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Internal Assessment Test 3 January - 2024

Sub:	**Mathematics - I**							Code:	BMATE101
Date:	02/01/2024	Duration:	90 mins	Max Marks:	50	Sem:	I	Branch:	**EEE Stream (CHEM Cycle only)**

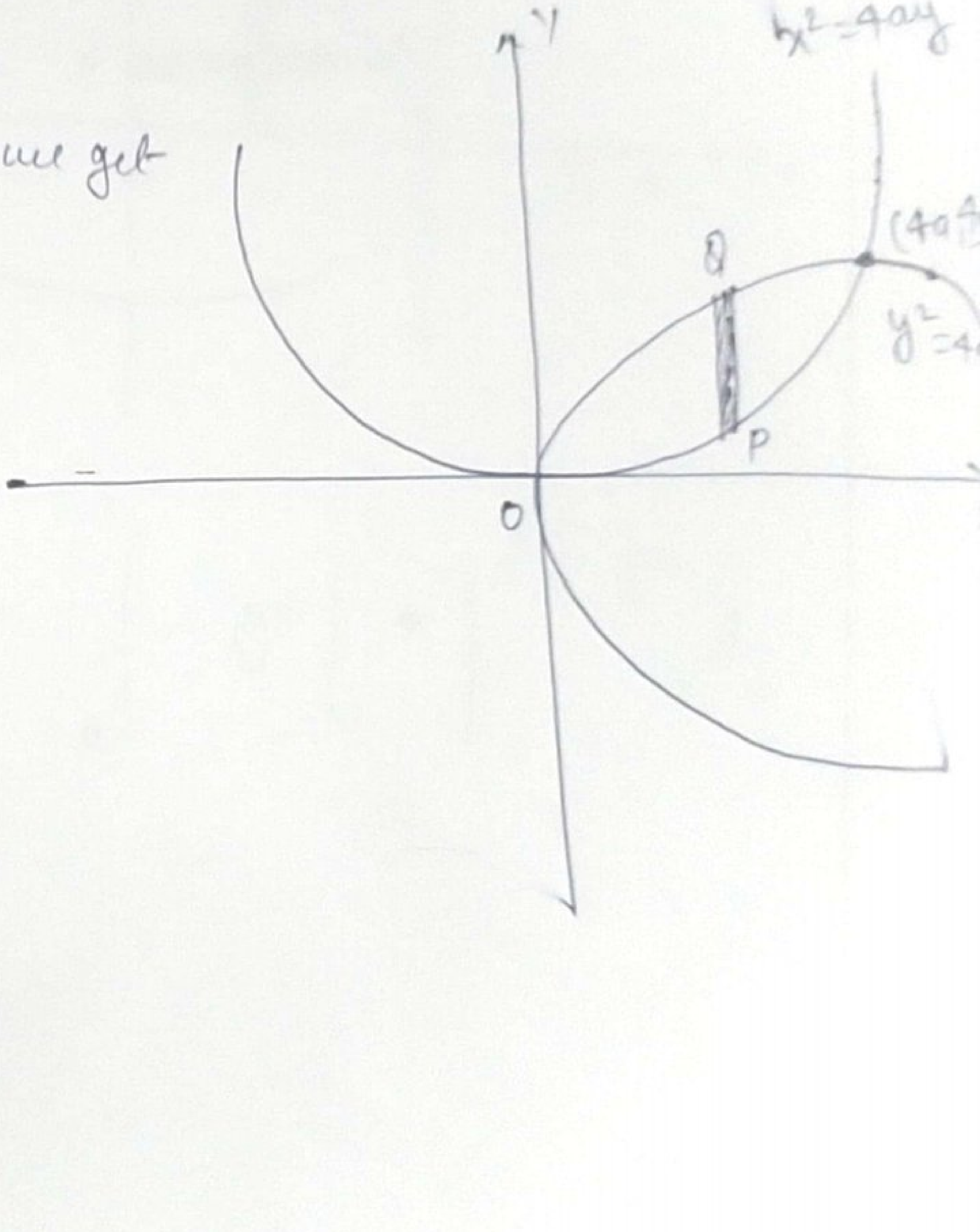
	Answer any 5 questions:	Marks	OBE CO	OBE RBT
1	Find the area bounded by the parabolas $x^2 = 4ay$ and $y^2 = 4ax$, using double integration.	[10]	CO4	L3
2	Evaluate $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(x^2+y^2+z^2)dx\,dy\,dz$.	[10]	CO4	L3
3	Derive the relation between beta and gamma function.	[10]	CO4	L3
4	Evaluate $\int_0^1\int_0^{\sqrt{(1-y^2)}}(x^2+y^2)dxdy$ by changing to polar coordinates.	[10]	CO4	L3

