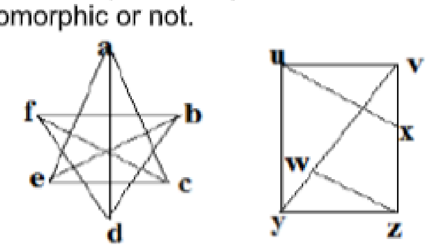
QUESTION PAPER

CMR INSTITUTE OF TECHNOLOGY		USN							CMRIT
		Internal Assessment Test I June 2024							
Sub:	**Graph Theory**						Code:	BCS405B	
Date:	09/09/2023	Duration:	90 mins	Max Marks:	50	Sem:	IV	Branch:	**CSE/IS**

Question 1 is compulsory and Answer any 6 from the remaining questions.		Marks	OBE CO	OBE RBT
1	Prove that a connected graph G is an Euler Graph if and only if all the vertices of G are of Even degree	[8]	CO1	L1,L3
2	Define the following 1. Simple graph 2. Complete graph 3. Bipartite graph 4. Complement of a graph	[7]	CO1	L3
3	Prove that two simple graphs G_1 and G_2 are isomorphic if and only if G^c_1 and G^c_2 are isomorphic.	[7]	CO1	L3
4	State and Prove Handshaking Theorem and Show that the maximum number of edges in a simple graph with n vertices is n(n – l)/2.	[7]	CO2	L3
5	Define a complement of a simple graph .let G be a simple graph of order n . If the size of G is 56 And size of is 80 what is n?	[7]	CO2	L3
6	Show that two graphs G_1 and G_2 are isomorphic if and only if G_1^c and G_2^c isomorphic	and	CO2	L1,L3
7	Find the complement of the complete bipartite graph $K_{3,3}$	[7]	CO2	L3
8	Define Graph isomorphism. Determine whether the following graphs are isomorphic or not. 	[7]	CO2	L1,L3

ANSWER KEY

1.

Definitions: If there is a circuit in a connected graph G that contains all the edges of G then that circuit is called an Euler circuit or an Eulerian line or an Eulerian tour in G.

A given connected graph G is an Euler graph if and only if all vertices of G are of even degree.

Proof: Suppose that G is an Euler graph. It therefore contains an Euler line (which is a closed walk). In tracing this walk we observe that every time the walk meets a vertex v it goes through two "new" edges incident on v—with one we "entered" v and with the other "exited." This is true not only of all intermediate vertices of the walk but also of the terminal vertex, because we "exited" and "entered" the same vertex at the beginning and end of the walk, respectively. Thus if G is an Euler graph, the degree of every vertex is even.

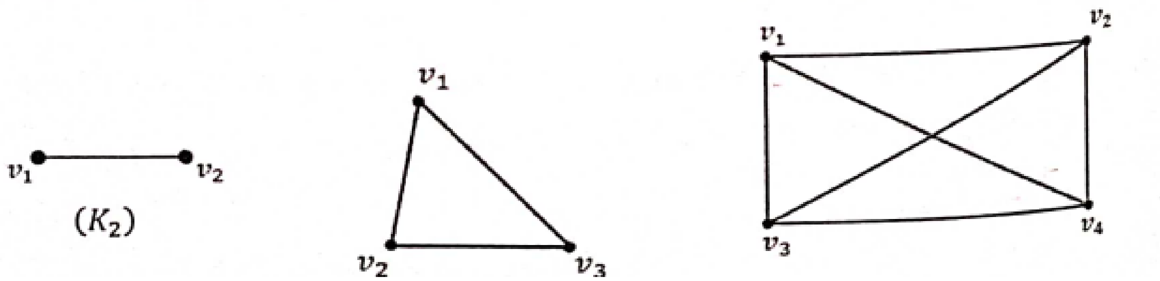
To prove the sufficiency of the condition, assume that all vertices of G are of even degree. Now we construct a walk starting at an arbitrary vertex v and going through the edges of G such that no edge is traced more than once. We continue tracing as far as possible. Since every vertex is of even degree, we can exit from every vertex we enter; the tracing cannot stop at any vertex but v. And since v is also of even degree, we shall eventually reach v when the tracing comes to an end. If this closed walk h we just traced includes all the edges of G, G is an Euler graph. If not, we remove from G all the edges in h and obtain a subgraph h' of G formed by the remaining edges. Since both G and h have all their vertices of even degree, the degrees of the vertices of h' are also even. Moreover, h' must touch h at least at one vertex a, because G is connected. Starting from a, we can again construct a new walk in graph h'. Since all the vertices of h' are of even degree, this walk in h' must terminate at vertex a; but this walk in h' can be combined with h to form a new walk, which starts and ends at vertex v and has more edges than h. This process can be repeated until we obtain a closed walk that traverses all the edges of G. Thus G is an Euler graph. ■

