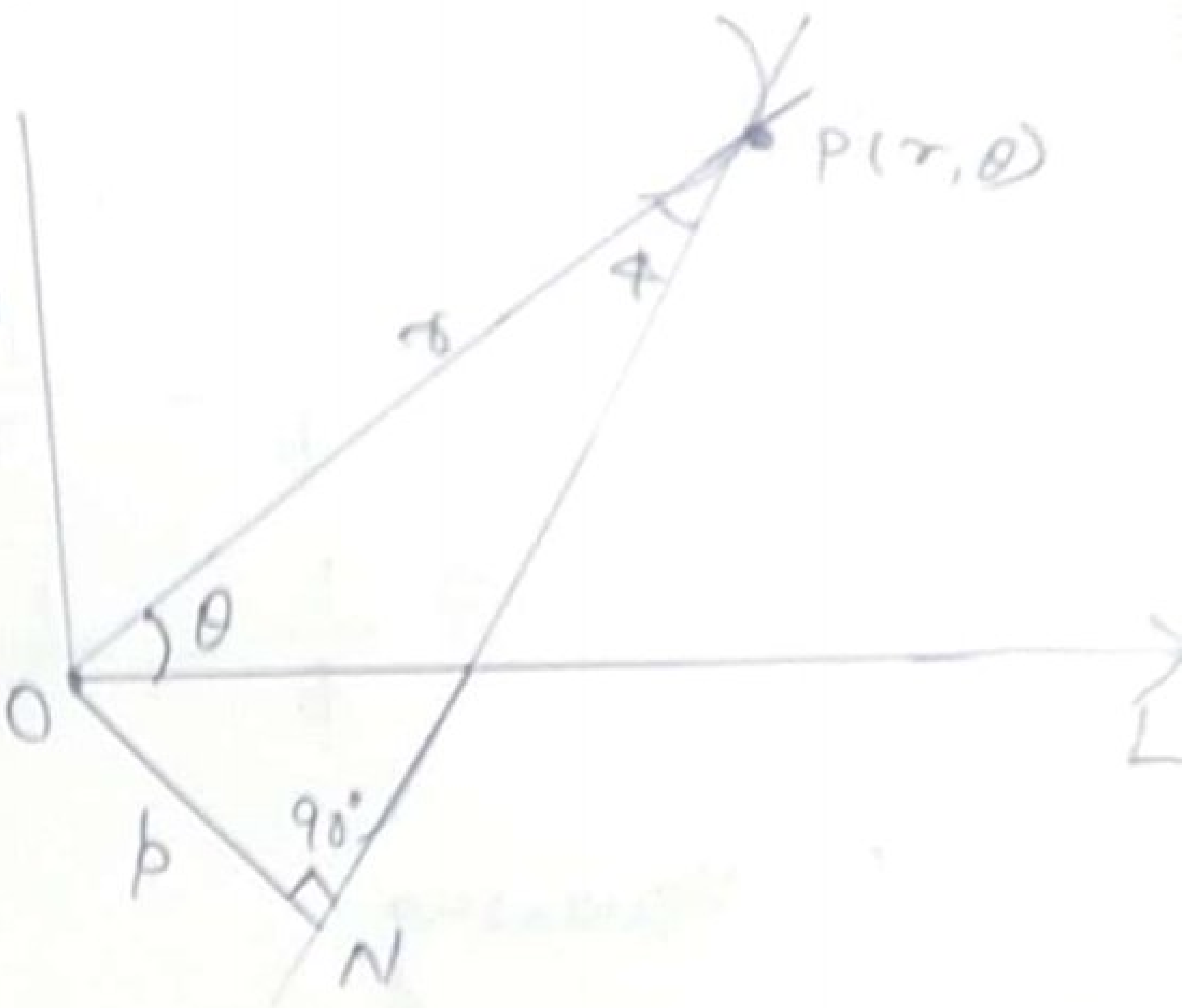
1) With usual notations prove that for the curve, $r=f(\theta)$,

$$\frac{1}{p^2}=\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2.$$



Sol: Let O be the pole and OL be the initial line. Let $P(r,\theta)$ be any point on the curve.

Therefore we have

$OP=r$ & $\angle LOP=\theta$.

Draw ON $\perp^{r}$ from the pole onto the tangent at P, and let ϕ be the angle made by the radius vector with the tangent.

