Model Question Paper-I with effect from 2022

USN | | | | | | | | | | |

Fourth Semester B.E Degree Examination
Complex Analysis, Probability & Linear Programming
(Mechanical Engg. Allied branches)-21MATME41

TIME: 03 Hours **Max. Marks: 100**

Note: Answer any **FIVE** full questions, choosing at least **ONE** question from each module.

Q.No.		Question	M	L	CO
		Module -1			
01	a	With usual notations, derive the Cauchy-Riemann equation in the Cartesian form	**06**	**L2**	**CO1**
	b	If f(z) is regular function of z, prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\lvert f(z)\rvert^2 = 4\lvert f'(z)\rvert^2$	**07**	**L2**	**CO1**
	c	Determine the analytical function whose real part is $y + e^x \cos y$	**07**	**L2**	**CO1**
		OR			
02	a	If $f(z)$ is an analytic function with constant modulus, show that $f(z)$ is constant.	**06**	**L2**	**CO1**
	b	Show that $w = \log z$ is analytic everywhere except at $z = 0$ and hence find its derivative.	**07**	**L2**	**CO1**
	c	Find the analytical function whose imaginary part is $e^{-x}(\,x\, siny - y\cos y)$	**07**	**L2**	**CO1**
		Module-2			
03	a	Discuss the transformation $w = e^z$	**06**	**L3**	**CO2**
	b	State and prove the Cauchy Integral theorem	**07**	**L2**	**CO2**
	c	Find the bilinear transformation which maps the points $z = 1, i, -1$ onto the points $\omega = i, 0, -i$	07	**L2**	**CO2**
		OR			
4	a	Find the bilinear transformation which maps $1,\ i, -1\ to\ 2,\ i, -2$ respectively.	**06**	**L2**	**CO2**
	b	Verify Cauchy's theorem for the integral of z^3 over the boundary of the rectangle with vertices $z = -1,\ 1,\ 1 + i,\ -1 + i$	**07**	**L2**	**CO2**
	c	Evaluate $\oint \frac{e^{-z}}{(z-1)(z-2)^2}\,dz$, over the curve $\lvert z\rvert = 3$	**07**	**L3**	**CO2**
		Module-3			
5	a	A random variable X has the following probability function: x: -2, -1, 0, 1, 2, 3 $P(x)$: 0.1, k, 0.2, 2k, 0.3, k Find the value of k and calculate the mean and variance	**06**	**L2**	**CO3**
	b	Find the mean and standard deviation of the Binomial distribution	**07**	**L2**	**CO3**
	c	In a certain factory turning out razor blades, there is a small chance of 0.002 for any blade to be defective. The blades are supplied in packets of 20. Use Poisson distribution to calculate the approximate number of packets containing no defective, one defective and two defective blades respectively in a consignment of 500 packets	**07**	**L3**	**CO3**

		OR			
6	a	The diameter of an electric cable is assumed to be a continuous variable with p.d.f $$f(x) = \begin{cases} 6x(1-x), & 0 \le x \le 1 \\ 0, & elsewhere \end{cases}$$ Verify that the above is a valid p.d.f. Also, find its mean and variance.	**06**	**L2**	**CO3**
	b	In a test on 2000 electric bulbs, it was found that the life of a particular make was normally distributed with an average life of 2040 hours and Standard deviation of 60 hours. Estimate the number of bulbs likely to burn for i. More than 2150 hours ii. Less than 1950 hours iii. Between 1920 and 2160 hours	**07**	**L3**	**CO3**
	c	The life of a T.V tube manufactured by a company is known to have a mean of 200 months. Assuming that the life has an exponential distribution, find the probability that the life of a tube manufactured by the company is i. Less than 200 months ii. Between 100 and 300 months iii. More than 200 months	**07**	**L3**	**CO3**
		Module-4			
7	a	Using Simplex method solve the L.P.P $Maximize\ Z = 3x_1 + 2x_2$, subject to: $2x_1 + x_2 \le 5$ $x_1 + x_2 \le 3$ $x_1, x_2 \ge 0$	**10**	**L3**	**CO4**
	b	Using Big –M method , solve the LPP $Minimize\ Z = 2x_1 + x_2$, subject to: $3x_1 + x_2 = 3$ $4x_1 + 3x_2 \ge 6$ $x_1 + 2x_2 \le 3$ $x_1, x_2 \ge 0$	**10**	**L3**	**CO4**
		OR			
8	a	Explain the canonical form and standard form of an LPP. Convert the following LPP to the standard form $Maximize\ Z = 3x_1 + 5x_2 + 7x_3$, subject to: $6x_1 - 4x_2 \le 5$ $3x_1 + 2x_2 + 5x_3 \ge 11$ $4x_1 + 3x_3 \le 2$ $x_1, x_2 \ge 0$ $x_1, x_2 \ge 0$	**10**	**L3**	**CO4**
	b	Use two –Phase method to solve the LPP $Maximize\ Z = 9x_1 + 3x_2$, subject to: $4x_1 + x_2 \le 8$ $2x_1 + x_2 \le 4$ $x_1, x_2 \ge 0$	**10**	**L3**	**CO4**
		Module-5			

9 a Solve the following transportation problem

		Destination				
		A	B	C	D	Availability
Source	I	21	16	25	13	**11**
	II	17	18	14	23	**13**
	III	33	27	18	41	**19**
	Requirements	**6**	**10**	**12**	**15**	**43**

10 L3 **CO5**

b Solve the assignment problem

		Machines			
		M_1	M_2	M_3	M_4
Jobs	J_1	2	3	4	5
	J_2	4	5	6	7
	J_3	7	8	9	8
	J_4	3	5	8	4

Assign the jobs to different machines so as to minimize the total cost

10 L3 **CO5**

OR

10 a Obtain an initial basic solution to the following transportation problem

		To				
		A	B	C	D	Availability
From	I	11	13	17	14	250
	II	16	18	14	10	300
	III	21	24	13	10	400
	Requirements	**200**	**225**	**275**	**250**	

10 L3 **CO5**

b Five men are available to do five different jobs. From past records, the time (in hours) that each man takes to do each job is given below

		Jobs				
		I	II	III	IV	V
Man	A	2	9	2	7	1
	B	6	8	7	6	1
	C	4	6	5	3	1
	D	4	2	7	3	1
	E	5	3	9	5	1

Find the assignment of men to jobs that will minimize the total time taken

10 L3 **CO5**

Bloom's Taxonomy Levels	**Lower-order thinking skills**		
	Remembering (knowledge): L_1	Understanding (Comprehension): L_2	Applying (Application): L_3
	Higher-order thinking skills		

	Analyzing (Analysis): L_4	Valuating (Evaluation): L_5	Creating (Synthesis): L_6

(*) Cauchy Reimann equation in Carterian form

(or)

1 a Derive Cauchy-Reimann equation in cartesian coordinates

(or)

With usual notations, derive the Cauchy-Reimann equation in the Carterian form

Statement :- If a function $f(z) = u(x,y) + iv(x,y)$ is analytic at the ~~function $w=f(z)$=~~ complex variable $z = x+iy$, ~~then~~ the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$ should exist and satisfy the equations

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

Proof:- Given $w = f(z) = u(x,y) + iv(x,y)$ — (1)

is analytic at $z = x+iy$

(1) $\Rightarrow f(x+iy) = u(x,y) + iv(x,y)$ — (2)

Differentiate (2) partially w.r.t 'x'

(2) $\Rightarrow f'(x+iy) \cdot \frac{\partial}{\partial x}(x+iy) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$\Rightarrow f'(x+iy)(1) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$\Rightarrow \boxed{f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}}$ — (3)

Similarly, Differentiate (2) partially w.r.t 'y'

(2) $\Rightarrow f'(x+iy) \cdot \frac{\partial}{\partial y}(x+iy) = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$

$\Rightarrow f'(x+iy)(0+i) = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$

$\Rightarrow f'(x+iy) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$

$\Rightarrow f'(z) = \frac{i}{i^2}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$

$\Rightarrow f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$

$\Rightarrow \boxed{f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}}$ — (4)

From eqn (3) and (4) we get. $\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$

$\therefore \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$

If $w = f(z)$ is regular function of z, show that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(|f(z)|)^2 = 4(|f'(z)|)^2$

<u>Proof</u>:- Given, $f(z) = u(x,y) + iv(x,y) = u + iv$ is regular

$\Rightarrow f(z)$ is differentiable at any point of $z = x + iy$

$\Rightarrow f'(z) = u_x + i v_x$

W.K.T, $|f(z)| = \sqrt{u^2+v^2}$

$$|f(z)|^2 = u^2 + v^2 = \phi \quad \text{---} \ (1)$$

$$\therefore \ |f'(z)| = \sqrt{u_x^2 + v_x^2}$$

$$|f'(z)|^2 = u_x^2 + v_x^2 \quad \text{---} \ (2)$$

Consider,

$$LHS = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \cdot |f(z)|^2$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \cdot \phi$$

$$LHS = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \quad \text{---} \ (3)$$