2.

Simple graph : A graph without any loops and parallel edges is called a *Simple graph.*

Complete graph : A simple graph of order $n \geq 2$ is said to be a ***complete graph*** if every vertex posses edges with all the other vertices and is denoted by K_n.

Examples: Complete graphs with vertices 2, 3, 4 respectively denoted by K_2, K_3, K_4



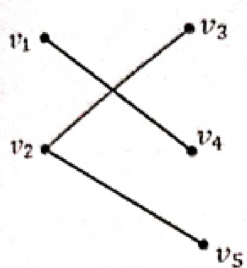
Bipartite and Complete Bipartite graph

A simple graph $G = (V, E)$ with vertex partition $V = \{V_1, V_2\}$ is called a ***Bipartite graph*** if every edge of E joins a vertex in V_1 to a vertex in V_2.

Here $V_1 \subseteq V$, $V_2 \subseteq V$, $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \phi$.

Example : $V = \{v_1, v_2, v_3, v_4, v_5\}$; $V_1 = \{v_1, v_2\}, V_2 = \{v_3, v_4, v_5\}$

G :

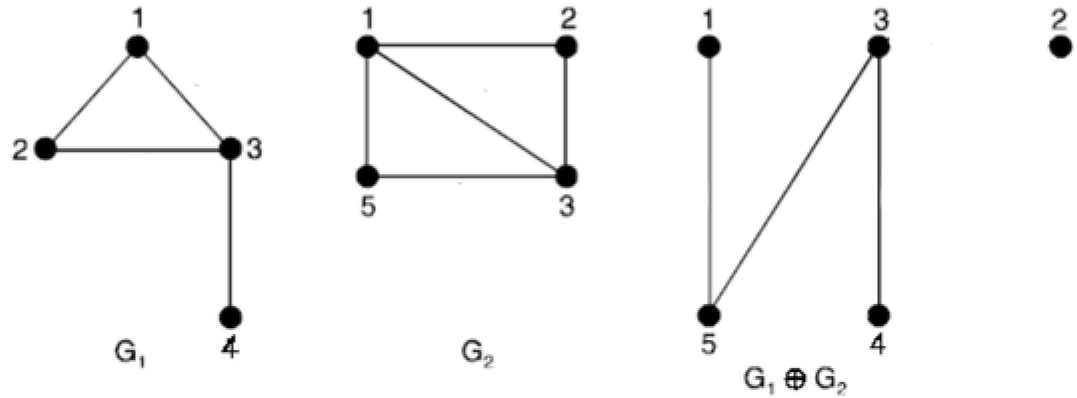


Every edge in G joins a vertex in V_1 and a vertex in V_2.

<u>Hamiltionian graph</u>**: A graph that contains a Hamiltonian cycle is called a Hamiltonian graph. A cycle that contains all the vertices of a graph is called a Hamiltonian cycle.**

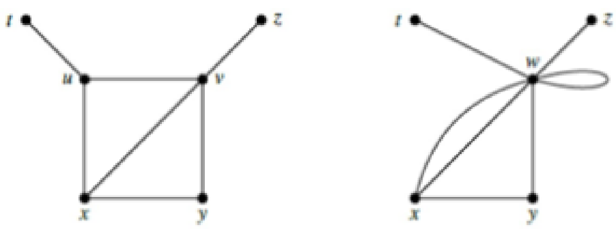
3.

