Internal Assessment Test II – May 2023

Sub: Finite Element Methods **Code:** 18ME61

Date: 23/ 05 / 2023 Duration: 90 mins Max Marks: 50 **Sem:** VI **Branch:** MECH

Note: Answer all questions.

		Marks	OBE CO	OBE RBT
1	Derive the stiffness matrix for truss element in terms of direction cosines.	**10**	CO4	L2
2	Derive Hermite shape function for beams.	**10**	CO3	L2
3	Determine the nodal displacements, stress on each element and reaction for the structure shown in fig 1. Take E = 2 × 10^5 MPa, A = 200mm².	**20**	CO4	L3

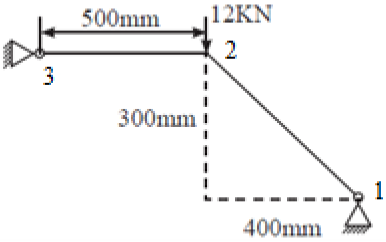

fig 1.

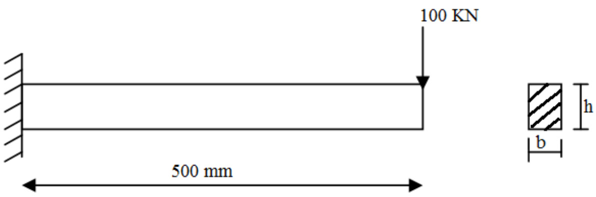

fig 2.

4	Calculate the deflection at the free end of the cantilever beam shown in fig 2. Take E = 2 × 10^5 MPa; b = 10 mm, d = 2b.	**10**	CO5	L3

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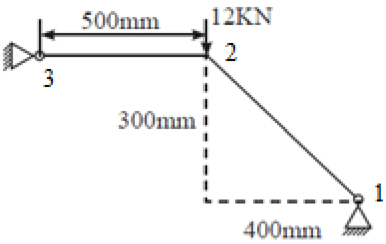

fig 1.

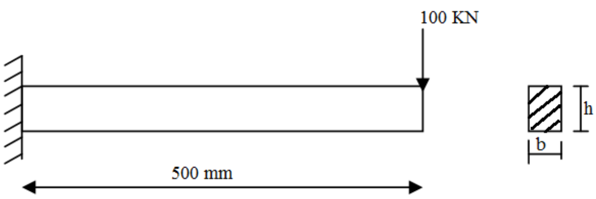

fig 2.

4	Calculate the deflection at the free end of the cantilever beam shown in fig 2. Take E = 2 × 10^5 MPa; b = 10 mm, d = 2b.	**10**	CO5	L3

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DERIVATION OF ELEMENTAL STIFFNESS MATRIX FOR A TRUSS ELEMENT

Relation b/w nodal displacement of a truss element in local coordinates & global coordinate is expressed as

$$q' = L q$$

Where $q' = \begin{bmatrix} q_1' \\ q_2' \end{bmatrix}$; $L = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$; $q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$

A truss element in local coordinate is equivalent to 1D bar element having the stiffness matrix

$$K' = \frac{EA}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Strain energy for a truss element in local coordinate is given by

$$U_e = \frac{1}{2} q'^T K' q' \rightarrow (1)$$

It is required to determine strain energy of truss element in global coordinates.

$$q' = L q$$

Sub. this in eqn (1) we get

$$U_e = \frac{1}{2} q^T L^T K' L q$$

$$U_e = \frac{1}{2} q^T \left(L^T K' L \right) q$$

$$U_e = \frac{1}{2} q^T k_e q$$

Where K_e - elemental stiffness matrix in global coordinates

$$K_e = L^T K' L$$

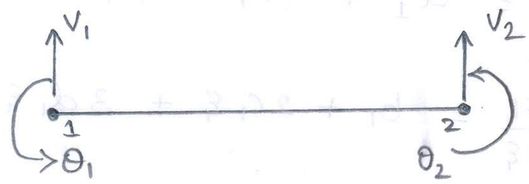
$$L = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix} \quad ; \quad L^T = \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix} \quad ; \quad K' = \frac{EA}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K_e = \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix} \frac{EA}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$$

$$K_e = \frac{EA}{l_e} \begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

Where l & m are direction cosines.

