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CMRIT

Internal Assessment Test III – Jan 2024

Sub:	Mathematics-I for Computer Science and Engineering Stream					Sub Code:	BMATS101	
Date:	02/01/2024	Duration:	90 mins	Max Marks:	50	Sem / Sec:	FIRST / I to L (CHEM CYCLE)	OBE

Question 1 is compulsory and answer any SIX questions from the rest.

		MARKS	CO	RBT
1.	Find the Orthogonal Trajectories of the family of curves $\frac{x^2}{a^2}+\frac{y^2}{b^2+\lambda}=1$.	[08]	CO3	L3
2.	Solve the equation $\frac{dy}{dx}-\frac{dx}{dy}=\frac{x}{y}-\frac{y}{x}$.	[07]	CO3	L3
3.	Solve the equation $x^3\frac{dy}{dx}-x^2y+y^4\cos x=0$.	[07]	CO3	L3
4.	Find all the solutions of $18x\equiv 30(mod\ 42)$	[07]	CO4	L3

5.	Solve the system of linear congruence by using CRT. $$x \equiv 3(mod5)$$ $$x \equiv 2(mod6)$$ $$x \equiv 4(mod7)$$	[07]	CO4	L3
6.	(i) Find the last digit of 7^{118}. (ii) Find the remainder when the number 3^{100000} is divided by 53.	[07]	CO4	L3
7.	Find the general solution of the equation $(px - y)(py + x) = 2p$ by reducing it into Clairaut's form by using the substitution $X = x^2, Y = y^2$.	[07]	CO3	L3
8.	Find the general solution of the linear Diophantine equation $70x + 112y = 168$.	[07]	CO4	L3

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IAT - 3 (CSE Stream)

CMR

1) Find the orthogonal trajectories of the family of curves

$\frac{x^2}{a^2}+\frac{y^2}{b^2+\lambda}=1$, where λ is the parameter.

Sol:

Given eq:-,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2+\lambda}=1 \quad \text{— (1)}$$

Diff. (1) w.r.t 'x', we get

$$\frac{2x}{a^2}+\frac{2y}{b^2+\lambda}\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \frac{2x}{a^2}=\frac{-2y}{b^2+\lambda}\frac{dy}{dx}$$

$$\Rightarrow \frac{1}{b^2+\lambda}=-\frac{x}{y}\cdot\frac{1}{a^2}\cdot\frac{dx}{dy} \quad \text{— (2)}$$

Sub. (2) in (1), we get

$$\frac{x^2}{a^2} + y^2\left(\frac{-x}{ya^2}\frac{dx}{dy}\right) = 1$$

$$\Rightarrow \frac{xy}{a^2}\frac{dx}{dy} = \frac{x^2}{a^2} - 1$$

$$\Rightarrow \frac{xy}{a^2}\frac{dx}{dy} = \frac{x^2-a^2}{a^2}$$

$$\Rightarrow xy\frac{dx}{dy} = x^2 - a^2$$

$$\Rightarrow \frac{1}{xy}\frac{dy}{dx} = \frac{1}{x^2-a^2}$$

Replace $\frac{dy}{dx}$ by $\frac{-dx}{dy}$, we get

$$\Rightarrow \frac{1}{xy}\frac{-dx}{dy} = \frac{1}{x^2-a^2}$$

$$\Rightarrow \frac{x^2-a^2}{x}dx = -y\,dy$$

$$\Rightarrow x\,dx - a^2\frac{dx}{x} + y\,dy = 0$$

on integrating,

$$\frac{x^2}{2} - a^2\log x + \frac{y^2}{2} = c \Rightarrow \boxed{x^2+y^2-2a^2\log x - c = 0}$$

