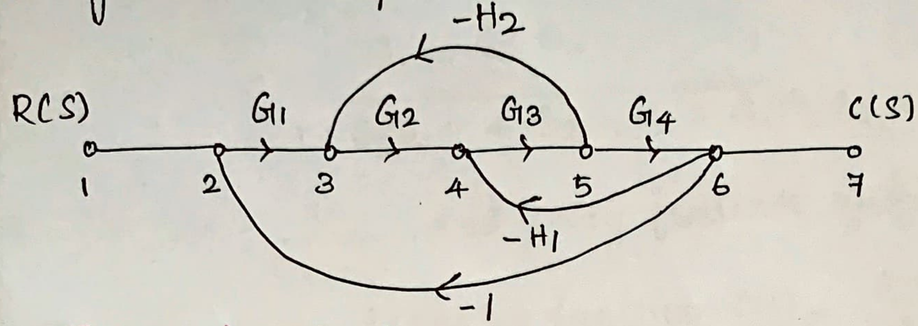
21EC43 - Circuits & Controls

Internal Assessment 3 - Sep 2023

Answer Key

1. Signal Flow Graph:

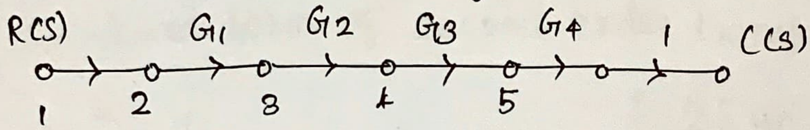


(2 Marks)

1. Forward path Gains:

There is only one forward path. $\therefore k=1$

Let the forward path gain be P_1.

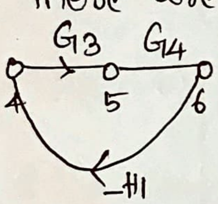


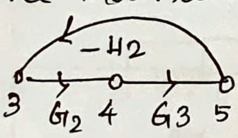
Gain of forward path-1 $P_1 = G_1 G_2 G_3 G_4$

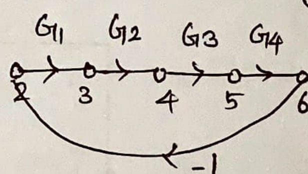
(2 Marks)

2. Individual Loop Gain

There are three individual loops. Let the loop gains be P_{11}, P_{21}, P_{31}







Loop1 Loop 2 Loop-3

Loop gain of individual loop-1 $P_{11} = -G_3 G_4 H_1$

Loop gain of individual loop-2 $P_{21} = -G_2 G_3 H_2$

Loop gain of individual loop-3 $P_{31} = -G_1 G_2 G_3 G_4$

(2 Marks)

3. Gain products of Two Non-touching Loops

There are no possible combinations of two non-touching loops, three non-touching loops, etc.

(2 Marks)

4. Calculation of Δ and Δ_K

$$\Delta = 1 - (P_{11} + P_{21} + P_{31})$$

$$= 1 - (-G_3 G_4 H_1 - G_2 G_3 H_2 - G_1 G_2 G_3 G_4)$$

$$= 1 + G_3 G_4 H_1 + G_2 G_3 H_2 + G_1 G_2 G_3 G_4$$

5. Transfer function, T (2 Marks)

By Mason's gain formula the transfer function, T is given by

$$T = \frac{C(S)}{R(S)} = \frac{1}{\Delta} \sum P_K \Delta_K = \frac{1}{\Delta} P_1 \Delta_1$$

$$= \frac{G_1 G_2 G_3 G_4}{1 + G_3 G_4 H_1 + G_2 G_3 H_2 + G_1 G_2 G_3 G_4}$$

2. $G(s) = \frac{10}{S(S+2)}$ and Unity feedback system

The closed loop transfer function, $\frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)}$.

$$= \frac{\frac{10}{S(S+2)}}{1 + \frac{10}{S(S+2)}} = \frac{10}{S^2+2S+10}$$

Standard form of second order transfer function $\frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2+2\zeta\omega_n S+\omega_n^2}$