$\therefore$ Consider $\frac{\partial \phi}{\partial x} = 2u \cdot u_x + 2v \cdot v_x$

$$\frac{\partial^2 \phi}{\partial x^2} = 2(u \cdot u_{xx} + u_x \cdot u_x) + 2(v \cdot v_{xx} + v_x v_x)$$

$$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} = 2(u_x^2 + u \cdot u_{xx} + v_x^2 + v \cdot v_{xx}) \quad \text{---} \ (4)$$

Consider,

$$\frac{\partial \phi}{\partial y} = 2u \cdot u_y + 2v \cdot v_y$$

$$\Rightarrow \frac{\partial^2 \phi}{\partial y^2} = 2(u \cdot u_{yy} + u_y \cdot u_y) + 2(v \cdot v_{yy} + v_y v_y)$$

$$\Rightarrow \frac{\partial^2 \phi}{\partial y^2} = 2(u_y^2 + u \cdot u_{yy} + v_y^2 + v \cdot v_{yy}) \quad \text{--- (5)}$$

Substitute eqn (4) and (5) in eqn (3)

i.e., $\text{LHS} = 2(u_x^2 + u \cdot u_{xx} + v_x^2 + v \cdot v_{xx} + u_y^2 + u \cdot u_{yy} + v_y^2 + v \cdot v_{yy})$

$= 2(u_x^2 + v_x^2 + u_y^2 + v_y^2 + u(u_{xx} + u_{yy}) + v(v_{xx} + v_{yy}))$

$= 2(u_x^2 + v_x^2 + u_y^2 + v_y^2 + u(0) + v(0))$ {Harmonic}

$= 2(u_x^2 + v_x^2 + (-v_x)^2 + (u_x)^2)$ {using CR-equation}

$= 2(2u_x^2 + v_x^2 + v_x^2)$

$= 2(2u_x^2 + 2v_x^2)$

$= 4(u_x^2 + v_x^2)$

$= 4(|f'(z)|)^2$ {using (2)}

$= \text{RHS}$

(14) Determine the analytical function whose real part is $(y + e^x \cos y)$

Sol:- Given, $u = y + e^x \cos y$

$\therefore u_x = e^x \cos y$, $u_y = 1 - e^x \sin y$

Consider, $f'(z) = u_x + i v_x$. But $v_x = -i u_y$ (by C-R equation)

$\therefore f'(z) = u_x - i u_y = e^x \cos y - i(1 - e^x \sin y)$ — (1)

Put $x = z$ and $y = 0$ to obtain $f'(z)$ as a function of z.

(1) $\Rightarrow f'(z) = e^z \cos(0) - i(1 - e^z \sin 0)$

$= e^z(1) - i(1 - 0)$

$= e^z - i$

$\therefore \int f'(z) dz = \int (e^z - i) dz$

$\Rightarrow f(z) = \int e^z dx - i \int 1 dz$

$\boxed{f(z) = e^z - iz + c}$

2 (a) If $f(z)$ is an analytic function with constant modulus, show that $f(z)$ is constant.

Sol:- Given $w = f(z) = u + iv$ is an analytic function

$\therefore f(z)$ is differentiable

$\Rightarrow f'(z) = u_x + i v_x$

We have, $f(z) = u + iv$

$\Rightarrow |f(z)| = \sqrt{u^2 + v^2}$ ——— (1)

$f'(z) = u_x + i v_x$

$\Rightarrow |f'(z)| = \sqrt{u_x^2 + v_x^2}$

and $|f'(z)|^2 = u_x^2 + v_x^2$ ——— (2)

Given, $|f(z)| = K$

$\Rightarrow \sqrt{u^2 + v^2} = K$ {using (1)}

$\Rightarrow u^2 + v^2 = K^2$ ——— (3)

Differentiate (3) partially w.r.t 'x'

(3) $\Rightarrow 2uu_x + 2vv_x = 0$

$\Rightarrow uu_x + vv_x = 0$ ——— (4)

Similarly, differentiate (3) partially w.r.t 'y'

(3) $\Rightarrow 2uu_y + 2vv_y = 0$

$\Rightarrow uu_y + vv_y = 0$ ——— (5)

Since $f(z)$ is analytic,

we have $u_x = v_y$ and $v_x = -u_y$ (C-R equation)

$\hookrightarrow u_y = -v_x$

$\therefore$ (5) $\Rightarrow u(-v_x) + v u_x = 0$

$\Rightarrow v u_x - u v_x = 0$ —— (6)

Equation $(4)^2$ + equation $(6)^2$ gives

$$(u u_x + v v_x)^2 + (v u_x - u v_x)^2 = 0$$

$$\Rightarrow u^2 u_x^2 + v^2 v_x^2 + \cancel{2 u v_x v v_x} + v^2 u_x^2 + u^2 v_x^2 - \cancel{2 v u_x u v_x} = 0$$

$$\Rightarrow u^2(u_x^2 + v_x^2) + v^2(u_x^2 + v_x^2) = 0$$

$$\Rightarrow (u^2 + v^2)(u_x^2 + v_x^2) = 0$$

$$\Rightarrow K^2 (u_x^2 + v_x^2) = 0 \quad \{\text{using } (3)\}$$

$$\Rightarrow K^2 \neq 0, \quad u_x^2 + v_x^2 = 0 \quad \{\text{Since } K \text{ is Constant}\}$$

$$\Rightarrow |f'(z)|^2 = 0$$

$$\Rightarrow |f'(z)| = 0$$

$$\therefore f(z) = c \quad \{\text{on integrating}\}$$

$\Rightarrow f(z)$ is constant

Problems on finding the derivative of an analytic function.

2 b

1. Show that $w = \log z$, $z \neq 0$ is analytic and hence find $\frac{dw}{dz}$.

Sol:- Given, $w = \log z$ —— (1)

Take, $z = re^{i\theta}$

$(1) \Rightarrow u + iv = \log(re^{i\theta})$

$\Rightarrow u + iv = \log r + \log(e^{i\theta})$

$\Rightarrow u + iv = \log r + i\theta \log(e)$

$\Rightarrow u + iv = \log r + i\theta$ (because $\log e = 1$)

$\therefore u = \log r$, $v = \theta$

$u_r = \frac{\partial u}{\partial r} = \frac{1}{r}$, $v_r = \frac{\partial v}{\partial r} = 0$

$u_\theta = \frac{\partial u}{\partial \theta} = 0$, $v_\theta = \frac{\partial v}{\partial \theta} = 1$

C-R equations in the polar form: $ru_r = v_\theta$ and $rv_r = -u_\theta$ are satisfied.

$\therefore$ $\boxed{w = \log z \text{ is analytic}}$

$\therefore f'(z) = \cancel{\frac{dw}{dz}}\ e^{-i\theta}(u_r + iv_r)$

$= e^{-i\theta}\left(\frac{1}{r} + i \cdot 0\right) = \frac{1}{re^{i\theta}}$

Since, $z = re^{i\theta}$, $\boxed{f'(z) = \frac{1}{z}}$

2 c

3 Find the analytical function whose imaginary part is $e^{-x}(x\sin y - y\cos y)$

Sol:- ~~Given, $v = e^{-x}(x\sin y - y\cos y) = e^{-x}x\sin y - e^{-x}y\cos y$~~

$\therefore$ ~~$v_x = e^{-x}(\sin y) - (e^{-x})y\cos y$~~

Given, $v = e^{-x}(x\sin y - y\cos y) = e^{-x}(x\sin y) - e^{-x}(y\cos y)$

$\therefore v_x = -e^{-x}(x\sin y) + e^{-x}(\sin y) + e^{-x}(y\cos y)$

i.e., $v_x = e^{-x}(\sin y + y\cos y - x\sin y)$ —— (1)

$v_y = e^{-x}(x\cos y) + (x\sin y)(0) - e^{-x}(y(-\sin y) + \cos y(1))$

$v_y = e^{-x}x\cos y + e^{-x}y\sin y - e^{-x}\cos y$

i.e., $v_y = e^{-x}(x\cos y + y\sin y - \cos y)$ —— (2)

Consider, $f'(z) = u_x + iv_x$ (but $u_x = v_y$ by C-R equation)

i.e., $f'(z) = v_y + iv_x$

i.e., $f'(z) = e^{-x}(x\cos y + y\sin y - \cos y) + ie^{-x}(\sin y + y\cos y - x\sin y)$

put $x = z$ and $y = 0$ to obtain $f'(z)$ as a function of z.