Let $ON=p$.

From the right angled triangle ONP,

$$\sin\phi=\frac{ON}{OP}=\frac{p}{r}$$

$$\Rightarrow p=r\sin\phi. \quad \text{——} (1)$$

Eqn (1) is the basic expression for the length of the perpendicular p.

We have,

$$\cot\phi = \frac{1}{r}\frac{dr}{d\theta} \quad — (2)$$

From (1), $\quad p = r\sin\phi$

$$\Rightarrow \frac{1}{p} = \frac{1}{r}\frac{1}{\sin\phi}$$

$$\Rightarrow \frac{1}{p} = \frac{1}{r}\operatorname{cosec}\phi$$

On squaring,

$$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left[1+\left(\frac{1}{r}\frac{dr}{d\theta}\right)^2\right] \quad (\text{from } (2))$$

$$\Rightarrow \boxed{\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2}$$

CMR

2) Find the angle between the curves, $r^2 \sin 2\theta = 4$ and $r^2 = 16 \sin 2\theta$.

Sol:-

Given curves,

$$r^2 \sin 2\theta = 4 \quad \text{--- (1)}$$

$$\& \; r^2 = 16 \sin 2\theta \quad \text{--- (2)}$$

Let ϕ_1 and ϕ_2 be the angles between the radius vector and the tangents of the given curves (1) & (2) respectively.

From (1),

$$r^2 \sin 2\theta = 4$$

Taking log on both sides, we get

$$\log r^2 + \log \sin 2\theta = \log 4$$

$$\Rightarrow 2 \log r + \log \sin 2\theta = \log 4$$

Diff. w.r.t 'θ', we get

$$2 \cdot \frac{1}{r} \frac{dr}{d\theta} + \frac{1}{\sin 2\theta} 2 \cos 2\theta = 0$$

$$\Rightarrow 2 \cdot \frac{1}{r} \frac{dr}{d\theta} = -2 \frac{\cos 2\theta}{\sin 2\theta}$$

$\Rightarrow \cot \phi_1 = -\cot 2\theta$

$\Rightarrow \cot \phi_1 = \cot(-2\theta)$

$\Rightarrow \boxed{\phi_1 = -2\theta}$

From (2),

$$r^2 = 16 \sin 2\theta$$

Taking log on both sides, we get

$$2 \log r = \log 16 + \log \sin 2\theta$$

Diff. w.r.t 'θ', we get

$$2 \cdot \frac{1}{r} \frac{dr}{d\theta} = 0 + \frac{1}{\sin 2\theta} 2\cos 2\theta$$

$\Rightarrow \cot \phi_2 = \cot 2\theta$

$\Rightarrow \boxed{\phi_2 = 2\theta}$

Thus, $|\phi_2 - \phi_1| = |2\theta - (-2\theta)| = |4\theta| = 4\theta$ — (3)

Now, sub. (2) in (1), we get

$$16 \sin 2\theta \cdot \sin 2\theta = 4$$

$\Rightarrow \sin^2 2\theta = \frac{4}{16} \Rightarrow \sin^2 2\theta = \frac{1}{4}$

$\Rightarrow \sin 2\theta = \frac{1}{2}$

$\Rightarrow \sin 2\theta = \sin \pi/6$

$\Rightarrow 2\theta = \pi/6$

$\Rightarrow \theta = \pi/12.$

From ③,

$|\phi_2 - \phi_1| = 4 \cdot \frac{\pi}{12}$

$\Rightarrow \boxed{|\phi_2 - \phi_1| = \pi/3}$

Thus the angle of intersection is $\pi/3$.

3) Find the pedal eqⁿ of the curve $r^m = a^m(\cos m\theta + \sin m\theta)$.