$\omega_n^2 = 10$ | $2\zeta\omega_n = 2$

$\omega_n = \sqrt{10} = 3.162$ rad/sec | $\zeta = \frac{2}{2\omega_n} = \frac{1}{3.162} = 0.316$

(2 Marks)

$$\theta = \tan^{-1}\left[\frac{\sqrt{1-\zeta^2}}{\zeta}\right] = \tan^{-1}\frac{\sqrt{1-0.316^2}}{0.316} = 1.249 \text{ rad.}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 3.162\sqrt{1-0.316^2} = 3 \text{ rad/sec.}$$

Rise time, $t_r = \frac{\pi-\theta}{\omega_d} = \frac{\pi-1.249}{3} = 0.63$ sec (2 Marks)

percentage overshoot, $\%M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100 = e^{-\frac{0.316\pi}{\sqrt{1-0.316^2}}} \times 100$

$$= 0.3512 \times 100 = 35.12\ \%$$

Peak time, $t_p = \frac{\pi}{\omega_d} = \frac{\pi}{3} = 1.047$ sec (2 Marks)

Time constant, $T = \frac{1}{\zeta\omega_n} = \frac{1}{0.316 \times 3.162} = 1$ sec (2 Marks)

$\therefore$ For 5% error, settling time $t_s = 3T = 3$ sec

For 2% error, Settling time, $t_s = 4T = 4$ sec

Peak overshoot $= \frac{35.12}{100} \times 12 = 4.2144$ units. (2 Marks)

3. $G(s) = \frac{K}{s(s+1)(s+2)}$. Find the range of K for stability $H(s) = 1$

The closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{K}{s(s+1)(s+2)+K}$

The characteristic equation is $s(s+1)(s+2) + K = 0$ (5 Marks)

$\therefore s(s^2+3s+2) + K = 0 \Rightarrow s^3 + 3s^2 + 2s + K = 0$.

$$\begin{array}{l|cc} s^3: & 1 & 2 \\ s^2: & 3 & K \\ s^1: & \frac{6-K}{3} & \\ s^0: & K & \end{array}$$

column - 1.

For the system to be stable there should not be any sign change in the elements of first column. Hence choose the value of K so that the first column elements are positive. (3 Marks)

From s^0 row, for the system to be stable, $K > 0$.

From s^1 row, the system to be stable, $\frac{6-K}{3} > 0$

For $\frac{6-K}{3} > 0$, the value of K should be less than 6. (2 Marks)

$\therefore$ The range of K for the system to be stable is $0 < K < 6$.

4. Time Domain Specification of a under damped second order system to a step input. (2 Marks)

Rise Time (t_r)

The Unit Step response of second order system for underdamped case is given by $c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$

At $t = t_r$, $c(t) = c(t_r) = 1$.

$$\therefore c(t) = 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_r + \theta) = 1$$

$$\therefore \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_r + \theta) = 0$$

since $e^{-\zeta\omega_n t_r} \neq 0$, the term, $\sin(\omega_d t_r + \theta) = 0$

When $\phi = 0, \pi, 2\pi, 3\pi, \ldots$ $\sin\phi = 0$

$\therefore \omega_d t_r + \theta = \pi$

$$\omega_d t_r = \pi - \theta$$

$$\boxed{t_r = \frac{\pi - \theta}{\omega_d}}$$

(3 Marks)

Peak time (t_p)

To find the expression for peak time t_p, differentiate $c(t)$ wrt t and equate to 0.

$$\left.\frac{d}{dt} c(t)\right|_{t=t_p} = 0$$

The Unit step response of under damped second order system is given by $c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$