BEAMS



V_1 → Displacement at node 1.

θ_1 → Slope $\frac{dy}{dx}$ at node 1.

V_2 → Displacement at node 2

θ_2 → slope $\frac{dy}{dx}$ at node 2.

V. IMP

DERIVATION OF SHAPE FUNCTION [HERMITE SHAPE FUNCTION]

Let us consider the shape function H_i as

$$H_i = a_i + b_i\xi + c_i\xi^2 + d_i\xi^3$$

	H_1	H_1'	H_2	H_2'	H_3	H_3'	H_4	H_4'
$\xi = -1$. (Node 1)	1	0	0	1	0	0	0	0
$\xi = 1$. (Node 2)	0	0	0	0	1	0	0	1

$$H_i' = \frac{dH}{d\xi}$$

Let shape function H_1 be

$$H_1 = a_1 + b_1\xi + c_1\xi^2 + d_1\xi^3$$

$$H_1' = \frac{dH_1}{d\xi} = b_1 + 2c_1\xi + 3d_1\xi^2$$

At node 1, $H_1 = 1$, $\xi = -1$.

$$1 = a_1 - b_1 + c_1 - d_1 \rightarrow (1)$$

At node 1, $H_1' = 0$, $\xi = -1$

$$0 = b_1 - 2c_1 + 3d_1 \rightarrow (2)$$

At node 2, $H_1 = 0$, $\xi = 1$.

$$0 = a_1 + b_1 + c_1 + d_1 \rightarrow (3)$$

At node 2, $H_1' = 0$, $\xi = 1$.

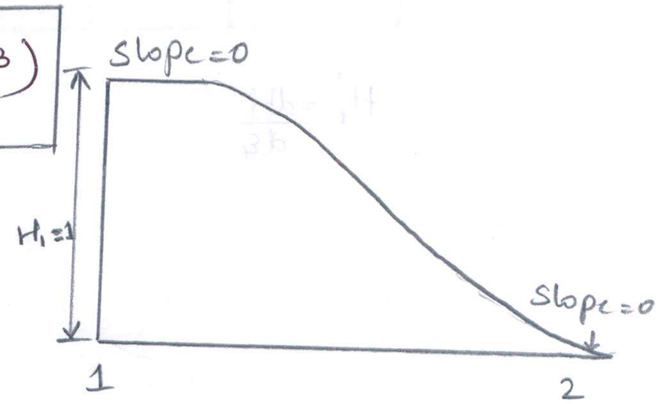
$$0 = b_1 + 2c_1 + 3d_1 \rightarrow (4)$$

Solving (1) (2) (3) & (4) we get

$$a_1 = \frac{1}{2} \; ; \quad b_1 = \frac{-3}{4} \; ; \quad c_1 = 0 \; ; \quad d_1 = \frac{1}{4}$$

$$H_1 = \frac{1}{2} - \frac{3}{4}\xi + \frac{1}{4}\xi^3$$

$$\boxed{H_1 = \frac{1}{4}\left(2 - 3\xi + \xi^3\right)}$$



Let the shape function H_2 be

$$H_2 = a_2 + b_2\xi + c_2\xi^2 + d_2\xi^3$$

$$H_2' = b_2 + 2c_2\xi + 3d_2\xi^2$$

At node 1, $H_2 = 0$, $\xi = -1$

$$0 = a_2 - b_2 + c_2 - d_2 \rightarrow (5)$$

At node 1, $H_2' = 1$, $\xi = -1$

$$01 = b_2 - 2c_2 + 3d_2 \rightarrow (6)$$

At node 2, $H_2 = 0$, $\xi = 1$.

$$0 = a_2 + b_2 + c_2 + d_2 \rightarrow (7)$$

At node 2, $H_2' = 0$, $\xi = 1$.