$\therefore f'(z) = e^{-z}(z\cos(0) + 0 - \cos(0)) + ie^{-z}(\sin(0) + (0)\cos(0) - z\sin(0))$

$f'(z) = e^{-z}(z-1)$ $\{$because $\cos 0 = 1$, $\sin(0) = 0\}$

Integrating by parts,

$$\int f'(z)\,dz = \int (z-1)e^{-z}\,dz + c$$

$$\Rightarrow f(z) = (z-1)\int e^{-z}\,dz - \int\left[\int e^{-z}\,dz\right]\frac{d}{dz}(z-1)\,dz + c$$

$$= (z-1)\frac{e^{-z}}{-1} - \int \frac{e^{-z}}{-1}(1)\,dz + c$$

$$= -(z-1)e^{-z} + \left(\frac{e^{-z}}{-1}\right) + c$$

$$= -ze^{-z} + e^{-z} - e^{-z} + c$$

$$\boxed{f(z) = -z\,e^{-z} + c}$$

Module-2

③ ⓐ Discuss the transformation $w = e^z$

Sol:- Given, $w = f(z) = e^z$ is analytic

$\therefore f(z)$ is differentiable

$\Rightarrow f'(z) = e^z, \ z \neq 0$

We have, $w = f(z) = u + iv = e^z$ —— ①

Let $z = x + iy$

$\therefore$ ① $\Rightarrow u + iv = e^{x+iy}$

$\Rightarrow u + iv = e^x \cdot e^{iy}$

$\Rightarrow u + iv = e^x(\cos y + i \sin y)$

$\Rightarrow u + iv = e^x \cos y + i e^x \sin y$

$\therefore u = e^x \cos y$ —— ② and $v = e^x \sin y$ —— ③

~~[illegible]~~

. Take Equation $(2)^2$ + equation $(3)^2$ to eliminate 'y'

i.e., $u^2 + v^2 = (e^x \cos y)^2 + (e^x \sin y)^2$

$\Rightarrow u^2 + v^2 = e^{2x}(\cos^2 y + \sin^2 y)$

$\Rightarrow u^2 + v^2 = e^{2x}$ —— ④

~~[illegible]~~

. Divide equation ③ from equation ② to eliminate 'x'

i.e., $\dfrac{v}{u} = \dfrac{e^x \sin y}{e^x \cos y} = \dfrac{\sin y}{\cos y} = \tan y$

$\Rightarrow v = u \cdot \tan y$ —— ⑤

Case (1) :- Let $x = c1 =$ constant, we have

$$(4) \Rightarrow u^2 + v^2 = e^{2c1} = r^2$$

$$\Rightarrow \boxed{u^2 + v^2 = r^2 \text{ (say)}} \quad — (6)$$

Eq (6) represents a circle having the centre as origin with radius 'r'.

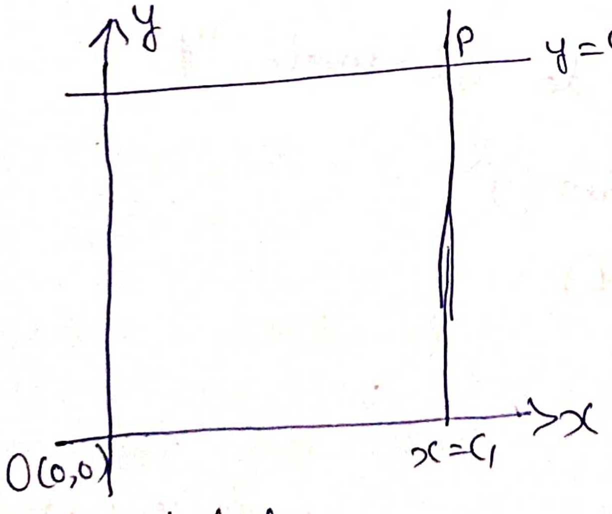
Case (2) :- Let $y = c2 =$ constant, we have

$$(5) \Rightarrow \quad v = u \cdot \tan y$$

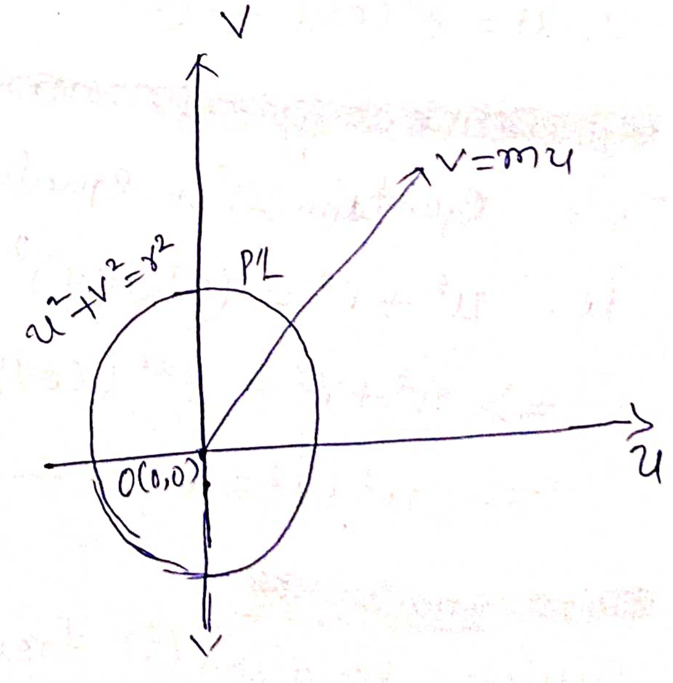
$$\Rightarrow \quad v = \tan c_2 \cdot u$$

$$\Rightarrow \boxed{v = m \cdot u} \quad — (7)$$, where $m = \tan c_2 =$ slope

Eq (7) represents a straight line passing through the origin having the slope 'm'.



'z' plane



3 (b) State and prove the Cauchy Integral theorem

Statement:- If a function $f(z)$ is analytic at all the points within and on a closed curve, then $\int_C f(z)dz = 0$.

Proof:- Given $w = f(z) = u + iv$ is an analytic function at all the points of z within and on a closed curve 'C'.

$\Rightarrow f(z)$ is differentiable

$$\therefore \int_C f(z)dz = \int_C (u+iv)(dx+idy)$$

$$\Rightarrow \int_C f(z)dz = \int_C (udx + uidy + ivdx - vdy)$$

$$\Rightarrow \int_C f(z)dz = \int_C [(udx - vdy) + i(udy + vdx)]$$

$$\Rightarrow \int_C f(z)dz = I_1 + I_2 \quad \text{--- (1)}$$

W.K.T from green's theorem

$$\int_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\therefore I_1 = \int_C (udx + (-v)dy) = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy \quad \text{--- (2)}$$

$$I_2 = \int_C (vdx + udy) = \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy \quad \text{--- (3)}$$

using (2) and (3) in (1), gives

$$(1) \Rightarrow \int_C f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy \quad \to (4)$$

Since $f(z)$ is analytic.

we have $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$\therefore$

$$(4) \Rightarrow \int_C f(z)\,dz = \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dx\,dy + i \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right) dx\,dy$$

$$\Rightarrow \int_C f(z)\,dz = 0$$

(3) (c) Find the bilinear transformation which maps the points $z = 1, i, -1$ onto the points $w = i, 0, -i$

Sol:- W.K.T $w = \dfrac{az+b}{cz+d}$ — (1)

- when $z = 1$, $w = i$, (1) becomes

$$i = \frac{a+b}{c+d} \Rightarrow a+b = ci + di$$

$$\Rightarrow a + b - ic - id = 0 \quad \text{—(2)}$$

- when $z = i$, $w = 0$, (1) becomes

$$0 = \frac{ai+b}{ci+d} \Rightarrow ai + b = 0 \quad \text{—— (3)}$$

• when $z=-1$, $w=-i$, (1) becomes

$$-i = \frac{-a+b}{-c+d} \Rightarrow -a+b = ic-id$$

$$\Rightarrow -a+b-ic+id=0 \quad \text{---} (4)$$

• Adding (2) and (4), gives

$$(a+b-ic-id)+(-a+b-ic+id) = 0$$

$$\Rightarrow a+b-ic-id-a+b-ic+id = 0$$

$$\Rightarrow \quad 2b-2ic=0$$

$$\Rightarrow b-ic=0 \quad \text{---} (5)$$

• We shall solve (3) and (5) by writing them in the form.

$$ia+1b+0c=0 \quad \text{---} (6)$$

$$0a+1b+(-i)c=0 \quad \text{---} (7)$$

• Applying the rule of cross multiplication we have,

$$\frac{a}{\begin{vmatrix} 1 & 0 \\ 1 & -i \end{vmatrix}} = \frac{-b}{\begin{vmatrix} i & 0 \\ 0 & -i \end{vmatrix}} = \frac{c}{\begin{vmatrix} i & 1 \\ 0 & 1 \end{vmatrix}}$$

i.e., $\frac{a}{-i} = -\frac{b}{-i^2} = \frac{c}{i}$

(or) $\frac{a}{-i} = \frac{b}{-1} = \frac{c}{i} = K$ (say)

· When $z = -1$, $w = -i$, ① becomes

$$-i = \frac{-a+b}{-c+d} \Rightarrow -a+b = ic+id$$
$$\Rightarrow -a+b-ic-id=0 \quad \text{——} \ (4)$$

· Subtract ② from ③

$$\Rightarrow a+b-ic-id-ai-b=0$$
$$\Rightarrow (1-i)a-ic-id=0 \quad \text{——} \ (5)$$

· Subtract ③ from ④

$$\Rightarrow ai+b-(-a+b-ic-id)=0$$
$$\Rightarrow ai+b+a-b+ic+id=0$$
$$\Rightarrow (1+i)a+ic+id=0 \quad \text{——} \ (6)$$

$$\therefore \frac{a}{-i}=k \Rightarrow \boxed{a=-ik}, \quad \frac{b}{-1}=k \Rightarrow \boxed{b=-k}, \quad \frac{c}{i}=k \Rightarrow \boxed{c=ik}$$

Substituting these in ②, we get

$$-ik-k-i(ik)-id=0$$
$$\Rightarrow -ik-k+k-id=0$$
$$\Rightarrow -i(k+d)=0$$
$$\Rightarrow k+d=0 \Rightarrow \boxed{d=-k}$$

Substituting the values of a, b, c, d in the bilinear transformation ①, we get

$$w = \frac{-ikz-k}{ikz-k} = \frac{-k(1+iz)}{-k(1-iz)}$$

Thus, $\boxed{w = \frac{1+iz}{1-iz}}$ is the required bilinear transformation

4 (a) Find the bilinear transformation which maps $1, i, -1$ to $2, i, -2$ respectively.