Differentiating $c(t)$ wrt t

$$\frac{d}{dt} c(t) = \frac{-e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} (-\zeta\omega_n) \sin(\omega_d t + \theta) + \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \cos(\omega_d t + \theta)\, \omega_d$$

put $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\therefore \frac{d}{dt} c(t) = \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} (\zeta\omega_n) \sin(\omega_d t + \theta) - \frac{\omega_n\sqrt{1-\zeta^2}}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos(\omega_d t + \theta)$$

$$= \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \left[\zeta \sin(\omega_d t + \theta) - \sqrt{1-\zeta^2} \cos(\omega_d t + \theta)\right]$$

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \left[\cos\theta \sin(\omega_d t + \theta) - \sin\theta \cos(\omega_d t + \theta)\right]$$

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \left[\sin(\omega_d t + \theta) - \theta\right]$$

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t)$$

at $t = t_p$, $\frac{d}{dt} c(t) = 0$

$$\therefore \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_p} (\sin\omega_d t_p) = 0$$

$e^{-\zeta\omega_n t_p} \neq 0$, the term $\sin(\omega_d t_p) = 0$

When $\phi = 0, \pi, 2\pi, 3\pi \ldots$, $\sin\phi = 0$

$\therefore \quad \omega_d t_p = \pi$

$$\boxed{t_p = \frac{\pi}{\omega_d}}$$

where $\omega_d = \omega_n \sqrt{1-\zeta^2}$ (2 Marks)

Peak Overshoot (M_p)

% Peak overshoot, $\% M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100$

where $c(t_p)$ = peak response at $t = t_p$.
$c(\infty)$ = final steady state value.

At $t(\infty)$, $c(t) = c(\infty) = 1$

At $t = t_p$, $c(t) = c(t_p) = 1 + e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$

Percentage peak overshoot $\% M_p = \frac{1 + e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} - 1}{1} \times 100$

$$\boxed{\% M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100}$$

(3 Marks)

Settling Time (t_s)

The response of second order system has two components. They are (1) Decaying exponential component, $\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}$

(2) Sinusoidal component, $\sin(\omega_d t + \theta)$.

In this the decaying exponential term dampens (or) reduces the oscillations produced by sinusoidal component.

For 2% tolerance error at $t = t_s$, $\frac{e^{-\zeta\omega_n t_s}}{\sqrt{1-\zeta^2}} = 0.02$

$-\zeta\omega_n t_s = \ln(0.02)$

$-\zeta\omega_n t_s = -4 \Rightarrow t_s = \frac{4}{\zeta\omega_n}$.

$$\boxed{\text{Settling time, } t_s = \frac{4}{\zeta\omega_n} = 4T \text{ (for 2\% error)}}$$

For 5% error, $e^{-\zeta\omega_n t_s} = 0.05$

$-\zeta\omega_n t_s = \ln(0.05) \Rightarrow -\zeta\omega_n t_s = -3 \Rightarrow t_s = \frac{3}{\zeta\omega_n}$

$$\boxed{\text{Settling time, } t_s = \frac{3}{\zeta\omega_n} = 3T \text{ for 5\% error}}$$

5. Characteristic equation

$$s^7 + 9s^6 + 24s^5 + 24s^4 + 24s^3 + 24s^2 + 23s + 15 = 0$$

(5 Marks)

s^7 :	1	24	24	23
s^6 :	9	24	24	15
s^6 :	3	8	8	5
s^5 :	1	1	1	
s^4 :	1	1	1	
s^3 :	0	0		
s^3 :	2	1		
s^2 :	0.5	1		
s^1 :	-3			
s^0 :	1			

column 1

The Auxillary polynomial is

$A = s^4 + s^2 + 1$ (2 Marks)

Differentiate A wrt s

$$\frac{dA}{ds} = 4s^3 + 2s$$

s^3 : 2 1

Put $s^2 = x$ in the auxillary eqn.

$$s^4 + s^2 + 1 = x^2 + x + 1 = 0.$$

The roots of quadratic are,

$$x = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$$

$$= 1\angle 120^\circ \text{ or } 1\angle -120^\circ$$ (3 Marks)

But $s^2 = x$, $\therefore s = \pm\sqrt{x} = \pm\sqrt{1\angle 120^\circ}$ or $\pm\sqrt{1\angle -120^\circ}$

$$= \pm\sqrt{1}\angle 120^\circ/2 \text{ or } \pm\sqrt{1}\angle -120^\circ/2$$

$$= \pm 1\angle 60^\circ \text{ or } \pm 1\angle -60^\circ$$

$$= \pm\, 0.5 + j0.866 \text{ or } \pm(0.5 - j0.866)$$

- The system is unstable.
- Two roots are lying on right half of s-plane and five roots are lying on left half of s-plane.