Solⁿ:- Given curve,

$$r^m = a^m(\cos m\theta + \sin m\theta) \quad \text{— ①}$$

Taking log on both sides, we get

$$m \log r = m \log a + \log(\cos m\theta + \sin m\theta)$$

Diff. w.r.t. 'θ', we get

$$m \cdot \frac{1}{r}\frac{dr}{d\theta} = m \cdot 0 + \frac{1}{\cos m\theta + \sin m\theta}(-m\sin m\theta + m\cos m\theta)$$

$$\Rightarrow m\left(\frac{1}{r}\frac{dr}{d\theta}\right) = m\left(\frac{\cos m\theta - \sin m\theta}{\cos m\theta + \sin m\theta}\right)$$

$$\Rightarrow \cot\phi = \frac{\cos m\theta\,(1-\tan m\theta)}{\cos m\theta\,(1+\tan m\theta)}$$

$$\Rightarrow \cot\phi = \frac{1-\tan m\theta}{1+\tan m\theta} = \cot\left(\frac{\pi}{4}+m\theta\right)$$

$$\Rightarrow \phi = \frac{\pi}{4}+m\theta \quad \text{— (2).}$$

Consider, $p = r\sin\phi$

$$\Rightarrow p = r\sin\left(\frac{\pi}{4}+m\theta\right) \qquad (\text{from } (2))$$

$$\Rightarrow p = r\left(\sin\frac{\pi}{4}\cos m\theta + \cos\frac{\pi}{4}\sin m\theta\right)$$

$$\Rightarrow p = r\left(\frac{1}{\sqrt{2}}\cos m\theta + \frac{1}{\sqrt{2}}\sin m\theta\right)$$

$$\Rightarrow p = \frac{r}{\sqrt{2}}(\cos m\theta + \sin m\theta)$$

$$\Rightarrow p = \frac{r}{\sqrt{2}}\cdot\frac{r^m}{a^m} \qquad (\text{from } (1))$$

$$\Rightarrow \boxed{\sqrt{2}\,p\,a^m = r^{m+1}}$$ is the required pedal eq- of the given curve (1).

CMR

4) Show that for the curve $r = a e^{\theta \cot \alpha}$ where a & α are constants, $\frac{\rho}{r}$ is a constant.

Sol^n:- Given curve,

$$r = a e^{\theta \cot \alpha} \quad - (1)$$

Taking log on both sides, we get

$$\log r = \log a + \log e^{\theta \cot \alpha}$$

$$\Rightarrow \log r = \log a + \theta \cot \alpha \log e$$

$$\Rightarrow \log r = \log a + \theta \cot \alpha \qquad [\text{since } \log e = 1]$$

Diff. w.r.t 'θ', we get

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \cot\alpha \cdot 1$$

$$\Rightarrow \frac{dr}{d\theta} = r \cot\alpha$$

$$\Rightarrow r_1 = r\cot\alpha \quad - (2)$$

Diff (2) w.r.t 'θ', we get

$$\frac{dr_1}{d\theta} = r_2 = \cot\alpha \frac{dr}{d\theta}$$

$\Rightarrow r_2 = r_1 \cot\alpha$ — (3)

$\Rightarrow r_2 = r \cot\alpha \cot\alpha$

$\Rightarrow r_2 = r \cot^2\alpha$ — (3)

The radius of curvature for a polar curve,

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$$

$$\Rightarrow \rho = \frac{(r^2 + r^2\cot^2\alpha)^{3/2}}{r^2 + 2r^2\cot^2\alpha - r \cdot r\cot^2\alpha}$$ (using (2) & (3))

$$\Rightarrow \rho = \frac{(r^2)^{3/2}(1+\cot^2\alpha)^{3/2}}{r^2 + r^2\cot^2\alpha}$$

$$\Rightarrow \rho = \frac{r^3 (\operatorname{cosec}^2\alpha)^{3/2}}{r^2(1+\cot^2\alpha)}$$

$$\Rightarrow \rho = r \cdot \frac{\operatorname{cosec}^3\alpha}{\operatorname{cosec}^2\alpha}$$

$$\Rightarrow \boxed{\frac{\rho}{r} = \operatorname{cosec}\alpha = \text{constant}}$$

5) Evaluate

(i) $\lim_{x \to 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$

Sol: Let $K = \lim_{x \to 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$ $[1^\infty \text{ form}]$

Taking log on both sides, we get

$$\log K = \lim_{x \to 0} \log\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$$

$$\Rightarrow \log K = \lim_{x \to 0} \frac{1}{x^2} \log\left(\frac{\tan x}{x}\right)$$

$$\Rightarrow \log K = \lim_{x \to 0} \frac{\log\left(\frac{\tan x}{x}\right)}{x^2} \quad \left[\frac{0}{0} \text{ form}\right]$$