Sol:- Let $w = \dfrac{az+b}{cz+d}$ be the required bilinear transformation ——— (1)

• When $z=1$, $w=2$, (1) becomes

$2 = \dfrac{a+b}{c+d} \Rightarrow 2c+2d = a+b$

$\Rightarrow a+b-2c-2d=0$ ——— (2)

• When $z=i$, $w=i$, (1) becomes

$i = \dfrac{ai+b}{ci+d} \Rightarrow i^2c+id = ai+b$

$\Rightarrow ia+b+c-id=0$ ——— (3)

• When $z=-1$, $w=-2$, (1) becomes

$-2 = \dfrac{-a+b}{-c+d} \Rightarrow 2c-2d = -a+b$

$\Rightarrow -a+b-2c+2d=0$ ——— (4)

• Adding (2) and (4), we get

$(a+b-2c-2d)+(-a+b-2c+2d)=0$

$\Rightarrow \cancel{a}+b-2c-\cancel{2d}-\cancel{a}+b-2c+\cancel{2d}=0$

$\Rightarrow 2b-4c=0$

$\Rightarrow b-2c=0$ ——— (5)

• Adding (3) and $i\times$(4), we get

$(ai+b+c-di)+i(-a+b-2c+2d)=0$

$\Rightarrow \cancel{ia}+b+c-id-\cancel{ia}+ib-2ic+2id=0$

$\Rightarrow (1+i)b+(1-2i)c+id=0$ —— 6

Let us solve 5 and 6 by writing them in the form,

$1b-2c+0d=0$ —— 7

$(1+i)b+(1-2i)c+id=0$ —— 8

Applying the rule of cross multiplication, we have.

$$\frac{b}{\begin{vmatrix}-2 & 0\\(1-2i) & i\end{vmatrix}}=\frac{-c}{\begin{vmatrix}1 & 0\\(1+i) & i\end{vmatrix}}=\frac{d}{\begin{vmatrix}1 & -2\\(1+i) & (1-2i)\end{vmatrix}}$$

$(1-2i)+2(1+i)$
$=1-\cancel{2i}+2+\cancel{2i}$
$=3$

$$\Rightarrow \frac{b}{-2i}=\frac{-c}{i}=\frac{d}{3}=K(\text{say})$$

$\Rightarrow \frac{b}{-2i}=K \Rightarrow \boxed{b=-2iK}$, $\frac{-c}{i}=K\Rightarrow \boxed{c=-iK}$, $\frac{d}{3}=K\Rightarrow\boxed{d=3K}$

With these values, 2 becomes.

$a-2iK+2iK-6=0 \Rightarrow \boxed{a=6}$

Thus by substituting the values of a, b, c, d in 1, we get

$\boxed{w=\frac{6z-2i}{-iz+3}}$ is the required bilinear transformation

(4)(b) Verify Cauchy's theorem for the integral of z^3 over the boundary of the rectangle with vertices $z=-1, 1, 1+i, -1+i$.

Sol:- Given, $f(z) = z^3$ over the boundary of the rectangle with vertices $z=-1, 1, 1+i, -1+i$

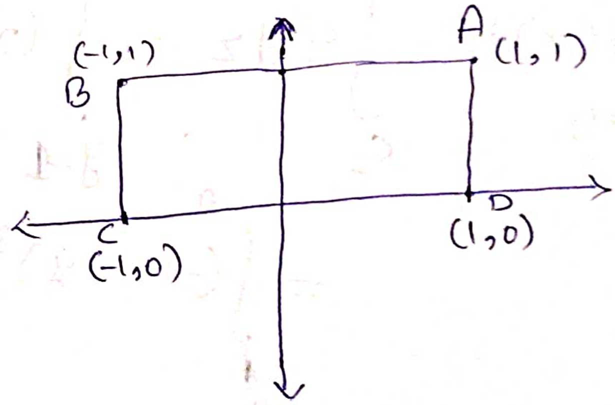
i.e., $-1+i0,\ 1+i0,\ 1+i,\ -1+i$

i.e., $(-1,0), (1,0), (1,1), (-1,1)$

We have.

$$z = x+iy$$

$$dz = dx + i\,dy$$



· Along AB:

$A(1,1) \longrightarrow B(-1,1)$

$x \to +1$ to -1, $y=1$, $dy=0$

$$\int_{C_1=AB} f(z)\,dz = \int z^3\,dz$$

$$= \int_{x=1}^{x=-1} (x+iy)^3 (dx + i\cancel{dy}^{\,0})$$

$$= \int_{+1}^{-1} (x+i)^3 dx = \int_{1}^{-1} (x^3 + 3x^2 i + 3x i^2 + i^3)\,dx$$

$$= \int_{1}^{-1} (x^3 + 3x^2 i - 3x - i)\,dx$$

$$= \left[\frac{x^4}{4} + 3i\frac{x^3}{3} - \frac{3x^2}{2} - ix\right]_1^{-1}$$

$$= \left(\frac{1}{4} - i - \frac{3}{2} - i\right) - \left(\frac{1}{4} + i - \frac{3}{2} - i\right)$$

$$= \frac{1}{4} - i - \frac{3}{2} - i - \frac{1}{4} - i + \frac{3}{2} + i$$

$$\int_{AB} f(z)dz = 0 \quad \text{—— (1)}$$

- Along BC

$B(-1,1) \to C(-1,0)$, $x=-1$, $y \to 1$ to 0

$dx = 0$

$$\int_{BC=C_2} f(z)dz = \int_{C_2} z^3 dz = \int_{y=1}^{0} (x+iy)^3 (dx + i\,dy)$$

$$= \int_1^0 (-1+iy)^3 \, i\,dy$$

$$= \int_1^0 \left[(-1)^3 + 3(-1)^2(iy) + 3(-1)(iy)^2 + (iy)^3\right] i\,dy$$

$$= \int_1^0 (-1 + 3yi + 3y^2 - iy^3)\, i\,dy$$

$$= \int_{+1}^0 (-i - 3y + 3y^2 i + y^3)\,dy$$

$$= \left[-iy - \frac{3y^2}{2} + 3i\frac{y^3}{3} + \frac{y^4}{4}\right]_1^0$$

$$= 0 - \left(-i - \frac{3}{2} + i + \frac{1}{4}\right) = -\left(\frac{-6+1}{4}\right) = \frac{5}{4}$$

$$\therefore \int_{BC=C_2} f(z)dz = \frac{5}{4} \quad \text{—— (2)}$$

Along CD:

$C(-1,0) \quad D(1,0),$

$x=0 \Rightarrow dx=0 \; ; \; y=0$

$y \to -1 \text{ to } 1$

$$\int_{CD} f(z)\,dz = \int_{-1}^{1} z^3\,dz = \int_{-1}^{1} (x+i\cancel{y}^{\,0})^3 (dx + i\,\cancel{dy}^{\,0})$$

$$= \int_{-1}^{1} x^3\,dx = \left[\frac{x^4}{4}\right]_{-1}^{1} = \left(\frac{1}{4}\right) - \left(\frac{1}{4}\right) = 0$$

$$\therefore \int_{CD=C_3} f(z)\,dz = 0 \quad \text{——— } (3)$$

Along DA:

$D(1,0) \quad A(1,1)$

$x=1 \Rightarrow dx=0$

$y \to 0 \text{ to } 1$

$$\int_{DA=C_4} f(z)\,dz = \int z^3\,dz = \int_{0}^{1} (x+iy)^3 (\cancel{dx}^{\,0} + i\,dy)$$

$$= \int_{0}^{1} (1+iy)^3 (i\,dy)$$

$$= \int_{0}^{1} \left(1^3 + 3(1)(iy) + 3(1)(iy)^2 + (iy)^3\right) i\,dy$$

$$= \int_{0}^{1} (i - 3y - 3y^2 i + y^3)\,dy$$

$$= \left[iy - \frac{3y^2}{2} - \frac{\cancel{3}y^3}{\cancel{3}} i + \frac{y^4}{4}\right]_0^1$$

$$= \left[\cancel{1} - \frac{3}{2} - \cancel{1} + \frac{1}{4}\right] - 0$$

$$= \frac{-6+1}{4} = -\frac{5}{4}$$

$$\therefore \int f(z)\,dz = \frac{-5}{4} \quad \text{---} \quad (4)$$

$C_4 = DA$

$$\therefore \int_C f(z)\,dz = \int_{C_1} z^3\,dz + \int_{C_2} z^3\,dz + \int_{C_3} z^3\,dz + \int_{C_4} z^3\,dz$$

$$= 0 + \cancel{\frac{5}{4}} + 0 - \cancel{\frac{5}{4}}$$

$$\int_C f(z)\,dz = 0$$

Hence Cauchy's theorem is verified

4 (c) Evaluate $\oint \frac{e^{-z}}{(z-1)(z-2)^2} dz$ over the curve $|z|=3$

Sol:- We shall first solve $\frac{1}{(z-2)^2(z-1)}$ into partial fractions.