5	Find the general and singular solution of the equation $(px-y)(py+x)=a^2p$, reducing into Clairaut's form, using the substitution $X=x^2, Y=y^2$.	[10]	CO3	L3
6	Solve $\frac{dy}{dx}+\frac{y}{x}=y^2x$.	[10]	CO3	L3
7	Find the orthogonal trajectories of the family of curves $\frac{x^2}{a^2}+\frac{y^2}{b^2+\alpha}=1$, where α is the parameter.	[10]	CO3	L3

1) Find the area bounded by the parabolas $y^2 = 4ax$ & $x^2 = 4ay$ by using double integration.

Sol: Given curves. $y^2 = 4ax$ & $x^2 = 4ay$

$$x = \frac{y^2}{4a}$$



Sub. x in $x^2 = 4ay$, we get

$$\left(\frac{y^2}{4a}\right)^2 = 4ay$$

$$\Rightarrow \frac{y^4}{16a^2} = 4ay$$

$$\Rightarrow y^3 = 64a^3$$

$$\Rightarrow y = 4a$$

$$x = \frac{y^2}{4a} = \frac{16a^2}{4a} = 4a$$

$x = 4a$, $y = 4a$

$(4a, 4a)$ is the pt. of intersection.

Let PQ be the vertical strip. Then we have

$y : \frac{x^2}{4a} - 2\sqrt{ax}$ & $x : 0 - 4a$

$$A = \int_{0}^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy \ dx = \int_{0}^{4a} \Big[y \Big]_{x^2/4a}^{2\sqrt{ax}} dx$$

$$= \int_{0}^{4a} \left[2\sqrt{ax} - \frac{x^2}{4a} \right] dx = 2\sqrt{a} \int_{0}^{4a} x^{\frac{1}{2}} dx - \frac{1}{4a} \int_{0}^{4a} x^2 dx$$

$$= 2\sqrt{a} \left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right]_{0}^{4a} - \frac{1}{4a} \left[\frac{x^3}{3} \right]_{0}^{4a}$$

$$= 2 \cdot \sqrt{a} \cdot \frac{2}{3} \left[x^{3/2} \right]_{0}^{4a} - \frac{1}{12a} \left[x^3 \right]_{0}^{4a}$$

$$= \frac{4}{3}\sqrt{a} \, (4a)^{3/2} - \frac{1}{12a} (4a)^3$$

$$= \frac{4}{3} \sqrt{a} \cdot \left(4^{1/2}\right)^3 \left(a^{1/2}\right)^3 - \frac{1}{12a} \times 64 a^3$$

$$= \frac{4}{3}\sqrt{a} \cdot 8 \cdot a\sqrt{a} - \frac{1}{3} \times 16 a^2$$

$$= \frac{32}{3} a^2 - \frac{16}{3} a^2 = \frac{16}{3} a^2.$$

2)
$$\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(x^2+y^2+z^2)\,dx\,dy\,dz$$

$$=\int_{-c}^{c}\int_{-b}^{b}\left[\frac{x^3}{3}\right]_{-a}^{a}+y^2[x]_{-a}^{a}+z^2[x]_{-a}^{a}\,dy\,dz$$

$$=\int_{-c}^{c}\int_{-b}^{b}\frac{a^3}{3}+\frac{a^3}{3}+y^2[a+a]+z^2[a+a]\,dy\,dz$$

$$=\int_{-c}^{c}\int_{-b}^{b}\frac{2a^3}{3}+2ay^2+2az^2\,dy\,dz$$

$$=\int_{-c}^{c}\frac{2a^3}{3}[y]_{-b}^{b}+2a\left[\frac{y^3}{3}\right]_{-b}^{b}+2az^2[y]_{-b}^{b}\,dz$$

$$=\int_{-c}^{c}\frac{2a^3}{3}\left[b+b\right]+2a\left[\frac{b^3}{3}+\frac{b^3}{3}\right]+2az^2\left[b+b\right]dz$$

$$=\int_{-c}^{c}\frac{4a^3b}{3}+2a\left[\frac{2b^3}{3}\right]+2az^2[2b]\,dz$$

$$=\frac{4a^3b}{3}[z]_{-c}^{c}+\frac{4ab^3}{3}[z]_{-c}^{c}+4ab\left[\frac{z^3}{3}\right]_{-c}^{c}$$