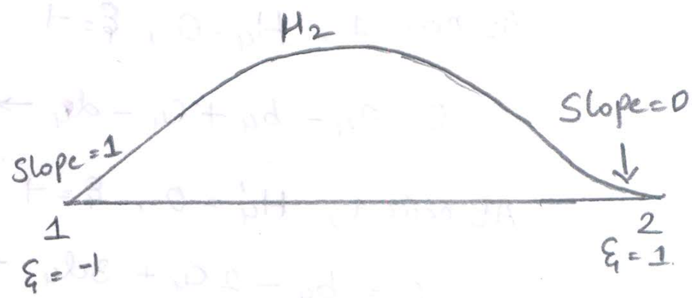
$$0 = b_2 + 2c_2 + 3d_2 \rightarrow (8)$$

Solving 5, 6, 7 & 8

$a_2 = \frac{1}{4}$, $b_2 = -\frac{1}{4}$; $c_2 = -\frac{1}{4}$; $d_2 = \frac{1}{4}$

$$H_2 = \frac{1}{4} - \frac{1}{4}\xi - \frac{1}{4}\xi^2 + \frac{1}{4}\xi^3$$

$$\boxed{H_2 = \frac{1}{4}\left(1 - \xi - \xi^2 + \xi^3\right)}$$



Let the shape function H_3 be

$$H_3 = a_3 + b_3\xi + c_3\xi^2 + d_3\xi^3$$

$$H_3' = b_3 + 2c_3\xi + 3d_3\xi^2$$

At node 1, $H_3 = 0$, $\xi = -1$

$$0 = a_3 - b_3 + c_3 - d_3 \rightarrow (9)$$

At node 1, $H_3' = 0$ $\xi = -1$

$$0 = b_3 - 2c_3 + 3d_3 \rightarrow (10)$$

At node 2, $H_3 = 1$, $\xi = 1$.

$$1 = a_3 + b_3 + c_3 + d_3 \rightarrow (11)$$

At node 2, $H_3 = 0$, $\xi = 1$.

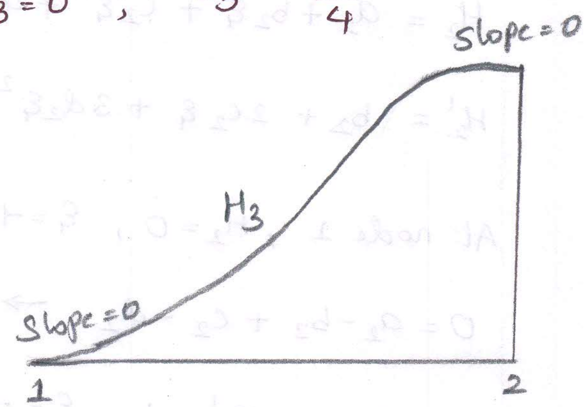
$$0 = b_3 + 2c_3 + 3d_3 \rightarrow (12)$$

Solving 9, 10, 11 & 12 we get

$a_3 = \frac{1}{2}$; $b_3 = \frac{3}{4}$; $c_3 = 0$; $d_3 = -\frac{1}{4}$

$$H_3 = \frac{1}{2} + \frac{3}{4}\xi - \frac{1}{4}\xi^3$$

$$\boxed{H_3 = \frac{1}{4}\left(2 + 3\xi - \xi^3\right)}$$



Let the Shape function H_4 be

$$H_4 = a_4 + b_4\xi + c_4\xi^2 + d_4\xi^3$$

$$H_4' = b_4 + 2c_4\xi + 3d_4\xi^2$$

At node 1, $H_4 = 0$, $\xi = -1$

$$0 = a_4 - b_4 + c_4 - d_4 \rightarrow (13)$$

At node 1, $H_4' = 0$, $\xi = -1$

$$0 = b_4 - 2c_4 + 3d_4 \rightarrow (14)$$

At node 2, $H_4 = 0$, $\xi = 1$

$$0 = b_4 + 2c_4 + 3d_4 \rightarrow (15)$$

At node 2, $H_4 = 01$ $\xi = 1$.