Solve the equation $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$

Let $\frac{dy}{dx} = p$

$$\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$$

$$\Rightarrow p - \frac{1}{p} = \frac{x}{y} - \frac{y}{x}$$

$$\Rightarrow \frac{p^2-1}{p} = \frac{x^2-y^2}{xy}$$

$$\Rightarrow (p^2-1)\,xy = (x^2-y^2)\,p$$

$$(p^2-1)\,xy = (x^2-y^2)\,p$$

$$\Rightarrow xyp^2 - (x^2-y^2)p - xy = 0$$

$$\Rightarrow xyp^2 - x^2p + y^2p - xy = 0$$

$$\Rightarrow xp(yp-x) + y(yp-x) = 0$$

$$\Rightarrow (yp-x)\,(xp+y) = 0$$

$yp - x = 0$ $\qquad\qquad$ $xp + y = 0$

$\Rightarrow x = yp$ $\qquad\qquad$ $\Rightarrow xp = -y$

$\Rightarrow \frac{x}{y} = \frac{dy}{dx}$ $\qquad\qquad$ $\Rightarrow x\frac{dy}{dx} = -y$

$\Rightarrow x\,dx = y\,dy$ $\qquad\qquad$ $\Rightarrow \frac{dx}{x} = -\frac{dy}{y}$

$\Rightarrow \frac{x^2}{2} = \frac{y^2}{2} + c$ $\qquad\qquad$ $\Rightarrow \log x = -\log y + c$

$\Rightarrow \frac{x^2}{2} - \frac{y^2}{2} - c = 0$ $\quad$ & $\quad \log x - \log y - c = 0$

The required solution is

$$\left(\frac{x^2}{2} - \frac{y^2}{2} - c\right)(\log x - \log y - c) = 0.$$

Solve the eq^n $x^3 \frac{dy}{dx} - x^2y + y^4 \cos x = 0$

$$x^3 \frac{dy}{dx} - x^2y = -y^4 \cos x \quad \text{— (1)}$$

Dividing by $\frac{1}{x^3y^4}$ both sides of the eq^n (1),

$$\frac{x^3}{x^3y^4} \frac{dy}{dx} - \frac{x^2y}{x^3y^4} = -\frac{y^4 \cos x}{x^3y^4}$$

$$\Rightarrow \frac{1}{y^4}\frac{dy}{dx} - \frac{1}{x} \cdot \frac{1}{y^3} = -\frac{\cos x}{x^3} \quad \text{— (2)}$$

Let $\frac{1}{y^3} = t \Rightarrow \frac{-3}{y^4}\frac{dy}{dx} = \frac{dt}{dx}$

$$\Rightarrow \frac{1}{y^4}\frac{dy}{dx} = \frac{-1}{3}\frac{dt}{dx}$$

Sub. $\frac{1}{y^4}\frac{dy}{dx}$ in (2), we get

$$-\frac{1}{3}\frac{dt}{dx} - \frac{1}{x}t = -\frac{\cos x}{x^3}$$

$$\Rightarrow \frac{dt}{dx} + \frac{3}{x}t = \frac{3\cos x}{x^3} \quad \text{— (3)}$$

Eq² ③ is a linear D.E in t.

$P = \frac{3}{x}$ & $Q = \frac{3\cos x}{x^3}$

$I.F = e^{\int P dx} = e^{3\int \frac{dx}{x}} = e^{3\log x} = e^{\log x^3}$

$I.F = x^3$.

Sol^n of ③,

$$t(IF) = \int Q(IF)\,dx + c$$

$$\Rightarrow t \cdot x^3 = \int \frac{3\cos x}{x^3} \times x^3\,dx + c$$

$$\Rightarrow t x^3 = 3\int \cos x\,dx + c$$

$$\Rightarrow t x^3 = 3\sin x + c$$

$$\Rightarrow \boxed{\frac{x^3}{y^3} = 3\sin x + c}$$

4) Find all the solutions of $18x \equiv 30 \pmod{42}$ — (1)

Sol: Compare the given eqn with $ax \equiv b \pmod{m}$

$$a = 18,\ b = 30,\ m = 42$$

$$(a, m) = (18, 42) = 6 = d$$

$$6 \mid 30$$

$\Rightarrow$ The given congruence has 6 incongruent solns

$A = \frac{a}{d} = \frac{18}{6} = 3$, $B = \frac{b}{d} = \frac{30}{6} = 5$, $M = \frac{m}{d} = \frac{42}{6} = 7$

Eqn (1) becomes

$$Ax \equiv B \pmod{M} \quad -(2)$$

$\Rightarrow (A, M) = (3, 7) = 1$

$\Rightarrow$ Eqn (2) has a unique soln.

$$3x \equiv 5 \pmod{7}$$

$\Rightarrow 3x - 5 = 7k \Rightarrow x = \frac{5+7k}{3}$

$k=0$, $x = 5/3 \notin \mathbb{Z}$

$k=1$, $x = 4 \in \mathbb{Z}$

let $x_0 = 4$ be a sol of eqn (1).