By L' Hospital rule,

$$\log K = \lim_{x \to 0} \frac{\frac{1}{\frac{\tan x}{x}} \times \left[\frac{x \sec^2 x - \tan x}{x^2}\right]}{2x}$$

$$\Rightarrow \log K = \lim_{x \to 0} \frac{x}{\tan x} \times \lim_{x \to 0} \frac{x \sec^2 x - \tan x}{2x^3}$$

$$\Rightarrow \log K = \lim_{x \to 0} \frac{x \sec^2 x - \tan x}{2x^3} \quad \left[\text{since } \lim_{x \to 0} \frac{x}{\tan x} = 1\right]$$

$\Rightarrow \log K = \frac{1}{2} \lim_{x \to 0} \frac{x \sec^2 x - \tan x}{x^3}$ $\left[\frac{0}{0} \text{ form}\right]$

Again apply L' Hospital rule,

$\log K = \frac{1}{2} \lim_{x \to 0} \frac{x \cdot 2\sec x \sec x \cdot \tan x + \cancel{\sec^2 x} - \cancel{\sec^2 x}}{3x^2}$

$\Rightarrow \log K = \frac{1}{2} \lim_{x \to 0} \frac{2x \sec^2 x \tan x}{3x^2}$

$\Rightarrow \log K = \frac{1}{3} \lim_{x \to 0} \sec^2 x \frac{\tan x}{x}$

$\Rightarrow \log K = \frac{1}{3} \times 1 \times 1 = \frac{1}{3}$

$\Rightarrow K = e^{1/3}$

(ii) $\lim_{x \to 0} (\cos x)^{\frac{1}{x^2}}$

Sol: Let $K = \lim_{x \to 0} (\cos x)^{\frac{1}{x^2}}$ (1^∞ form)

Taking log on both sides, we get

$\log K = \lim_{x \to 0} \log(\cos x)^{\frac{1}{x^2}}$

$\Rightarrow \log K = \lim_{x \to 0} \frac{1}{x^2} \log(\cos x)$

$$\Rightarrow \log K = \lim_{x\to 0} \frac{\log(\cos x)}{x^2} \quad \left[\frac{0}{0} \text{ form}\right]$$

Applying log on both sides, we get

$$\Rightarrow \log K = \lim_{x\to 0} \frac{\frac{1}{\cos x}(-\sin x)}{2x}$$

$$\Rightarrow \log K = -\frac{1}{2} \lim_{x\to 0} \frac{1}{\cos x} \times \frac{\sin x}{x}$$

$$\Rightarrow \log K = \frac{-1}{2}\left[\lim_{x\to 0} \sec x \times \lim_{x\to 0} \frac{\sin x}{x}\right]$$

$$\Rightarrow \log K = \frac{-1}{2} \times 1 \times 1 = -\frac{1}{2}$$

$$\Rightarrow K = e^{-1/2}$$

$$\Rightarrow \lim_{x\to 0} (\cos x)^{\frac{1}{x^2}} = e^{-1/2}$$

$$\Rightarrow \boxed{\lim_{x\to 0} (\cos x)^{\frac{1}{x^2}} = 1/\sqrt{e}}$$

6) Expand $\sqrt{1+\sin 2x}$ by Maclaurin's series up to the term containing x^5.

Sol^n:- Let $y = f(x) = \sqrt{1+\sin 2x}$, $f(0) = \sqrt{1+\sin 0}$

$$\Rightarrow \boxed{f(0) = 1 = y(0)}$$

$y = \sqrt{1+\sin 2x}$

$\Rightarrow y = \sqrt{\sin^2 x + \cos^2 x + 2\sin x \cos x}$ $[\text{Since } \sin^2 x + \cos^2 x = 1]$

$\Rightarrow y = \sqrt{(\sin x + \cos x)^2}$

$\Rightarrow y(x) = \sin x + \cos x.$ $y(0) = 1$

$y_1(x) = \cos x - \sin x$, $y_1(0) = 1$

$y_2(x) = -\sin x - \cos x$, $y_2(0) = -1$

$\Rightarrow y_2(x) = -y(x)$

$y_3(x) = -y_1(x)$, $y_3(0) = -y_1(0) = -1$

$y_4(x) = -y_2(x)$, $y_4(0) = -y_2(0) = 1$

$y_5(x) = -y_3(x)$, $y_5(0) = -y_3(0) = 1$

From Maclaurin's series expansion,

$\cancel{y(x) = y(0) + x y_1'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0)}$

$$y(x) = y(0) + x\, y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \frac{x^4}{4!} y_4(0) + \frac{x^5}{5!} y_5(0) + \cdots$$

$$\Rightarrow \boxed{y(x) = \sqrt{1+\sin 2x} = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} \cdots}$$

7) Show that the radius of curvature at any point of the cycloid $x = a(\theta + \sin\theta)$ and $y = a(1-\cos\theta)$ is $4a\cos\frac{\theta}{2}$.

