = 2.5$$

ii) Step, $e_{ss} = \frac{1}{1+K_p} = 0.$

Ramp, $e_{ss} = \frac{1}{K_v} = 0$

Parabola, $e_{ss} = \frac{1}{K_a} = \frac{1}{2.5} = 0.4$

c.

5.c) $s^2+6s+25 \Leftrightarrow s^2+2\delta\omega_n s+\omega_n^2.$

$\omega_n^2 = 25$
$\omega_n = 5$ rad/sec

$2\delta\omega_n = 6$
$\delta = \frac{6}{2\times 5} = 0.6$

Delay Time, $t_d = \frac{1+0.7\delta}{\omega_n} =$

Rise Time, $t_r = \frac{\pi-\theta}{\omega_d}$

$$\theta = \tan^{-1}\frac{\sqrt{1-\delta^2}}{\delta} =$$

$$\omega_d = \omega_n\sqrt{1-\delta^2}$$

$\therefore t_r =$

Peak Time, $t_p = \frac{\pi}{\omega_d} =$

Max overshoot, $\% M_p = e^{-\delta\pi/\sqrt{1-\delta^2}} \times 100$

$=$

Settling Time 2% error, $t_s = \frac{4}{\delta\omega_n} =$

Question 6

a. Difficulty in constructing the Routh array: For complex systems with many equations, manually constructing the Routh array can be tedious and error-prone. Even with computer assistance, it can become time-consuming.
Remedy: Utilize computational software or programming languages with built-in functions to automatically generate the Routh array and analyze the stability.
Inability to handle systems with uncertain parameters or time delays: The Routh-Hurwitz criteria are only applicable to systems with known and constant coefficients. If the system parameters are uncertain or time-delay effects are present, the criteria become less effective or may not be applicable.
Remedy: Explore other stability analysis methods suitable for systems with uncertain parameters or time delays, such as Lyapunov stability analysis or robust control techniques.
Difficulty in dealing with non-polynomial systems: The Routh-Hurwitz criteria are designed for polynomial systems. If the system contains non-polynomial terms or fractional powers, the criteria cannot be directly applied.
Remedy: Linearize the system around an operating point to approximate it with a polynomial form and then apply the Routh-Hurwitz criteria for stability analysis.
Precision issues with floating-point calculations: In some cases, especially for large systems or high-order polynomials, the precision of floating-point calculations may introduce numerical errors and inaccuracies in the Routh-Hurwitz analysis.
Remedy: Utilize numerical methods with higher precision or consider symbolic computation techniques to improve the accuracy of the Routh-Hurwitz analysis.

b.

6. b) Routh table,

s^4	1	4	8
s^3	2	6	
s^2	1	8	
s^1	-10		
s^0	8		

∴ unstable system

c.

6.c)
$$\frac{C(s)}{R(s)} = \frac{\frac{K(s+13)}{s(s+3)(s+7)}}{1 + \frac{K(s+13)}{s(s+3)(s+7)}}$$

$$\frac{C(s)}{R(s)} = \frac{K(s+13)}{s(s+3)(s+7) + K(s+13)}$$

characteristic equation,

$$s(s+3)(s+7) + K(s+13) = 0.$$

$$s^3 + 10s^2 + s(21+K) + 13K = 0$$

Routh table,

s^3	1	$(21+K)$
s^2	10	$13K$
s^1	$\frac{210-3K}{10}$	
s^0	$13K$	

s^0 row $\Rightarrow 13K > 0$

$K > 0.$

s^1 row $\Rightarrow \frac{210-3K}{10} > 0$

$K < 70.$

$\therefore$ Range of K for stable system is,

$$0 < K < 70$$

Question 7

a. i) Angle of Asymptote:

$$\phi_a = \frac{\pm(2q+1) \times 180°}{n-m}, n \neq m$$

where $q = 0,1,2, \ldots, n-m-1$, n and m are numbers of finite poles and zeros

ii) Break away points:

- The breakaway or break-in points on the root loci are the points where the loci meet and then depart or the points where the loci meet and then arrive, respectively.
- These points correspond to multiple-order roots of the characteristic equations.
- The breakaway/break-in points on the root loci must satisfy the criterion (necessary criterion but not sufficient):

$$\frac{dK}{ds} = 0$$

- In order to apply the criterion ,it is necessary to rearrange the characteristic equation to isolate the gain K

b. Root locus

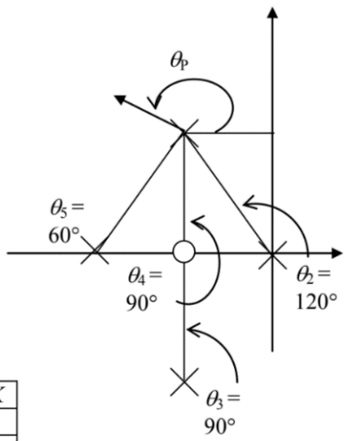
Start points ($K = 0$); $s = 0, -2, -1 \pm j1$
End points ($K = \infty$); $s = -1$ and $s = \infty, \infty, \infty$
Number of branch = degree of the characteristic equation = 4
Root loci on real axis: $-\infty$ to -2, -1 to 0

Angle of asymptotes = $(2q+1)\times180°/(n-m) = (2q+1)\times180°/(3) = 60°, 180°, 300°$
($q = 0, 1, 2; n = 4, m = 1$)

Intersection of asymptotes,

$$-\sigma_A = \frac{\sum(\text{real parts of poles}) - \sum(\text{real parts of zeros})}{n-m} = \frac{-2-1-1+1}{3} = -1$$

Angle of departure:
$\theta_P = 180° + \theta_4 - (\theta_2 + \theta_3 + \theta_5) = 0^\circ$



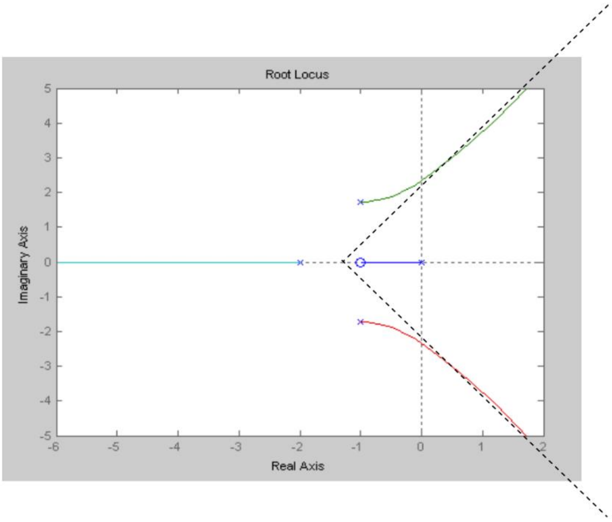
Intersection with imaginary axis:

$$KG(s)H(s) = \frac{K(s+1)}{s(s+2)(s^2+2s+4)}$$

$1+GH = s^4+4s^3+8s^2+s(K+8)+K = 0$

s^4	1	8	K
s^3	4	K+8	
s^2	(24-K)/4	K	
s^1	(192-K^2)/(24-K)		
s^0	K		

From s row, K = 13.856;
The auxiliary equation is $2.535s^2 + 13.856 = 0$; $s = \pm j2.33$