And the remaining sol^n are obtained by

$$x_0+\frac{m}{d},\ x_0+\frac{2m}{d},\ x_0+\frac{3m}{d},\ x_0+\frac{4m}{d},\ x_0+\frac{5m}{d}$$

$$= 4+\frac{42}{6},\ 4+\frac{2\times 42}{6},\ 4+\frac{3\times 42}{6},\ 4+\frac{4\times 42}{6},\ 4+\frac{5\times 42}{6}$$

$$= 11,\ 18,\ 25,\ 32,\ 39.$$

Thus the 6 sol^ns of eq^n ① are

$x \equiv 4 \pmod{42}$, $x \equiv 11 \pmod{42}$, $x \equiv 18 \pmod{42}$,

$x \equiv 25 \pmod{42}$, $x \equiv 32 \pmod{42}$, $x \equiv 39 \pmod{42}$.

-5 Solve the system of linear congruence by using CRT

$$x \equiv 3 \pmod 5$$
$$x \equiv 2 \pmod 6$$
$$x \equiv 4 \pmod 7$$

Sol^n: Here $b_1 = 3$, $b_2 = 2$, $b_3 = 4$

$m_1 = 5$, $m_2 = 6$, $m_3 = 7$

We verify that $(5,6)=1$, $(6,7)=1$ & $(5,7)=1$.

$$M = m_1 m_2 m_3 = 5\times 6\times 7 = 210$$

$M_K = \frac{M}{m_K} = \frac{210}{m_K}$, for $K = 1, 2, 3$.

$M_1 = \frac{210}{5} = 42$, $M_2 = \frac{210}{6} = 35$, $M_3 = \frac{210}{7} = 30$.

Consider, $M_K X \equiv 1 \pmod{m_K}$ for $K = 1, 2, 3$ & $X = x, y, z$.

$42x \equiv 1 \pmod 5$, $35y = 1 \pmod 6$, $30z = 1 \pmod 7$.

$x = 3$, $y = 5$, $z = 4$ are satisfied.

Thus a unique soln of the system is

$X \equiv b_1 M_1 x + b_2 M_2 y + b_3 M_3 z$ modulo M

$\Rightarrow X \equiv 1208 \pmod{210}$

$\Rightarrow X \equiv 158 \pmod{210}$ is the unique soln.

Find the last digit of 7^{118}.

$$7^{118} = 7^{4\times 29+2}$$
$$= 7^{4K+2} \equiv 9 \pmod{10}$$

$$4\overline{)118}(29$$
$$8$$
$$\overline{38}$$
$$36$$
$$\overline{2}$$

$\Rightarrow 7^{118} \equiv 9 \pmod{10}$.

$\Rightarrow$ 9 is the last digit.

Find the remainder when the number 3^{100000} is divided by 53.

By Fermat's little theorem,

$$a^{p-1} \equiv 1 \pmod{p}$$

Here $a = 3$, $p = 53$.

$$52\overline{)100000}(1923$$
$$52$$
$$\overline{480}$$
$$468$$
$$\overline{120}$$
$$104$$
$$\overline{160}$$
$$156$$
$$\overline{4}$$

$\Rightarrow 3^{52} \equiv 1 \pmod{53}$

$\Rightarrow (3^{52})^{1923} \equiv (1)^{1923} \pmod{53}$

$\Rightarrow 3^{99996} \equiv 1 \pmod{53}$

$\Rightarrow 3^{99996} \cdot 3^{4} \equiv 3^{4} \pmod{53}$

$\Rightarrow 3^{100000} \equiv 81 \pmod{53}$

$\Rightarrow 3^{100000} \equiv 28 \pmod{53}$

$\Rightarrow$ 28 is the remainder when 3^{100000} is divided by 53.

7) Find the general soln of the equation $(px-y)(py+x)=2p$ by reducing it into Clairaut's form by using the substitution $X=x^2$, $Y=y^2$.