$$=\frac{4a^3b}{3}[c+c]+\frac{4ab^3}{3}[c+c]+4ab\left[\frac{c^3}{3}+\frac{c^3}{3}\right]$$

$$\frac{8a^3bc}{3} + \frac{8ab^3c}{3} + \frac{8abc^3}{3}$$

$$\frac{8abc}{3}\left[a^2+b^2+c^2\right]$$

$$\boxed{\therefore \int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(x^2+y^2+z^2)\,dx\,dy\,dz = \frac{8abc}{3}\left[a^2+b^2+c^2\right]}$$

3 Relation b/w beta and gamma function

Prove:

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

Proof:-

We know that

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta \rightarrow (i)$$

By the definition of gamma function

$$\Gamma m = 2\int_0^{\infty} e^{-x^2} x^{2m-1} \, dx \rightarrow ①$$

$$\Gamma n = 2\int_0^{\infty} e^{-y^2} y^{2n-1} \, dy \rightarrow ②$$

$$\Gamma m+n = 2\int_0^{\infty} e^{-r^2} r^{2(m+n)-1} \, dr \rightarrow ③$$

By multiplying ① and ②; We get

$$\Gamma m \, \Gamma n = 4\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} x^{2m-1} y^{2n-1} \, dx \, dy$$

From polar Coordinates

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$x^2+y^2 = r^2$$

$$dx\,dy = r\,dr\,d\theta$$

$$\therefore \Gamma m\,\Gamma n = 4\int_0^{\infty}\int_0^{\infty} e^{-r^2}\, r^{2m-1}\cos^{2m-1}\theta\; r^{2n-1}\sin^{2n-1}\theta\; r.\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\cos^{2m-1}\theta\,\sin^{2n-1}\theta\,d\theta \times 2\int_0^{\infty} e^{-r^2}\, r^{2(m+n)-1}\,dr$$