Let, $$\frac{1}{(z-2)^2(z-1)} = \frac{A}{(z-2)} + \frac{B}{(z-2)^2} + \frac{C}{(z-1)}$$

$$\Rightarrow \quad 1 = A(z-2)(z-1) + B(z-1) + C(z-2)^2$$

- put $z=2$: $1 = B \quad \therefore \boxed{B=1}$
- put $z=1$: $1 = C \quad \therefore \boxed{C=1}$
- put $z=0$: $1 = A(-2)(-1) + B(-1) + C(-2)^2$

$1 = 2A - 1 + 4$

$1 = 2A + 3 \Rightarrow 1-3 = 2A \Rightarrow \boxed{A = -\frac{2}{2} = -1}$

Now, $$\frac{1}{(z-2)^2(z-1)} = \frac{-1}{(z-2)} + \frac{1}{(z-2)^2} + \frac{1}{(z-1)}$$

Multiplying by e^{-z} and integrating w.r.t 'z' over c we have,

$$\int_c \frac{e^{-z}}{(z-1)(z-2)^2} dz = -1 \int_c \frac{e^{-z}}{(z-2)} dz + \int_c \frac{e^{-z}}{(z-2)^2} dz + \int_c \frac{e^{-z}}{(z-1)} dz \quad \text{--- (1)}$$

- The points $z=a=1$, $z=a=2$ lies inside the circle $|z|=3$.

• We shall consider Cauchy's integral formula in the forms,

$$\int_C \frac{f(z)}{(z-a)}\,dz = 2\pi i\, f(a) \text{ and } \int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

Taking $f(z) = e^{-z}$ we obtain $f'(z) = -e^{-z}$

Now, $$\int_C \frac{e^{-z}}{(z-1)}\,dz = 2\pi i f(1) = 2\pi i\, e^{-1} = \frac{2\pi i}{e} \quad \text{--- (2)}$$

$$\int_C \frac{e^{-z}}{(z-2)^2}\,dz = \frac{2\pi i}{1!} f'(2) = \frac{2\pi i}{1!}(-e^{-2}) = -\frac{2\pi i}{e^2} \quad \text{--- (3)}$$

$$\int_C \frac{e^{-z}}{(z-2)}\,dz = 2\pi i f(2) = 2\pi i\, e^{-2} = \frac{2\pi i}{e^2} \quad \text{--- (4)}$$

Substituting the results (2), (3) and (4) in the RHS of (1) we have,

$$\int_C \frac{e^{-z}}{(z-1)(z-2)^2}\,dz = (-1)\left(\frac{2\pi i}{e^2}\right) + \left(-\frac{2\pi i}{e^2}\right) + \left(\frac{2\pi i}{e}\right)$$

$$= -\frac{2\pi i}{e^2} - \frac{2\pi i}{e^2} + \frac{2\pi i}{e}$$

$$= -\frac{4\pi i}{e^2} + \frac{2\pi i}{e} = \frac{2\pi i}{e}\left(1 - \frac{2}{e}\right)$$

$$\therefore \boxed{\int_C \frac{e^{-z}}{(z-1)(z-2)^2}\,dz = 2\pi i\left(\frac{1}{e} - \frac{2}{e^2}\right)}$$

Module - 3

5) a) A random variable X has the following probability function:

x	-2	-1	0	1	2	3
$P(x)$	0.1	K	0.2	$2K$	0.3	K

Find the value of K and calculate the mean and variance

Sol :- • $P(x) \geq 0$

• $\sum P(x) = 1$

i.e, $0.1 + K + 0.2 + 2K + 0.3 + K = 1$

i.e., $4K + 0.6 = 1$

i.e., $4K = 1 - 0.6$

$$\boxed{K = \frac{0.4}{4} = 0.1}$$

• The value of K is 0.1

x	$P(x)$	$xP(x)$	$x^2P(x)$
-2	0.1	-0.2	0.4
-1	0.1	-0.1	0.1
0	0.2	0	0
1	0.2	0.2	0.2
2	0.3	0.6	1.2
3	0.1	0.3	0.9
		$\sum xP(x) = 0.8$	$\sum x^2P(x) = 2.8$

• Mean $= \sum xP(x) = 0.8$

• variance $V(x) = \sum x^2 P(x) - [\sum xP(x)]^2 = 2.8 - (0.8)^2 = \underline{\underline{2.16}}$

5) b) Find the mean and Standard deviation of the Binomial distribution

Sol: If 'p' is the probability of success and 'q' is the probability of failure, the probability of 'x' success out of n trials is given by

$$P(x) = {}^{n}C_{x}\, p^{x} q^{n-x} \quad \text{--- (1)}$$

We know that the mean of the discrete probability distribution is

$$\text{Mean}(\mu) = \sum_{x=0}^{n} x P(x) \quad \text{--- (2)}$$

$$\text{i.e., } \mu = \sum_{x=0}^{n} x \cdot {}^{n}C_{x}\, p^{x} q^{n-x}$$

$$= \sum_{x=0}^{n} x \cdot \frac{n!}{x!(n-x)!} p^{x} q^{n-x}$$

$$= \sum_{x=0}^{n} \cancel{x} \cdot \frac{n!}{(n-x)!\,\cancel{x}(x-1)!} p^{x} q^{n-x}$$

$$= \sum_{x=0}^{n} \frac{n(n-1)!}{[(n-1)-(x-1)]!\,(x-1)!} p \cdot p^{x-1} \cdot q^{[(n-1)-(x-1)]}$$

$${}^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

$$5! = 5 \times 4!$$

$$x! = x(x-1)!$$

$$n! = n(n-1)!$$

$$= np\,(p+q)^{n-1}$$

$$= np\,(1)^{n-1} \qquad (p+q=1)$$

$$= np(1)$$

$$\text{Mean} = np$$

Variance $(v) = \sum_{x=0}^{n} x^2 p(x) - \mu^2$

Now,

$$\sum_{x=0}^{n} x^2 p(x) = \sum_{x=0}^{n} [x + (x-1)x]\, p(x)$$

$$= \sum_{x=0}^{n} x\, p(x) + \sum_{x=0}^{n} (x-1)x\, p(x)$$

$$= np + \sum_{x=0}^{n} (x-1)x \cdot {}^nC_x\, p^x q^{n-x}$$

$$= np + \sum_{x=0}^{n} x(x-1) \cdot \frac{n!}{(n-x)!\, x!}\, p^x \cdot q^{n-x}$$

$$= np + \sum_{x=0}^{n} \cancel{x(x-1)} \frac{n!}{(n-x)!\, \cancel{x(x-1)}(x-2)!}\, p^x \cdot q^{n-x}$$

$$= np + \sum_{x=0}^{n} \frac{n(n-1)(n-2)!}{[(n-2)-(x-2)]!\,(x-2)!}\, p^x \cdot p^{x-2} q^{(n-2)-(x-2)}$$

$$= np + n(n-1)p^2 \sum \frac{(n-2)!}{[(n-2)-(x-2)]!\,(x-2)!}\, p^{x-2} q^{(n-2)-(x-2)}$$

$$= np + n(n-1)p^2 (p+q)^{n-2} \qquad (\because p+q=1)$$

$$= np + n^2p^2 - np^2 (1)^{n-2}$$

$$= np + n^2p^2 - np^2$$

$$\boxed{\begin{matrix} p+q=1 \\ q = 1-p \end{matrix}}$$

Variance $(v) = np + n^2p^2 - np^2 - (np)^2$

$$= np + \cancel{n^2p^2} - np^2 - \cancel{n^2p^2}$$

$$= np(1-p)$$

$$\boxed{v = npq}$$

5 c

Sol:-. poisson distribution, $P(x) = \frac{m^x e^{-m}}{x!}$

- $p = 0.002$, $n = 500$
- Mean(μ) $= m = np = 500 \times 0.002 = 1$

We have to find,

(i) $P(X=0)$ (ii) $P(X=1)$ and (iii) $P(X=2)$

(i) $P(X=0) = \frac{m^x e^{-m}}{x!} = \frac{(1)^0 e^{-1}}{0!} = \frac{1 e^{-1}}{1} = \frac{1}{e} = 0.36787$

(ii) $P(X=1) = \frac{m^x e^{-m}}{x!} = \frac{(1)^1 e^{-1}}{1!} = \frac{e^{-1}}{1} = \frac{1}{e} = 0.36787$

(iii) $P(X=2) = \frac{(1)^2 e^{-2}}{2!} = \frac{e^{-2}}{2} = \frac{1}{2e^2} = 0.06766$

6 a

Sol:- We have, $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)dx = 1$

Here $f(x) = 6x(1-x)$, $0 \leq x \leq 1$

$$\therefore \int_0^1 (6x - 6x^2)dx = \frac{6x^2}{2}\Big|_0^1 - \frac{6x^3}{3}\Big|_0^1 = 3[1-0] - 2[1-0] = 3-2 = 1$$

Hence the given function is a valid p.d.f.