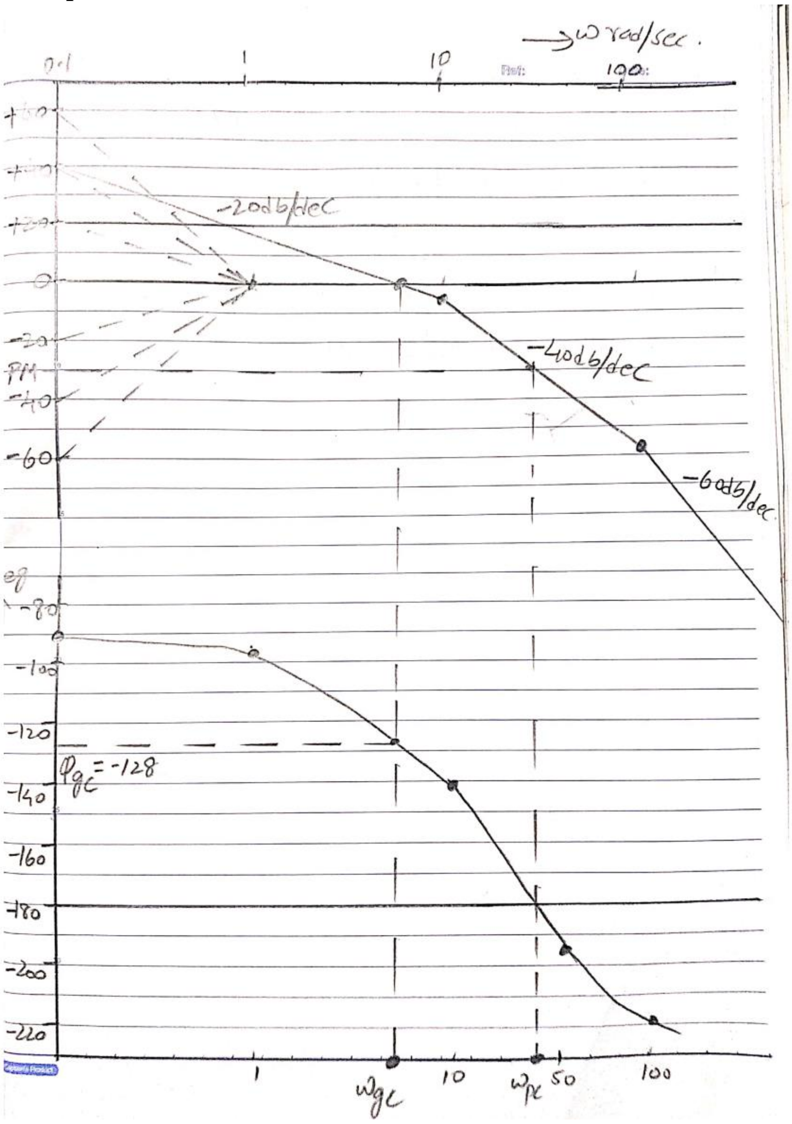


Question 8

a. i) Gain Margin:
Gain margin is a measure used in control systems engineering to quantify the relative stability of a system with feedback control. It indicates how much additional gain (amplification) a system can tolerate before it becomes unstable. Gain margin is an important parameter to assess the robustness of a control system.

ii) Phase Margin:
Phase margin is another critical parameter used in control systems engineering to assess the stability and robustness of a system with feedback control. It quantifies how far the system's phase shift is from -180 degrees at the frequency where the magnitude of the system's open-loop transfer function is unity (1 or 0 dB). Phase margin is particularly important because it provides information about the system's ability to handle phase shifts and phase delays without becoming unstable.

b. Bode plot:



From Bode plot,

i) Gain cross over freq, $\omega_{gc} = 7$ rad/sec.

Phase Margin, $\gamma = 180^\circ + \phi_{gc}$

$= 180 - 128 = 52^\circ$.

ii) Phase cross over freq, $\omega_{pc} = 40$ rad/sec

Gain Margin, $|G(j\omega)| = 30$db.

Question 9

a. Nyquist stability criteria:

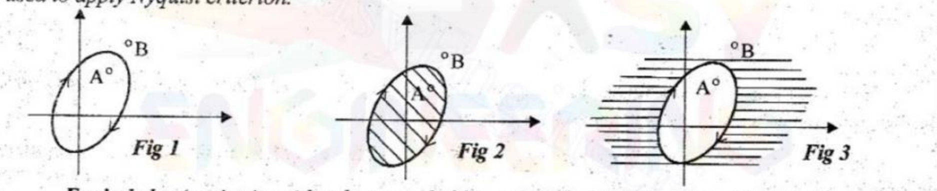
- Let F(s) be a function of s, which is expressed as a ratio of two polynomials in s.

$$F(s) = \frac{(s-z_1)(s-z_2)\ ......\ (s-z_m)}{(s-p_1)(s-p_2)\ ......\ (s-p_n)}$$

- For every point s in the s-pane at which F(s) is analytic, there exists a corresponding point F(s) in the F(s)-plane.
- Hence the function F(s) maps the points in the s-plane into the F(s)-plane.

Concept of encircled and enclosed

It is important to distinguish between the concept of encircled and enclosed which are frequently used to apply Nyquist criterion.

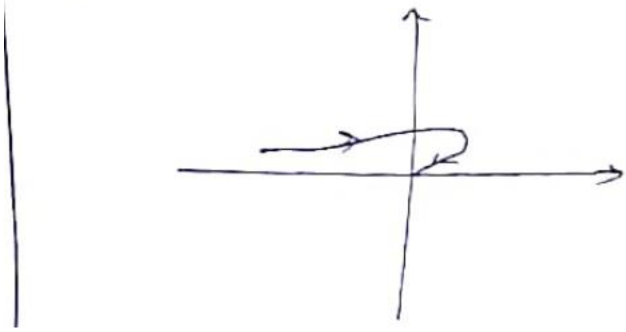


Encircled : *A point is said to be encircled by a closed path if it is found inside the path. With reference to fig 1, the point A is encircled in the clockwise direction and the point B is not encircled.*

Enclosed : *Any point or region is said to be enclosed by a closed path, if it is found to lie to the right of the path when the path is traversed in the prescribed direction. The shaded regions in fig 2 and 3 are the regions enclosed by the closed path. With reference to fig 2, the point A is enclosed by closed path and the point B is not enclosed. With reference to fig 3 the point A is not enclosed by closed path but point B is enclosed.*

b.