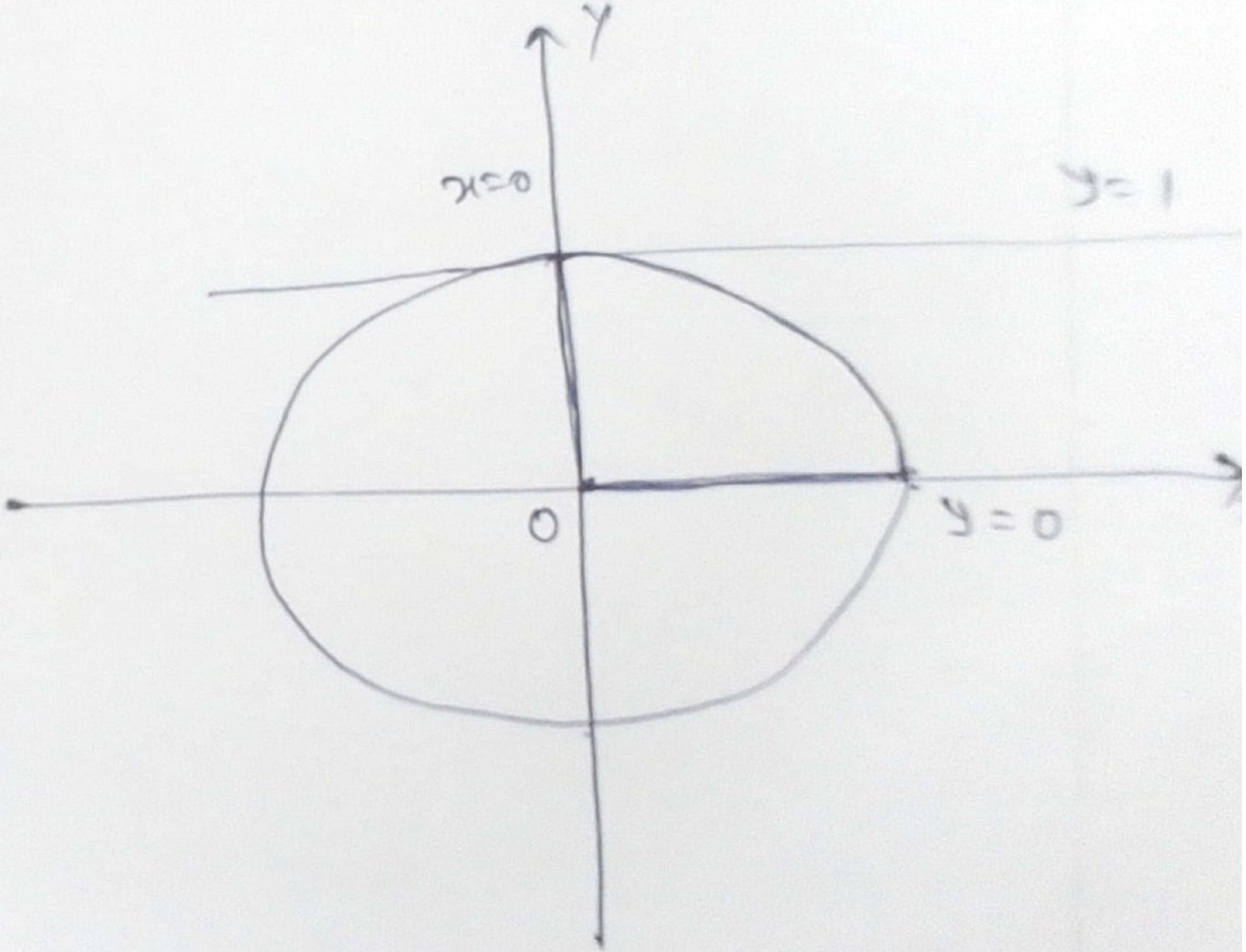
From Eq (i) and (3); We get

$$\Gamma m\,\Gamma n = \beta(n,m)\times\Gamma m+n$$

$$\boxed{\therefore \beta(n,m) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}}$$

$- \frac{\quad}{2} \cdot \pi/2 = \pi/4$

Evaluate $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2+y^2)\,dx\,dy$ by changing into polar coordinates.



$y : 0 - 1$

$x : 0 - \sqrt{1-y^2}$

$x^2+y^2=1$

$\theta : 0 - \pi/2$

$r : 0 - 1$

$$I = \int_0^{\frac{\pi}{2}} \int_0^1 r^2 \cdot r \, dr \, d\theta = \left[\frac{r^4}{4}\right]_0^1 \left[\theta\right]_0^{\pi/2}$$

$$= \frac{1}{4} \times \pi/2 = \pi/8 .$$

5) $(Px-y)(Py+x)=a^2P \rightarrow ①$ $\Rightarrow$ Given $X=x^2$, $Y=y^2$

Diff w.r.t x for $X=x^2$ $x=\sqrt{X}$ $y=\sqrt{Y}$

$$\frac{dX}{dx}=2x$$

$$dx=\frac{dX}{2x}$$

Diff w.r.t y for $Y=y^2$

$$\frac{dY}{dy}=2y$$

$$\frac{dY}{2y}=dy$$

We know that $p=\frac{dy}{dx}=\frac{dY}{2y}\times\frac{2x}{dX}=\frac{x}{y}\frac{dY}{dX}=\frac{\sqrt{X}}{\sqrt{Y}}P$

Substitute $x=\sqrt{X}$, $y=\sqrt{Y}$ and $p=\sqrt{\frac{X}{Y}}P$

$$\left(\sqrt{\frac{X}{Y}}P\times\sqrt{X}-\sqrt{Y}\right)\left(\sqrt{\frac{X}{\cancel{Y}}}P\cancel{\sqrt{Y}}+\sqrt{X}\right)=a^2\sqrt{\frac{X}{Y}}P$$

$$\left(\frac{XP-Y}{\cancel{\sqrt{Y}}}\right)\sqrt{X}(P+1)=a^2\frac{\cancel{\sqrt{X}}}{\cancel{\sqrt{Y}}}P$$

$$(XP-Y)(P+1)=a^2P$$

$$XP-Y=\frac{a^2P}{P+1}$$

$$Y=XP+\frac{a^2P}{P+1}$$

Q $\frac{dy}{dx} + \frac{y}{x} = xy^2$

It is in form of Bernoulli's form $\frac{dy}{dx} + Py = Qy^n$

$$y^{-2}\frac{dy}{dx} + y^{-1}x^{-1} = x \rightarrow (1).$$

Let $y^{-1} = t$

diff w.r.t 'x'

$$-1\,y^{-2}\frac{dy}{dx} = \frac{dt}{dx}$$

$$y^{-2}\frac{dy}{dx} = -\frac{dt}{dx}$$

Sub in eq (1).