• Mean $= \mu = \int_{-\infty}^{\infty} x \cdot f(x)dx = \int_0^1 x \cdot (6x - 6x^2)dx$

$$= \int_0^1 (6x^2 - 6x^3)dx = \frac{6x^3}{3}\Big|_0^1 - \frac{6x^4}{4}\Big|_0^1$$

$$= 2[1-0] - \frac{3}{2}[1-0]$$

$$= 2 - \frac{3}{2}$$

$$= \frac{4-3}{2} = \frac{1}{2}$$

• Variance $= V = \int_{-\infty}^{\infty} x^2 f(x)\, dx - (\mu)^2 = \int_0^1 x^2 (6x - 6x^2)dx - \left(\frac{1}{2}\right)^2$

$$= \int_0^1 (6x^3 - 6x^4)dx - \frac{1}{4} = \frac{6x^4}{4}\Big|_0^1 - \frac{6x^5}{5}\Big|_0^1 - \frac{1}{4}$$

$$= \frac{3}{2}(1-0) - \frac{6}{5}(1-0) - \frac{1}{4}$$

$$= \frac{30 - 24 - 5}{20} = \frac{1}{20}$$

$\therefore$ Variance $= \frac{1}{20}$

(6)(b)

Sol:- By data $\mu = 2040$, $\sigma = 60$

We have, s.n.v, $z = \frac{x-\mu}{\sigma} = \frac{x-2040}{60}$

(i) To find $P(x > 2150)$

If $x = 2150$, $z = \frac{2150-2040}{60} = 1.833$

$$
\begin{aligned}
P(x > 2150) &= P(z > 1.83) \\
&= P(z \geq 0) - P(0 < z < 1.83) \\
&= 0.5 - \phi(1.83) \\
&= 0.5 - 0.4664 \\
&= 0.0336
\end{aligned}
$$

$\longrightarrow 2000 \times 0.0336 = 67.2 \approx \underline{68}$

(ii) To find $P(x < 1950)$

If $x = 1950$, $z = \frac{1950-2040}{60} = -1.5$

$$
\begin{aligned}
P(x < 1950) &= P(z < -1.5) \\
&= P(z > 1.5) \\
&= P(z \geq 0) - P(0 < z < 1.5) \\
&= 0.5 - \phi(1.5) \\
&= 0.5 - 0.4332 \\
&= 0.0668
\end{aligned}
$$

$\therefore$ number of bulbs that are likely to last for more than 1950 hours is

$= 2000 \times 0.0668 = 133.6 \approx \underline{134}$

(iii) To find $P(1920 < x < 2160)$

If $x = 1920$, $Z = \frac{1920-2040}{60} = -2$

If $x = 2160$, $Z = \frac{2160-2040}{60} = 2$

$$P(1920 < x < 2160) = P(-2 < Z < 2)$$
$$= 2P(0 < Z < 2)$$
$$= 2\phi(2)$$
$$= 2(0.4772)$$
$$= \underline{\underline{0.9544}}$$

$\therefore$ number of bulbs that are likely to last between 1920 and 2160 hours is

$= 2000 \times 0.954 = \underline{\underline{1908}}$

6 c

We have $f(x) = \alpha e^{-\alpha x}$, $x > 0$; Mean $= \frac{1}{\alpha}$

By data, mean $= 200$, $200 = \frac{1}{\alpha} \Rightarrow \boxed{\alpha = \frac{1}{200}}$

Hence, $f(x) = \frac{1}{200} e^{-x/200}$

(i) $P(x < 200) = \int_0^{200} f(x)dx = \int_0^{200} \frac{1}{200} e^{-x/200} = \frac{1}{200}\left[-\frac{e^{-x/200}}{(1/200)}\right]_0^{200}$

$= -\left[e^{-1} - e^{0}\right] = e^{0} - \frac{1}{e}$

$= \left(1 - \frac{1}{e}\right) = 0.6321$

(ii) $P(100 < x < 300) = \int_{100}^{200} \frac{1}{200} e^{-x/200} dx = -\left[e^{-x/200}\right]_{100}^{200}$

$= -\left[e^{-1} - e^{-1/2}\right] = \frac{1}{e^{1/2}} - \frac{1}{e} = 0.2386$

(iii) $P(x > 200) = 1 - P(x \leq 200)$

$= 1 - \int_0^{200} \frac{1}{200} e^{-x/200}$

$= 1 - \left[1 - \frac{1}{e}\right] = 1 - 0.6321 = 0.3679$

7 (a)

(2) Using simplex method solve the L.P.P.

Maximize $Z = 3x_1 + 2x_2$

Subject to $2x_1 + x_2 \leq 5$

$x_1 + x_2 \leq 3$

$x_1, x_2 \geq 0$

Solution:-

→ Introduce 2 slack variables S_1 and S_2, rewrite the given L.P.P. in the standard form.

i.e., Maximize, $Z = 3x_1 + 2x_2 + 0S_1 + 0S_2$ → objective function

subject to, $2x_1 + x_2 + S_1 + 0S_2 = 5$

$x_1 + x_2 + 0S_1 + S_2 = 3$

$x_1, x_2, S_1, S_2 \geq 0$

R_1 → first row

R_2 → second row

- Total number of variables = 4
- Basic variables = 2
- Number of non-basic variables = 4 − 2 = 2

			3	2	0	0	
NZV	C_B	X_B	x_1	x_2	S_1	S_2	Ratio
R_1→ S_1	0	5	$\boxed{2}$	1	1	0	$\frac{5}{2} = 2.5$ → (PR)
R_2→ S_2	0	3	1	1	0	1	$\frac{3}{1} = 3$
$z_j - c_j$	$Z = C_B X_B =$ 0		$\Delta_1 = -3$ (PC)	$\Delta_2 = -2$	$\Delta_3 = 0$	$\Delta_4 = 0$	

- $Z = C_B X_B = (0\times5 + 0\times3) = 0$; $\Delta_1 = (0\times2 + 0\times1) - 3 = -3$; $\Delta_2 = (0\times1 + 0\times1) - 2 = -2$;

$\Delta_3 = (0\times1 + 0\times0) - 0 = 0$; $\Delta_4 = (0\times0 + 0\times1) - 0 = 0$

- pivot key = 2
- Outgoing variable = S_1
- Incoming variable = x_1

Apply $R_1' \to \frac{R_1}{2}$

NZV	C_B	X_B	$\overset{3}{x_1}$	$\overset{2}{x_2}$	$\overset{0}{S_1}$	$\overset{0}{S_2}$	Ratio
R_1' x_1	3	$\frac{5}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{5}{2}\times\frac{2}{1}=5$
R_2' S_2	0	$\frac{1}{2}$	0	$\boxed{\frac{1}{2}}$	$-\frac{1}{2}$	1	$\frac{1}{2}\times\frac{2}{1}=1 \to$ PR
$Z_j - C_j$	$Z=C_BX_B=$	$\frac{15}{2}$	$\Delta_1 = 0$	$\Delta_2 = -\frac{1}{2}$	$\Delta_3 = \frac{3}{2}$	$\Delta_4 = 0$	

Apply $R_2' \to R_2 - R_1'$ PC

- $Z = C_BX_B = \left(3\times\frac{5}{2} + 0\times\frac{1}{2}\right) = \frac{15}{2}$; $\Delta_1 = (3\times1 + 0\times0) - 3 = 0$
- $\Delta_2 = \left(3\times\frac{1}{2} + 0\times\frac{1}{2}\right) - 2 = \frac{3}{2} - 2 = -\frac{1}{2}$; $\Delta_3 = \left(3\times\frac{1}{2} + 0\times -\frac{1}{2}\right) - 0 = \frac{3}{2}$
- $\Delta_4 = (3\times0 + 0\times1) - 0 = 0$
- Pivot Key $= \frac{1}{2}$
- Outgoing variable $= S_2$; Incoming variable $= x_2$
- Apply $R_2'' \to 2R_2'$

NZV	C_B	X_B	$\overset{3}{x_1}$	$\overset{2}{x_2}$	$\overset{0}{S_1}$	$\overset{0}{S_2}$	Ratio
R_1'' x_1	3	2	1	0	1	-1	
R_2'' x_2	2	1	0	1	-1	2	
$Z_j - C_j$	$Z=C_BX_B=$	8	$\Delta_1 = 0$	$\Delta_2 = 0$	$\Delta_3 = 1$	$\Delta_4 = 1$	

- Apply $R_1'' \to R_1' - \frac{R_2''}{2}$
- $Z = C_BX_B = (3\times2 + 2\times1) = 8$; $\Delta_1 = (3\times1 + 2\times0) - 3 = 0$