6. $G(S) = \dfrac{K}{S(S^2+4S+13)}$, $H(S) = 1$.

(1 Mark)

Step 1: To locate poles and zeros

The poles of open loop transfer fn. are the roots of the eqn. $S(S^2+4S+13) = 0$.

The roots of the quadratic are $S = \dfrac{-4 \pm \sqrt{4^2-4\times13}}{2} = -2 \pm j3$

The poles are lying at $S = \overset{P_1}{0}, \overset{P_2}{-2+j3}$ and $\overset{P_3}{-2-j3}$.

(1 Mark)

Step 2: To find the root locus on real axis

There is only one pole on real axis at the origin. Hence if we choose any test point on the negative real axis then to the right of that point the total number of real poles and zeros is one, which is an odd number. Hence the entire negative real axis will be part of root locus.

(3 Marks)

Step 3: To find angles of asymptotes and centroid

Since there are 3 poles, the number of root locus branches are three.

$$\text{Angles of asymptotes} = \frac{\pm 180^\circ (2q+1)}{n-m} \qquad q = 0, 1, \ldots n-m$$

Here $n = 3$ and $m = 0$. $\therefore q = 0, 1, 2, 3$.

When $q = 0$, Angles $= \pm \dfrac{180^\circ}{3} = \pm 60^\circ$

When $q = 1$, Angles $= \pm \dfrac{180^\circ \times 3}{3} = \pm 180^\circ$

When $q = 2$, Angles $= \pm \dfrac{180^\circ \times 5}{3} = \pm 300^\circ = \mp 60^\circ$

When $q = 3$, Angles $= \pm \dfrac{180^\circ \times 7}{3} = \pm 420^\circ = \pm 60^\circ$

$$\text{Centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m}$$

$$= \frac{0 - 2 + j3 - 2 - j3 - 0}{3}$$

$$= \frac{-4}{3} = -1.33.$$

(2 Marks)

Step 4: To find the breakaway and breakin points.

The closed loop transfer function $\dfrac{C(S)}{R(S)} = \dfrac{G(S)}{1+G(S)} = \dfrac{K}{S(S^2+4S+13)+K}$

The characteristic equation is $S(S^2+4S+13) + K = 0$.

$\therefore K = -S^3 - 4S^2 - 13S$

On differentiating the eqn. of k wrt s we get,

$$\frac{dk}{ds} = -(3s^2 + 8s + 13)$$

Put $\frac{dk}{ds} = 0 \Rightarrow 3s^2 + 8s + 13 = 0$

$$s = \frac{-8 \pm \sqrt{8^2 - 4 \times 13 \times 3}}{2 \times 3} = -1.33 \pm j1.6$$

Check for k: When $s = -1.33 + j1.6$ the value of k is given by

$$k = -[(-1.33 + j1.6)^3 + 4(-1.33 + j1.6)^2 + 13(-1.33 + j1.6)]$$

$\neq$ positive and real.

When $s = -1.33 - j1.6$, the value of k is not equal to real and positive.

Since the value of k, for $s = -1.33 \pm j1.6$ are not real and positive. These points are not an actual breakaway or breakin points. These points are not an actual breakaway or breakin point.

The root locus has neither breakaway nor breakin point. (2 Marks)

Step 5: To find the angle of departure

Let us consider the complex pole P_2. Draw vectors from all other poles to the pole P_2.

Let the angles of these vectors be θ_1 and θ_2.