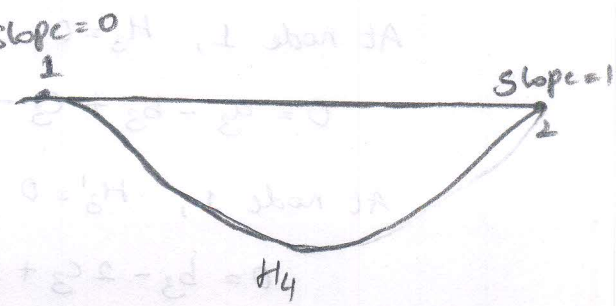
$$01 = b_4 + 2c_4 + 3d_4 \rightarrow (16)$$

Solving 13 14 15 & 16 we get

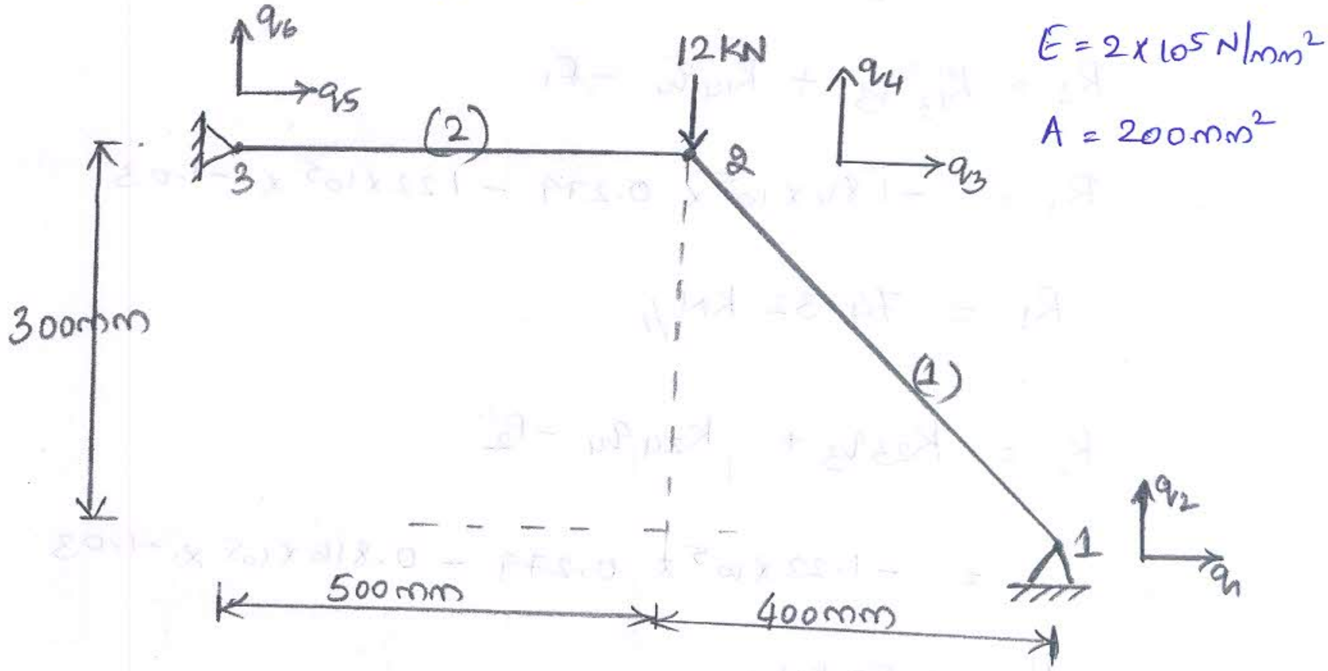
$a_4 = \frac{-1}{4}$; $b_4 = -\frac{1}{4}$; $c_4 = \frac{1}{4}$; $d_4 = \frac{1}{4}$

$$H_4 = -\frac{1}{4} - \frac{1}{4}\xi + \frac{1}{4}\xi^2 + \frac{1}{4}\xi^3$$

$$\boxed{H_4 = \frac{1}{4}\left(-1 - \xi + \xi^2 + \xi^3\right)}$$



(P) Determine the nodal displacements, stress in each element & reaction at supports for the truss shown.