Solⁿ:- Given curve,

$$x = a(\theta + \sin\theta) \quad \& \quad y = a(1-\cos\theta)$$

Given eq: of curve is in parametric form in θ.

The radius of curvature is given by

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}, \text{ where } y_1 = \frac{dy}{dx} \ \& \ y_2 = \frac{d^2y}{dx^2}.$$

Now,

$$x = a(\theta + \sin\theta) \qquad y = a(1-\cos\theta)$$

$$\frac{dx}{d\theta} = a(1+\cos\theta) \qquad \frac{dy}{d\theta} = a(0+\sin\theta)$$

$$\Rightarrow \frac{dx}{d\theta} = a\, 2\cos^2\theta/2 \qquad \frac{dy}{d\theta} = 2a\sin\theta/2 \cos\theta/2$$

$$y_1 = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2a\sin\theta/2\cos\theta/2}{2a\cos^2\theta/2} = \tan\theta/2$$

$$\Rightarrow \boxed{y_1 = \frac{dy}{dx} = \tan\theta/2}$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\tan\theta/_2\right)$$

$$= \frac{d}{d\theta}\left(\tan\theta/_2\right)\frac{d\theta}{dx}$$

$$= \sec^2\theta/_2 \times \frac{1}{2} \times \frac{1}{a(1+\cos\theta)} \qquad \left(\text{since } \frac{dx}{d\theta} = a(1+\cos\theta)\right)$$

$$\Rightarrow y_2 = \frac{1}{2}\,\frac{\sec^2\theta/_2}{2a\cos^2\theta/_2} = \frac{1}{4a}\sec^4\theta/_2$$

$$\Rightarrow \boxed{y_2 = \frac{1}{4a}\sec^4\theta/_2}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+\tan^2\theta/_2)^{3/2}}{\frac{1}{4a}\sec^4\theta/_2}$$