$\theta_1 = 180° - \tan^{-1}\left(\frac{3}{2}\right) = 123.7°$; $\theta_2 = 90°$

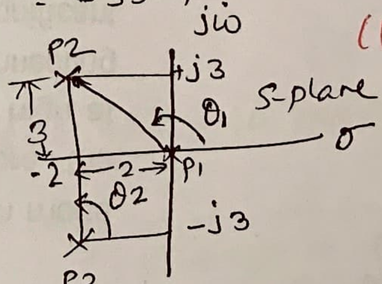
Angle of departure from the complex pole $P_2 = 180° - (\theta_1 + \theta_2)$

$= 180° - (123.7° + 90°)$

$= -33.7°$

Angle of departure at pole $P_3 = +33.7°$.

(1 Mark)



Step 6: To find the crossing point on imaginary axis

The characteristic eqn is given by

$$s^3 + 4s^2 + 13s + K = 0$$

Put $s = j\omega$, $(j\omega)^3 + 4(j\omega)^2 + 13(j\omega) + K = 0 \Rightarrow -j\omega^3 - 4\omega^2 + 13j\omega + K = 0$

On equating img part to zero, we get

$-\omega^3 + 13\omega = 0$

$-\omega^3 = -13\omega$

$\omega^2 = 13 \Rightarrow \omega = \pm\sqrt{13} = \pm 3.6$

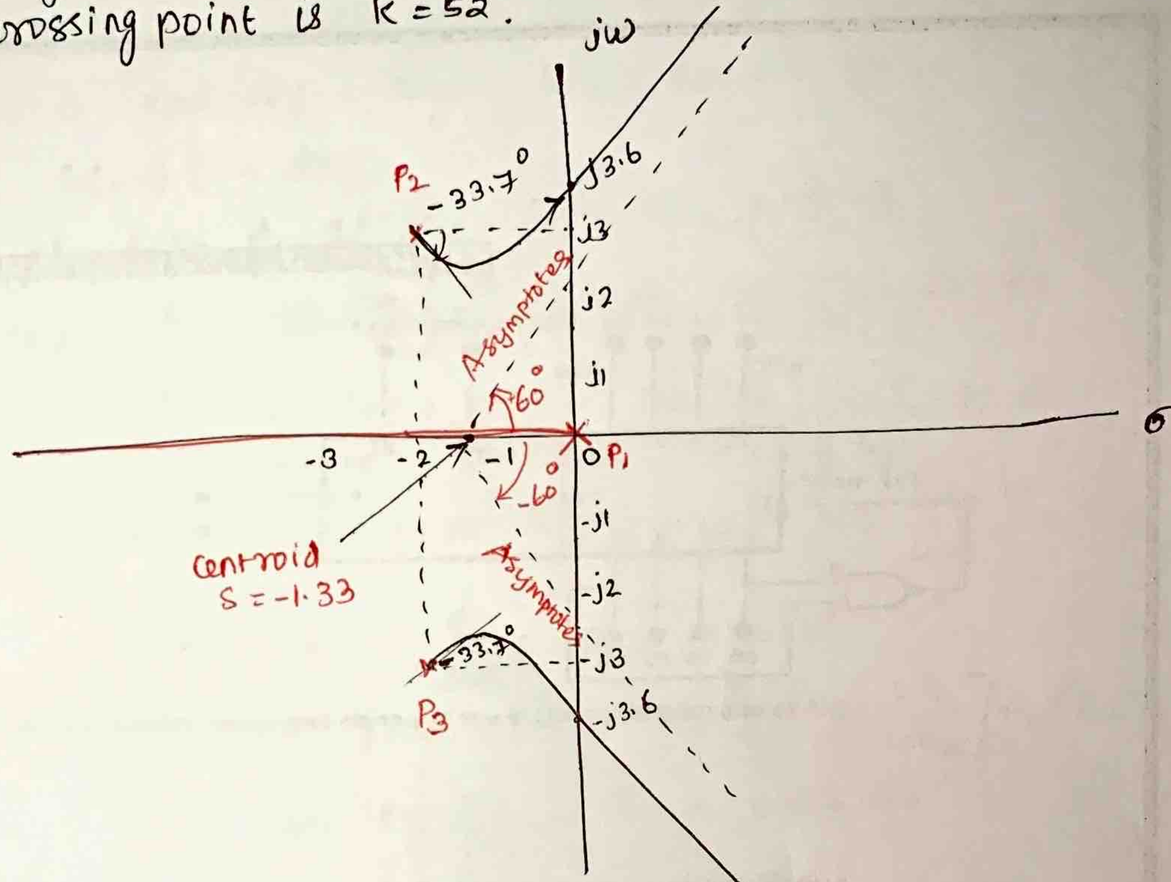
On equating real part to zero, we get

$-4\omega^2 + K = 0$

$K = 4\omega^2$

$= 4 \times 13 = 52$

The crossing point of root locus is $\pm j\,3.6$. The value of K at this crossing point is $K = 52$.



7. Bode plot

$$G(S) = \frac{10}{S(1+0.4S)(1+0.1S)}$$

The Sinusoidal transfer function of $G(j\omega)$ is obtained by replacing S by $j\omega$ in the given transfer function.

(1 Mark)

$$\therefore\ G(j\omega) = \frac{10}{j\omega(1+0.4j\omega)(1+0.1j\omega)}$$

(4 Marks)

Magnitude plot :

The corner frequencies are $\omega_{c1} = \frac{1}{0.4} = 2.5$ rad/sec and $\omega_{c2} = \frac{1}{0.1} = 10$ rad/sec.

Table 1

Term	Corner frequency rad/sec	slope db/dec	change in slope db/dec