Sol:

$X=x^2$ & $Y=y^2$

$\Rightarrow \frac{dX}{dx}=2x$ & $\frac{dY}{dy}=2y$

We know that, $p=\frac{dy}{dx}$

$\Rightarrow p=\frac{dy}{dY}\cdot\frac{dY}{dX}\frac{dX}{dx}$

$\Rightarrow p=\frac{1}{2y}\frac{dY}{dX}2x$

$\Rightarrow p=\frac{x}{y}\frac{dY}{dX}$

$\Rightarrow p=\sqrt{\frac{X}{Y}}\frac{dY}{dX}$

$\Rightarrow p=\sqrt{\frac{X}{Y}}P$, where $P=\frac{dY}{dX}$.

Sub $X=x^2$, $Y=y^2$ & $p=\sqrt{\frac{X}{Y}}P$ in the given differential eqn,

$(px-y)(py+x)=2p$

$$\Rightarrow \left(\sqrt{\frac{x}{y}}\, p \cdot \sqrt{x} - \sqrt{y}\right)\left(\sqrt{\frac{x}{y}}\, p \cdot \sqrt{y} + \sqrt{x}\right) = 2\sqrt{\frac{x}{y}}\, p$$

$$\Rightarrow \left(\frac{xp}{\sqrt{y}} - \sqrt{y}\right)\left(\sqrt{x}\,p + \sqrt{x}\right) = 2\sqrt{\frac{x}{y}}\, p$$

$$\Rightarrow \left(\frac{xp - y}{\sqrt{y}}\right)\sqrt{x}\,(p+1) = 2\frac{\sqrt{x}}{\sqrt{y}}\, p$$

$$\Rightarrow (xp-y)(p+1) = 2p$$

$$\Rightarrow xp - y = \frac{2p}{p+1} \Rightarrow y = xp - \frac{2p}{p+1} \quad \text{--- (1)}$$

Eq^n (1) is in Clairauts form.

Therefore the general sol^n of eq^n (1) is

$$y = xc - \frac{2c}{c+1}$$

$$\Rightarrow \boxed{y^2 = x^2 c - \frac{2c}{c+1}}$$

Q) Find the general solⁿ of the linear Diophantine eqⁿ

$$70x + 112y = 168.$$

Solⁿ: Given eqⁿ, $70x + 112y = 168$ — ①

Compare eqⁿ ① with $ax + by = c$.

$$a = 70 \quad , b = 112 \quad , \quad c = 168$$

$(a,b) = (70, 112) = 14 = d$ (say)

$70 = 2 \times 5 \times 7$

$112 = 2^4 \times 7$

$168 = 12 \times 14$

$14 \mid 168 \Rightarrow d \mid c.$

$\Rightarrow$ Solⁿ exists.

Divide eqⁿ ① by 14, we get

$$5x + 8y = 12 \quad — ②$$

$8 = 5 \times 1 + 3 \Rightarrow 3 = 8 - 5 \times 1$

$5 = 3 \times 1 + 2 \Rightarrow 2 = 5 - 3 \times 1$

$3 = 2 \times 1 + 1 \Rightarrow 1 = 3 - 2 \times 1$

$1 = 3 - 2 \times 1$

$\Rightarrow 1 = 3 - (5 - 3 \times 1) \times 1 = 3 - 5 \times 1 + 3 \times 1$

$\Rightarrow 1 = -5 \times 1 + 3 \times 2 = -5 \times 1 + (8 - 5 \times 1) \times 2$

$\Rightarrow 1 = -5 \times 1 + 8 \times 2 - 5 \times 2 \Rightarrow 1 = -5 \times 3 + 8 \times 2$

$\Rightarrow 1 = 5 \times (-3) + 8 \times (2)$

$\Rightarrow 12 = 5 \times (-36) + 8 \times 24$

$\Rightarrow x_0 = -36$ & $y_0 = 24$ is a particular soln

General soln is given by

$$x = x_0 - \frac{b}{d} m \quad \& \quad y = y_0 + \frac{a}{d} m, \quad m \in \mathbb{Z}$$

$\Rightarrow x = -36 - \frac{112}{14} m \quad \& \; y = 24 + \frac{70}{14} m$

$\Rightarrow x = -36 - 8m \quad \& \; y = 24 + 5m.$