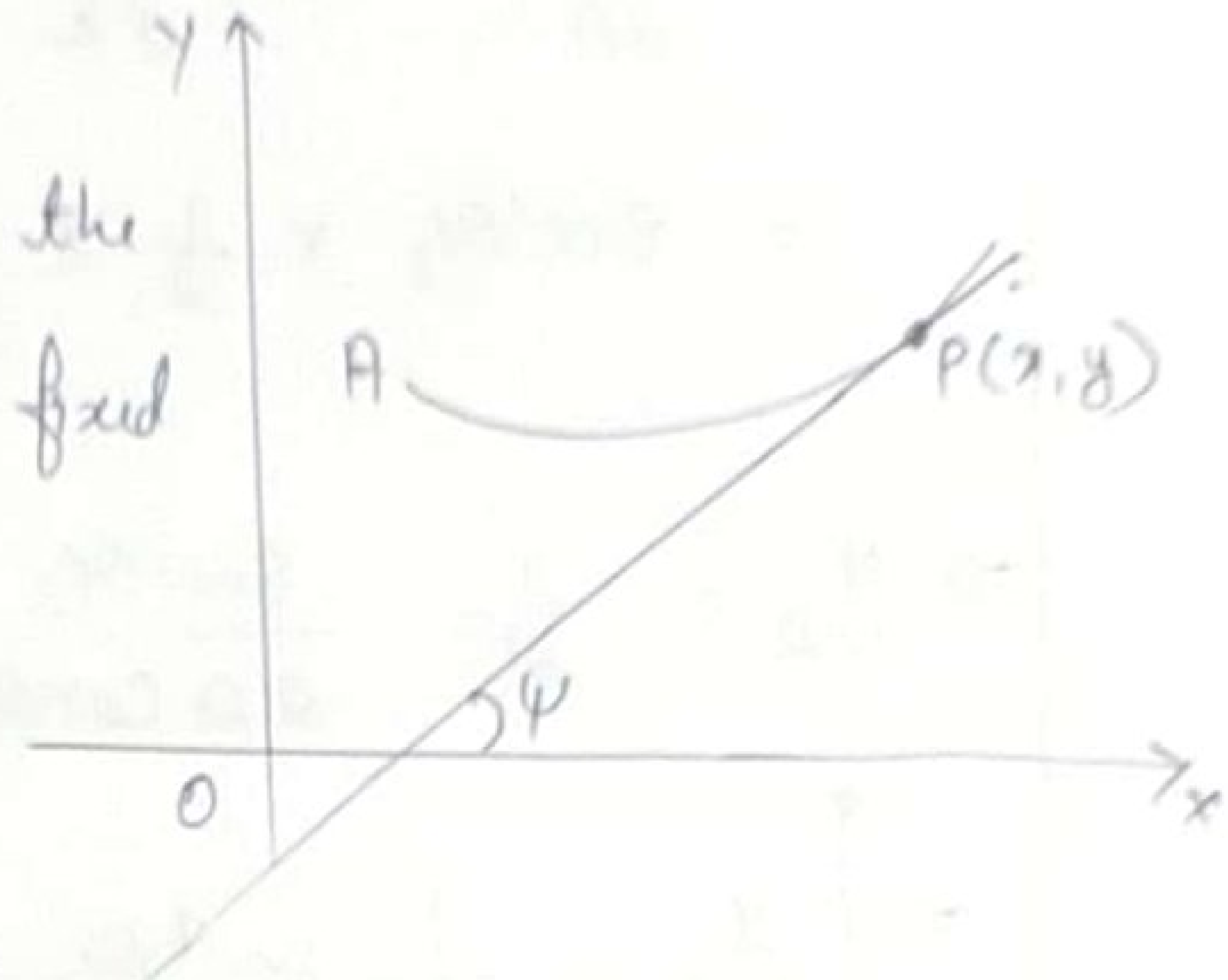
$$= \frac{(\sec^2\theta/_2)^{3/2}}{\frac{1}{4a}\sec^4\theta/_2} = 4a\,\frac{\sec^3\theta/_2}{\sec^4\theta/_2}$$

$$= 4a\,\frac{1}{\sec\theta/_2}$$

$$\Rightarrow \boxed{\rho = 4a\cos\theta/_2}$$

8) Derive an expression for radius of curvature for a cartesian curve $y=f(x)$, $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$, where $y_1 = \frac{dy}{dx}$ & $y_2 = \frac{d^2y}{dx^2}$.

Solⁿ:- Let $y=f(x)$ be the eqⁿ of the Cartesian curve and A be a fixed point on it.



Let $P(x,y)$ be a point on the curve such that $\widehat{AP} = s$

Let ψ be the angle made by the tangent at P with the x-axis.

Then we know that $\tan\psi = \frac{dy}{dx}$.

Diff. w.r.t 's', we get

$$\sec^2\psi \frac{d\psi}{ds} = \frac{d}{ds}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dx}\right) \times \frac{dx}{ds}$$

$$\Rightarrow \sec^2\psi \frac{d\psi}{ds} = \frac{d^2y}{dx^2} \frac{dx}{ds} \quad \text{--- (1)}$$

But we have

$\frac{dx}{ds} = \cos\psi$ & by defⁿ of radius of curvature

$$\frac{d\psi}{ds} = \frac{1}{\rho}.$$

From ①,

$$\sec^2\psi \times \frac{1}{\rho} = \frac{d^2y}{dx^2} \times \cos\psi$$

$$\Rightarrow \sec^2\psi \cdot \frac{1}{\rho} = \frac{1}{\sec\psi} \frac{d^2y}{dx^2}$$

$$\Rightarrow \sec^3\psi \cdot \frac{1}{\rho} = y_2$$

$$\Rightarrow \rho = \frac{\sec^3\psi}{y_2} = \frac{(\sec^2\psi)^{3/2}}{y_2}$$

$$\Rightarrow \rho = \frac{(1+\tan^2\psi)^{3/2}}{y_2}$$

$$\Rightarrow \boxed{\rho = \frac{(1+y_1^2)^{3/2}}{y_2}} \quad \left(\text{since } \frac{dy}{dx} = \tan\psi\right).$$