$\frac{10}{j\omega}$	-	-20	
$\frac{1}{1+0.4j\omega}$	$\omega_{c1} = \frac{1}{0.4} = 2.5$	-20	$-20-20=-40$
$\frac{1}{1+0.1j\omega}$	$\omega_{c2} = \frac{1}{0.1} = 10$	-20	$-40-20=-60$

choose a low frequency ω_l such that $\omega_l < \omega_{c_1}$ and choose a high frequency ω_h such that $\omega_h > \omega_{c_2}$.

Let $\omega_l = 0.1$ rad/sec and $\omega_h = 50$ rad/sec.

Let $A = |G(j\omega)|$ in db.

Let us calculate A at ω_l, ω_{c_1}, ω_{c_2} and ω_h.

At $\omega = \omega_l$, $A = 20 \log \left|\frac{10}{j\omega}\right| = 20 \log \left|\frac{10}{0.1}\right| = 40$ db

At $\omega = \omega_{c_1}$, $A = 20 \log \left|\frac{10}{j\omega}\right| = 20 \log \left(\frac{10}{2.5}\right) = 12$ db

At $\omega = \omega_{c_2}$, $A = \left[\text{Slope from } \omega_{c_1} \text{ to } \omega_{c_2} \times \log \frac{\omega_{c_2}}{\omega_{c_1}}\right] + A_{(\text{at } \omega = \omega_{c_1})}$

$A = -40 \log \frac{10}{2.5} + 12 = -12$ db

At $\omega = \omega_h$, $A = \left[\text{Slope from } \omega_{c_2} \text{ to } \omega_h \times \log \frac{\omega_h}{\omega_{c_2}}\right] + A_{(\text{at } \omega = \omega_{c_2})}$

$= -60 \log \frac{50}{10} - 12 = -54$ db.

Phase plot :

The phase angle $G(j\omega)$ as a function of ω is given by (3 Marks)

$\phi = -90^\circ - \tan^{-1} 0.4\omega - \tan^{-1} 0.1\omega$

ω	ϕ (deg)
0.1	-92
1	-118
2.5	-150
4	-170
10	-210
20	-236

(2 Marks)

Gain Cross-over Frequency = 5 rad/sec

phase cross-over frequency = 5 rad/sec.