Sol: Node Data

Node No	x (mm)	y (mm)
1	900	0
2	500	300
3	0	300

Element Connectivity table

Element No	Initial Node (IN)	Final Node (FN)	length of element 'le'	l	m
1	1	2	500	-0.8	0.6
2	2	3	500	-1	0

$$l_e = \sqrt{(x_{FN}-x_{IN})^2 + (y_{FN}-y_{IN})^2} \; ; \quad l = \frac{x_{FN}-x_{IN}}{l_e} \; ; \quad m = \frac{y_{FN}-y_{IN}}{l_e}$$

Elemental stiffness matrix

$$K = \frac{EA}{l_e}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

$$= \frac{2\times10^5 \times 200}{500}\begin{bmatrix} 0.64 & -0.48 & -0.64 & 0.48 \\ -0.48 & 0.36 & 0.48 & -0.36 \\ -0.64 & 0.48 & 0.64 & -0.48 \\ 0.48 & -0.36 & -0.48 & 0.36 \end{bmatrix}$$

$$= 10^5 \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0.512 & -0.384 & -0.512 & 0.384 \\ -0.384 & 0.288 & 0.384 & -0.288 \\ -0.512 & 0.384 & 0.512 & -0.384 \\ 0.384 & -0.288 & -0.384 & 0.288 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

For element 2

$$K = \frac{2\times10^5 \times 200}{500}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= 10^5 \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0.8 & 0 & -0.8 & 0 \\ 0 & 0 & 0 & 0 \\ -0.8 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

Global stiffness matrix

$$K = 10^5 \begin{bmatrix} 0.512 & -0.384 & -0.512 & 0.384 & 0 & 0 \\ -0.384 & 0.288 & 0.384 & -0.288 & 0 & 0 \\ -0.512 & 0.384 & 1.312 & -0.384 & -0.8 & 0 \\ 0.384 & -0.288 & -0.384 & 0.288 & 0 & 0 \\ 0 & 0 & -0.8 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Equilibrium eqn $[K][q] = [F]$

$$10^5 \begin{bmatrix} 0.512 & -0.384 & -0.512 & 0.384 & 0 & 0 \\ -0.384 & 0.288 & 0.384 & -0.288 & 0 & 0 \\ -0.512 & 0.384 & 1.312 & -0.384 & -0.8 & 0 \\ 0.384 & -0.288 & -0.384 & 0.288 & 0 & 0 \\ 0 & 0 & -0.8 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -12\times10^3 \\ 0 \\ 0 \end{bmatrix}$$

$$10^5 \begin{bmatrix} 1.312 & -0.384 \\ -0.384 & 0.288 \end{bmatrix} \begin{bmatrix} q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -12\times10^3 \end{bmatrix}$$

$q_3 = -0.2\,mm$; $q_4 = -0.68\,mm$

$q_1 = q_2 = q_5 = q_6 = 0$

Stresses

For element 1

$$\sigma_1 = E \cdot \frac{1}{l_e} \begin{bmatrix} -l & -m & l & m \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

$$\sigma_1 = 2\times10^5 \times \frac{1}{500} \begin{bmatrix} 0.8 & -0.6 & -0.8 & 0.6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -0.2 \\ -0.68 \end{bmatrix}$$