$\Delta_2 = (3\times0 + 2\times1) - 2 = 0$; $\Delta_3 = (3\times1 + 2\times(-1)) - 0 = 1$; $\Delta_4 = (3\times-1 + 2\times2) - 0 = 1$

As all the indicators are non-negative, $\boxed{Z_{max} = 8}$ and it occurs at $\boxed{x_1 = 2}$ and $\boxed{x_2 = 1}$

(7) (b)

(ii) Using Big-M method, solve the LPP

Minimize $Z = 2x_1 + x_2$,

Subject to : $3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \leq 3$

$x_1, x_2 \geq 0$

Sol:- We know minimization = -(maximization)

- ~~As the~~ As the first constraint is of the type '=', we introduce an A.V a_1

$$3x_1 + x_2 + a_1 = 3$$

- As the second constraint is of the type '$\geq$', we introduce surplus variable s_1 and hence an AV a_2.

$$4x_1 + 3x_2 - s_1 + a_2 = 6$$

- As the third constraint is of the type '$\leq$', we introduce slack variable s_2

$$x_1 + 2x_2 + s_2 = 3$$

- As two A.V's a_1 and a_2 are introduced, the new OF is

$$\text{Maximize } z' = -2x_1 - x_2 - Ma_1 - Ma_2$$

			-2	-1	0	0	$-M$	$-M$	Min
NZV	C_B	X_B	x_1	x_2	S_1	S_2	a_1	a_2	Ratio
R_1 a_1	$-M$	3	$\boxed{3}$	1	0	0	1	0	$\frac{3}{3}=1 \rightarrow$ PR
R_2 a_2	$-M$	6	4	3	-1	0	0	1	$\frac{6}{4}=1.5$
R_3 S_2	0	3	1	2	0	1	0	0	$\frac{3}{1}=3$
Z_j-C_j	$Z'=$	$-9M$	$-7M+2$	$-4M+1$	M	0	0	0	
			$\hookrightarrow$ PC						

- pivot element $=3$, OGV $=a_1$, ICV $=x_1$
- Apply $R_1' \rightarrow \frac{R_1}{3}$, $R_2' \rightarrow R_2 - 4R_1'$, $R_3' \rightarrow R_3 - R_1'$
- As the OGV is an A.V, we shall drop the column corresponding to a_1 in the next simplex table.

			-2	-1	0	0	$-M$	Min
NZV	C_B	X_B	x_1	x_2	S_1	S_2	a_2	ratio
R_1' x_1	-2	1	1	$1/3$	0	0	0	$1/(1/3)=3$
R_2' a_2	$-M$	2	0	$\boxed{5/3}$	-1	0	1	$2/(5/3)=\frac{6}{5} \rightarrow$ PR
R_3' S_2	0	2	0	$5/3$	0	1	0	$2/(5/3)=\frac{6}{5}$
Z_j-C_j	$Z'=-2-2M$		0	$\frac{-5M}{3}+\frac{1}{3}$	M	0	0	
				$\hookrightarrow$ PC				

- Pivot element $=\frac{5}{3}$, OGV $=a_2$, ICV $=x_2$
- Apply, $R_2'' \rightarrow \frac{3}{5} \times R_2'$, $R_1'' \rightarrow R_1' - \frac{1}{3}R_2''$, $R_3'' \rightarrow R_3' - \frac{5}{3}R_2''$

As the OG.V is an A.V, we can drop the column corresponding to a_2 in the next simplex table

	NZV	C_B	X_B	-2 x_1	-1 x_2	0 s_1	0 s_2
R_1'''	x_1	-2	$3/5$	1	0	$1/5$	0
R_2'''	x_2	-1	$6/5$	0	1	$-3/5$	0
R_3'''	s_2	0	0	0	0	1	1
	z_j-c_j	$z' = -12/5$		0	0	$1/5$	0

As all the indicators are non-negative, simplex method is complete

$$\therefore Z'_{max} = -\frac{12}{5} \text{ at } x_1 = \frac{3}{5} \text{ and } x_2 = \frac{6}{5}$$

$$\Rightarrow Z_{min} = \frac{12}{5} \text{ at } x_1 = \frac{3}{5} \text{ and } x_2 = \frac{6}{5}$$

8 (a)

(*) Canonical form and standard form

After the formulation of L.P.P., the next step is to obtain its solution. But before any method is used to find its solution, the problem must be presented in a suitable form. As such, we explain its following two forms:

1. Canonical form
2. Standard form

1. Canonical form

The general L.P.P. can always be expressed in the form:

Maximize $Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

Subject to the constraints

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \le b_i \; ; \; i = 1, 2, \ldots, m$$

$$x_1, x_2, \ldots, x_n \ge 0,$$

by making some elementary transformations. This form of the L.P.P. is called its canonical form and has the following characteristics:

(i) objective function is of maximization type

(ii) All constraints are of $(\le)$ type

(iii) All variables are non-negative

The Canonical form is a format for a L.P.P. which finds its use in the Duality theory.

(2) Standard form

The general L.P.P. can also be put in the following form:

Maximize $Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

Subject to the constraints

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \; ; \; i = 1, 2, \ldots m$$

$$x_1, x_2, \ldots, x_n \geq 0$$

This form of the L.P.P. is called its standard form and has the following characteristics:

(i) objective function is of maximization type

(ii) All constraints are expressed as equations

(iii) Right hand side of each constraints is non-negative

(iv) All variables are non-negative

e.g. a variable x_i can be written as

$$x_i = x_i' - x_i'' \quad \text{where } x_i' \geq 0, \; x_i'' \geq 0$$

Example 1:- Convert the following L.P.P. to the standard form:

8 (a)

Maximize $Z = 3x_1 + 5x_2 + 7x_3$,

Subject to $6x_1 - 4x_2 \leq 5$,

$3x_1 + 2x_2 + 5x_3 \geq 11$,

$4x_1 + 3x_3 \leq 2$,

and $x_1, x_2 \geq 0$.

Solution:- • As x_3 is unrestricted

• Let $x_3 = x_3' - x_3''$ where $x_3', x_3'' \geq 0$.

• Now the given constraints can be expressed as

$6x_1 - 4x_2 \leq 5$,

$3x_1 + 2x_2 + 5x_3' - 5x_3'' \geq 11$,

$4x_1 + 3x_3' - 3x_3'' \leq 2$,

and $x_1, x_2, x_3', x_3'' \geq 0$

• Introducing the slack/surplus variables, the problem in the standard form becomes.

Maximize $Z = 3x_1 + 5x_2 + 7x_3' - 7x_3''$

Subject to $6x_1 - 4x_2 + S_1 = 5$

$3x_1 + 2x_2 + 5x_3' - 5x_3'' - S_2 = 11$

$4x_1 + 3x_3' - 3x_3'' + S_3 = 2$

and $x_1, x_2, x_3', x_3'', S_1, S_2, S_3 \geq 0$

Solution.

(8) Use two-phase method to solve the LPP

(8)(b) Maximize $Z = 9x_1 + 3x_2$

subject to: $4x_1 + x_2 \leq 8$

$2x_1 + x_2 \leq 4$

$x_1, x_2 \geq 0$

Sol:- • Introduce two slack variables s_1 and

i.e, Maximize, $Z = 9x_1 + 3x_2 + 0s_1 + 0s_2$

STC $4x_1 + x_2 + s_1 = 8$

$2x_1 + x_2 + s_2 = 4$

$x_1, x_2, s_1, s_2 \geq 0$

put $x_1 = x_2 = 0$

$s_1 = 8$ feasible

$s_2 = 0$ feasible

Phase-I

- Assign a cost '0' to all the variables

NZV	C_B	X_B	$\overset{0}{x_1}$	$\overset{0}{x_2}$	$\overset{0}{S_1}$	$\overset{0}{S_2}$
S_1	0	8	4	1	1	0
S_2	0	4	2	1	0	1
$Z_j - C_j$	$Z^* = 0$		0	0	0	0

Since $z^* = 0$ (maximum) and no AV's appears in the optimum basis. In this case problem has a solution and to find the solution, go to phase 2.

Phase-II

Consider the final simplex table obtained at the end of phase I. Assign the actual cost of the OF.