9.b) Nyquist Plot



Question 10

a. Lead compensation:

STEP1: For the specified error constant determine the open loop gain necessary

STEP 2: With this value of K draw the magnitude and phase Bode plot and determine the phase margin and the GCF of the uncompensated system.

STEP 3: First design the lag section to provide only partial compensation of phase margin. Choose GCF such that it is higher than the GCF if the system is fully lag compensated.

STEP 4: determine the value of β, such that the high frequency attenuation provided by the Lag network is equal to the magnitude of the uncompensated system at this frequency.

STEP 5: calculate the value of 1such that the upper cut off frequency of lag network is below the gain crossover frequency

STEP 6: calculate the lower cut-off frequency ω1=1/β
So the lag network design is over. Find the transfer function, Draw the magnitude and phase bode plot of the lag compensated system and determine the GCF and phase margin,

STEP 7: For the lead section independent value of α cannot be chosen. So select α=1/β and calculate the maximum lead provided by the lead section by

$$\varphi_m = sin^{-1}\left[\frac{1-\alpha}{1+\alpha}\right]$$

STEP 8: Choose the compensated cross over frequency to coinside with ωm(corresponding to φm).
This is the frequency where lag section has a gain of -10 log β db.

$$\omega_m = \frac{1}{\tau_2\sqrt{\alpha}}$$

Calculate □2 and α□2 and write the lead compensator transfer function.

STEP 9: Combine the transfer function of the lag and lead sections to get the lag-lead compensator transfer function. Draw the bode plot and determine the phase margin.

Step 10: If the specifications are not met , redesign the system by modifying the values of β and □ .

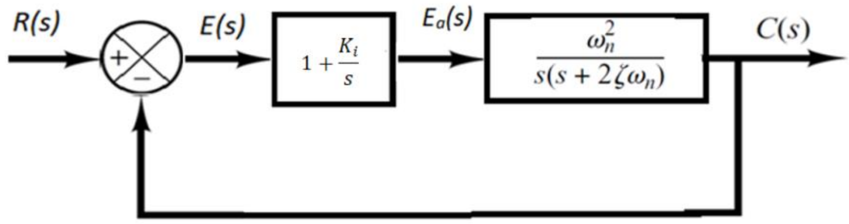
b. i) PI Controller:

Proportional plus Integral controller (PI controller)

In this control the actuating signal consists of proportional error signal added with Integral error signal

$$e_a(t) = e(t) + K_i \int e(t)dt$$

$$E_a(s) = E(s) + K_i \frac{E(s)}{s}$$



For a system without Integral controller the steady state error for ramp input is $= \frac{2\xi}{\omega_n}$

For a system with Integral controller the steady state error ramp input for is =0

Thus Integral controller improves the steady state performance, But it may lead to oscillatory response

ii) PD Controller:

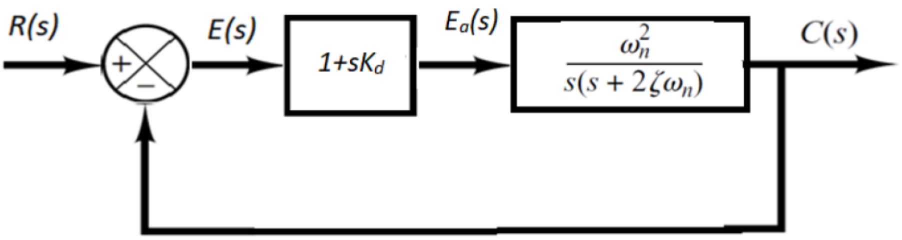
The actuating signal consists of proportional error signal and derivative of the error signal.

$$e_a(t) = e(t) + K_d \frac{de(t)}{dt}$$

Laplace transform of the above equation gives

$$E_a(s) = E(s) + sK_d E(s)$$

$$E_a(s) = [1 + sK_d]E(s)$$



Steady state error is same as that of second order system without derivative controller

Damping increases with ω_n remaining fixed, the system settling time reduces

The damping ratio increases, the maximum overshoot reduces