$\sigma_1 = -99.9\ N/mm^2$

$$\sigma_2 = E \cdot \frac{1}{l_e}\begin{bmatrix} -l & -m & l & m \end{bmatrix}\begin{bmatrix} q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix}$$

$$= 2 \times 10^5 \times \frac{1}{500}\begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} -0.2 \\ -0.68 \\ 0 \\ 0 \end{bmatrix}$$

$$\sigma_2 = -80 \ N/mm^2 //$$

<u>Reaction</u> $[R] = [K][Q] - [F]$

$$R_1 = K_{11}\cancel{q_1} + K_{12}\cancel{q_2} + K_{13} q_3 + K_{14} q_4 + K_{15}\cancel{q_5} + K_{16}\cancel{q_6} - \cancel{F_1}$$

$$= (-0.512 \times 10^5 \times -0.2) + (0.384 \times 10^5 \times -0.68)$$

$$R_1 = -15.872 \ KN //$$

$$R_2 = K_{23} q_3 + K_{24} q_4 - \cancel{F_2}$$

$$= (0.384 \times 10^5 \times -0.2) + (-0.288 \times 10^5 \times -0.68)$$

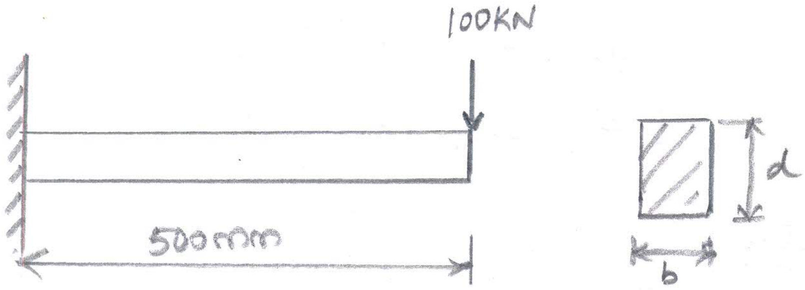
$$R_2 = 11.904 \ KN //$$

$$R_5 = K_{53} q_3 + K_{54} q_4 - \cancel{F_5}$$

$$= (0.8 \times 10^5 \times -0.2) + 0$$

$$R_5 = -16 \ KN //$$

(P) Calculate the deflection at free end of the Cantilever beam as shown in fig.

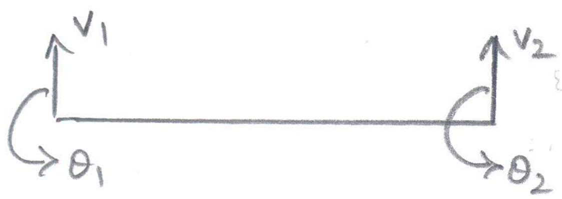


$E = 2 \times 10^5$ MPa

$b = 10$ mm

$d = 2b$

Sol:



$$I = \frac{bd^3}{12} = \frac{10 \times 20^3}{12} = 6.667 \times 10^3 \ mm^4$$

$$= 6.66 \times 10^3 \times 10^{-12}$$

$$I = 6.66 \times 10^{-9} \ m^4$$

$$E = 2 \times 10^5 \times 10^6 = 2 \times 10^{11} \ N/m^2$$

$$\frac{EI}{l_e^3} = \frac{2 \times 10^{11} \times 6.66 \times 10^{-9}}{0.5^3} = 10.656 \times 10^3$$

Elemental stiffness matrix

$$K = \frac{EI}{l_e^3} \begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}$$

$$K = 10.656 \times 10^3 \begin{bmatrix} 12 & 3 & -12 & 3 \\ 3 & 1 & -3 & 0.5 \\ -12 & -3 & 12 & -3 \\ 3 & 0.5 & -3 & 1 \end{bmatrix}$$

Load vector

$$f_e = \begin{bmatrix} 0 \\ 0 \\ 100 \times 10^3 \\ 0 \end{bmatrix}$$

Equilibrium eqn.

$$10.656 \times 10^3 \begin{bmatrix} 12 & 3 & -12 & 3 \\ 3 & 1 & -3 & 0.5 \\ -12 & -3 & 12 & -3 \\ 3 & 0.5 & -3 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ \theta_1 \\ V_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -100 \times 10^3 \\ 0 \end{bmatrix}$$

$\boxed{V_2 = -3.128 \text{ m}}$; $\boxed{\theta_2 = -9.38 \text{ rad}}$

Deflection at node 2 $V_2 = -3.128$ m

Slope at node 2 $\theta_2 = -9.38$ rad //