	NZV	C_B	X_B	$\overset{9}{x_1}$	$\overset{3}{x_2}$	$\overset{0}{S_1}$	$\overset{0}{S_2}$	Ratio	
R_1	S_1	0	8	4	1	1	0	$\frac{8}{4} = 2$	
R_2	S_2	0	4	2	1	0	1	$\frac{4}{2} = 2$	→ PR
	$Z_j - C_j$	$Z^* = C_B X_B = 0$		$\Delta_1 = -9$	$\Delta_2 = -3$	$\Delta_3 = 0$	$\Delta_4 = 0$		
				↳ PC					

Pivot key = 2

OBV = S_2

ICV = x_1

Apply $R_2' \rightarrow \frac{R_2}{2}$

$R_1' \rightarrow R_1 - 4R_2'$

NZV	C_B	X_B	$\overset{9}{x_1}$	$\overset{3}{x_2}$	$\overset{0}{S_1}$	$\overset{0}{S_2}$
S_1	0	0	0	-1	1	-2
x_1	9	2	1	$\frac{1}{2}$	0	$\frac{1}{2}$
$Z_j - C_j$	$Z^* = C_B X_B = 18$		$\Delta_1 = 0$	$\Delta_2 = \frac{3}{2}$	$\Delta_3 = 0$	$\Delta_4 = \frac{9}{2}$

- Since all $\Delta_j \geq 0$, optimal ~~solution~~ basic feasible Solution is obtained.
- Therefore, the solution is $\boxed{\text{Max } z^* = 18}$ at $\boxed{x_1 = 2}$ and $\boxed{x_2 = 0}$

9 (a)

	A	B	C	D	Availability	Iteration - 1
I	21	16	25	13 $\boxed{11}$	11 (0)	3
II	17	18	14	23	13	3
III	33	27	18	41	19	9
Requirements	6	10	12	$\cancel{15}$ (4)	$\boxed{43}$	
Iteration-1	4	2	4	10		

min(11, 15) = 11, Cut I - Row.

	A	B	C	D	Availability	Iteration - 2
II	17	18	14	23 $\boxed{4}$	$\cancel{13}$ (9)	3
III	33	27	18	41	19	9
Requirements	6	10	12	$\cancel{4}$ (0)		
Iteration-2	16	9	4	18		

min (13, 4) = 4, Cut D - Column

	A	B	C	Availability	Iteration - 3
II	17 $\boxed{6}$	18	14	$\cancel{9}$ $\cancel{6}$ (3)	3
III	33	27	18	19	9
Requirements	$\cancel{6}$ (0)	10	12		
Iteration-3	16	9	4		

min (13, 6) = 6, Cut A - Column

	B	C		Iteration 4
II	18	14	3	4
III	27	8 [12]	19 (7)	9
	10	12 (0)		
Iteration-4	9	4		

There is a tie in selecting the largest difference of both the rows and columns, choose the '9' where the maximum quantity can be assigned considering minimum cost as well.

min (19, 12) = 12, cut C - column

	B	
II	18 [3]	3 (0)
III	27 [7]	7 (0)
	10	

$\therefore$ Total cost = $(13\times11)+(23\times4)+(17\times6)+(18\times12)+(18\times3)+(27\times7)$

$= \underline{\underline{1,012}}$

Probleme on Hungarian Method

9 (b)

(1) Solve the assignment problem

		Machines			
		M_1	M_2	M_3	M_4
Jobs	J1	2	3	4	5
	J2	4	5	6	7
	J3	7	8	9	8
	J4	3	5	8	4

Assign the jobs to different machines so as to minimize the total cost.

Sol:- The given problem is balanced with 4 job and 4 machines.

$$A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 4 & 5 & 6 & 7 \\ 7 & 8 & 9 & 8 \\ 3 & 5 & 8 & 4 \end{bmatrix} \begin{matrix} (-2) \\ (-4) \\ (-7) \\ (-3) \end{matrix}$$

→ ~~subtract~~ ~~the~~

→ choose the least element in each row and subtract it from all the elements of the row

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 2 & 5 & 1 \end{bmatrix}$$
$$\begin{matrix} (-0) & (-1) & (-2) & (-1) \end{matrix}$$

→ choose the least element in each column and subtract it from all the elements of that column. Step 2 has to be performed from the table obtained in step 1.

$$A = \begin{bmatrix} \boxed{0} & \cancel{0} & \cancel{0} & 2 \\ \cancel{0} & \cancel{0} & \boxed{0} & 2 \\ \cancel{0} & \boxed{0} & \cancel{0} & \cancel{0} \\ \cancel{0} & 1 & 3 & \boxed{0} \end{bmatrix} \begin{matrix} \leftarrow \\ \leftarrow \\ \leftarrow \\ \leftarrow \end{matrix}$$

Total cost $= 2 + 6 + 8 + 4$

$= \underline{\underline{20}}$

10 a

Sol:- Σa_i (Availability) $= 250 + 300 + 400 = 950$

Σb_j (Requirements) $= 200 + 225 + 275 + 250 = 950$

As $\Sigma a_i = \Sigma b_j$. The transportation problem is balanced.

	A	B	C	D	Availability	Iteration-I
I	11	13 $\boxed{225}$	17	14	~~250~~ (25)	2
II	16	18	14	10	300	4
III	21	24	13	10	400	3
Requirements	200	~~225~~ (0)	275	250		
Iteration-I	5	5	1	4		

min (250, 225) = 225, Cut B- Column

	A	C	D		Iteration-II
I	11 $\boxed{25}$	17	14	~~25~~ (0)	3
II	16	14	10	300	4
III	21	13	10	400	3
	~~200~~ 175	275	250		
Iteration-II	5	1	4		

min (25, 200) = 25, Cut I - Row

	A	C	D		Iteration-III
II	16 $\boxed{175}$	14	10	~~300~~ 125	4
III	21	13	10	400	3
	~~175~~ (0)	275	250		
Iteration-III	(5)	1	0		

min(300, 175) = 175, cut A-column.

	C	D		Iteration-IV
II	14	10 $\boxed{125}$	125 (0)	(4)
III	13	10	400	3
	275	~~250~~ (125)		
Iteration-IV	1	0		

min(125, 250) = 125, cut II - Row

	C	D	
III	13 $\boxed{275}$	10 $\boxed{125}$	400
	~~275~~ (0)	~~125~~ (0)	

Total cost = (13×225) + (11×25) + (16×175) + (10×125) + (13×275) + (10×125)

= 12,075

10 (b)

Sol: The given problem is balanced with 5 men and 5 jobs

→ Choose the least element in each row and Subtract it from all the elements of the row.

$$A = \begin{bmatrix} 2 & 9 & 2 & 7 & 1 \\ 6 & 8 & 7 & 6 & 1 \\ 4 & 6 & 5 & 3 & 1 \\ 4 & 2 & 7 & 3 & 1 \\ 5 & 3 & 9 & 5 & 1 \end{bmatrix} \begin{matrix} (-1) \\ (-1) \\ (-1) \\ (-1) \\ (-1) \end{matrix} \Rightarrow A = \begin{bmatrix} 1 & 8 & 1 & 6 & 0 \\ 5 & 7 & 6 & 5 & 0 \\ 3 & 5 & 4 & 2 & 0 \\ 3 & 1 & 6 & 2 & 0 \\ 4 & 2 & 8 & 4 & 0 \end{bmatrix}$$

→ Choose the least element in each column and subtract it from all the elements of that column. Step 2 has to be performed from the matrix obtained in Step 1

$$A = \begin{array}{c} \begin{bmatrix} 1 & 8 & 1 & 6 & 0 \\ 5 & 7 & 6 & 5 & 0 \\ 3 & 5 & 4 & 2 & 0 \\ 3 & 1 & 6 & 2 & 0 \\ 4 & 2 & 8 & 4 & 0 \end{bmatrix} \\ (-1) \quad (-1) \quad (-1) \quad (-2) \quad (-0) \end{array} \Rightarrow A = \begin{bmatrix} \boxed{0} & 7 & \cancel{0} & 4 & \cancel{0} \\ 4 & 6 & 5 & 3 & \boxed{0} \\ 2 & 4 & 3 & \boxed{0} & \cancel{0} \\ 2 & \boxed{0} & 5 & \cancel{0} & \cancel{0} \\ 3 & 1 & 7 & 2 & \cancel{0} \end{bmatrix} \begin{matrix} \\ \leftarrow \\ \leftarrow \\ \leftarrow \\ \\ \end{matrix}$$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \uparrow$

: N = 4 (No of horizontal / vertical lines)

n = 5 (order of the matrix)

- Here, $N<n$, then follow the below mentioned procedure

→ Determine the smallest uncovered element 'x'.

(a) write uncovered value = uncovered value $- x$

(b) Intersection value = Intersection value $+ x$

(c) Line value (other value) as same.

- The smallest element from the last matrix is 2 i.e., $x = 2$.

- Modified matrix

$$A = \begin{bmatrix} \cancel{0} & 9 & \boxed{0} & 6 & 2 \\ 2 & 6 & 3 & 3 & \boxed{0} \\ \cancel{0} & 4 & 1 & \boxed{0} & \cancel{0} \\ \cancel{0} & \boxed{0} & 3 & \cancel{0} & \cancel{0} \\ 1 & 1 & 5 & 2 & \cancel{0} \end{bmatrix} \leftarrow$$

Now, $N = 4$, $n = 5$, $N < n$ and $x = 1$

∴ modified matrix

$$A = \begin{bmatrix} \cancel{0} & 9 & \boxed{0} & 6 & 3 \\ 1 & 5 & 2 & 2 & \boxed{0} \\ \cancel{0} & 4 & 1 & \boxed{0} & 1 \\ \cancel{0} & \boxed{0} & 3 & \cancel{0} & 1 \\ \boxed{0} & \cancel{0} & 4 & 1 & \cancel{0} \end{bmatrix}$$

← ∴ Total time taken

$= 2+1+3+2+5$

$=$ <u>13 hours</u>