$$-\frac{dt}{dx} + x^{-1}t = x$$

$$\frac{dt}{dx} - x^{-1}t = -x$$

It is in linear differential Eq $\Rightarrow$ $\frac{dy}{dx} + Py = Q$

$$IF = e^{\int P dx}$$

$$\therefore IF = e^{\int -\frac{1}{x}dx}$$

$$= e^{-\log x}$$

$$= e^{\log_e x^{-1}} = x^{-1} = \frac{1}{x}$$

The Solution of Equation in terms t is given by

$$t(IF) = \int Q(IF)\,dx + c$$

$$t\left(\frac{1}{x}\right) = \int -x\left(\frac{1}{x}\right)dx + c$$

$$\frac{1}{y}\cdot\frac{1}{x} = -x + c$$

$$\frac{1}{xy} = -x + c \Rightarrow xy = \frac{1}{-x+c} \Rightarrow y = \frac{1}{x(-x+c)}$$

$\therefore$ The required Solution is $y = \dfrac{1}{x(-x+c)}$

7)
$$\frac{x^2}{a^2} + \frac{y^2}{b^2+\alpha} = 1 \rightarrow (1)$$

diff w.r.t x

$$\frac{2x}{a^2} + \frac{2y}{b^2+\alpha} \cdot \frac{dy}{dx} = 0 \rightarrow (2)$$

From (1) $$\frac{y^2}{b^2+\alpha} = 1 - \frac{x^2}{a^2} = \frac{a^2-x^2}{a^2}$$

$$\frac{b^2+\alpha}{y^2} = \frac{a^2}{a^2-x^2}$$

$$b^2+\alpha = \frac{a^2y^2}{a^2-x^2} \rightarrow (3)$$

Sub eq ③ in eq ②; We get

$$\frac{2x}{a^2} + \frac{2y(a^2-x^2)}{a^2y^2}\frac{dy}{dx} = 0$$

$$\frac{2}{a^2}\left(x + \frac{a^2-x^2}{y}\frac{dy}{dx}\right) = 0$$

$$x + \frac{a^2-x^2}{y}\frac{dy}{dx} = 0$$

$$\frac{a^2-x^2}{y}\frac{dy}{dx} = -x$$

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$$\frac{a^2-x^2}{y}\left(-\frac{dx}{dy}\right) = -x$$

$$\Rightarrow \frac{a^2-x^2}{x}\,dx = y\,dy$$

$$\Rightarrow a^2\frac{dx}{x} - x\,dx = y\,dy$$

On integrating,

$$a^2\log x - \frac{x^2}{2} = \frac{y^2}{2} + \frac{c}{2}$$

$$\Rightarrow 2a^2\log x - x^2 = y^2 + c$$

$$\Rightarrow \boxed{y^2 = 2a^2\log x - x^2 - c}$$

(6) $XP - Y = \frac{a^2P}{P+1}$

$$\Rightarrow Y = XP - \frac{a^2P}{P+1}$$

This is in the clairaut's form.

The general sol^n is

$$Y = cX - \frac{a^2c}{c+1}$$

$$\Rightarrow \boxed{y^2 = cx^2 - \frac{a^2c}{c+1}} \quad — (2)$$

Diff. (2) w.r.t 'c' partially, we get

$$0 = x^2 - a^2\left[\frac{(c+1)\cdot 1 - c(1+0)}{(c+1)^2}\right]$$

$$\Rightarrow x^2 = \frac{a^2}{(c+1)^2}(c+1-c)$$

$$\Rightarrow x^2 = \frac{a^2}{(c+1)^2} \Rightarrow (c+1)^2 = \frac{x^2}{a^2}$$

(6) $\Rightarrow x^2 = \frac{a^2}{(c+1)^2}$

$\Rightarrow (c+1)^2 = \frac{a^2}{x^2}$

$\Rightarrow c+1 = a/x.$

from (2),

$$y^2 = x^2 \cdot \left(\frac{a}{x}-1\right) - \frac{a^2 \cdot \left(\frac{a}{x}-1\right)}{(a/x)}$$

$$= \frac{x^2(a-x)}{x} - \frac{a^2 x}{a} \cdot \frac{a-x}{x}$$

$$= x(a-x) - a(a-x)$$

$$= ax - x^2 - a^2 + ax$$

$$= -(x^2 - 2ax + a^2)$$

$$y^2 = -(x-a)^2$$

$$\Rightarrow \boxed{(x-a)^2 + y^2 = 0}$$