Definition: The ringsum of two graphs G_1 and G_2 is another graph G, written by $G = G_1 \oplus G_2$, with vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \oplus E(G_2) = E(G_1 \cup E(G_2)) - E(G_1 \cap E(G_2))$.



Definition: (Fusion of Vertices) A pair of vertices u and v are said to be fused (or merged) together if the two vertices are together replaced by a single vertex w such that every edge incident with either u or v is incident with the new vertex w.

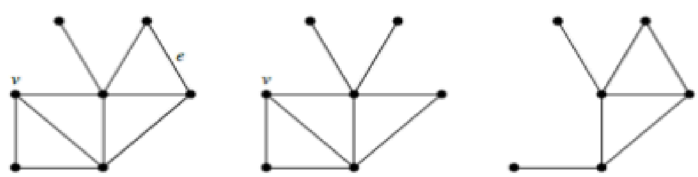
Note that the fusion of two vertices does not change the number of edges, but reduces the number of vertices by 1.



Definition: (Edge Deletion in Graphs) If e is an edge of G, then $G - e$ is the graph obtained by removing the edge e of G. The subgraph of G thus obtained is called an edge-deleted subgraph of G. Clearly, $G - e$ is a spanning subgraph of G.

Definition: (Vertex Deletion in Graphs) If v is a vertex of G, then $G - v$ is the graph obtained by removing the vertex v and all edges G that are incident on v. The subgraph of G thus obtained is called a vertex-deleted subgraph of G. Clearly, $G - v$ will not be a spanning subgraph of G.

The following figure illustrates the edge deletion and the vertex deletion of a graph G.



(a) G (b) $G - e$ (c) $G - v$

4.

- **Sum of degree of vertices property or Hand shaking property.**

If $G = (V, E)$ is a nondirected graph with vertices $v_1, v_2, \ldots v_n$ then

$$\sum_{i=1}^{n} \deg(v_i) = 2|E| \qquad \ldots\ldots\ldots\ldots(1)$$

Further if $G = (V, E)$ is a directed graph then

$$\sum_{i=1}^{n} \deg^{+}(v_i) = |E| = \sum_{i=1}^{n} \deg^{-}(v_i) \qquad \ldots\ldots\ldots\ldots(2)$$

edge counted twice.

The truth of (1) is evident as we have, $e = v_i v_j = v_j v_i$ in an undirected graph. Every edge is counted twice, once at each end. Accordingly the property is also called *hand shaking property,* since two hands are essential in a hand shake.

Ex – 13. *Prove the following for the graph $G = (V, E)$:*

(i) $\sum_{v \in V} \deg(v) = 2|E|$

(ii) *The number of vertices of odd degree must be even.* **[Dec 2010, Jan 2014,17]**

➤ (i) Every edge (a, b) is counted twice one at each end and hence prefixing 2 for $\sum \deg(v)$ is evident. Also $\sum \deg(v) = 2|E|$

Thus we have,

$$\sum_{v \in V} \deg(v) = 2|E|$$

(ii) Let $V = \{v_1, v_2, v_3, \ldots v_n\}$ and $V_1 \subseteq V$, $V_2 \subseteq V$ where

$V_1 = \{v_1, v_2, v_3, \ldots v_r\}$ be odd degree vertices set.

$V_2 = \{v_{r+1}, v_{r+2}, \ldots v_n\}$ be even degree vertices set.

We have, $\sum_{i=1}^{n} \deg(v_i) = 2|E| = 2m$ (say)

or $\sum_{i=1}^{r} \deg(v_i) + \sum_{i=r+1}^{n} \deg(v_i) = 2m$

$$\Rightarrow \sum_{i=1}^{r} deg(v_i) = 2m - \sum_{i=r+1}^{n} \deg(v_i)$$

But $v_{r+1}, v_{r+2}, \ldots v_n$ are even degree vertices and hence their sum of degrees is also even say $2m_1$.

$$\therefore \sum_{i=1}^{r} \deg(v_i) = 2m - 2m_1 = 2(m - m_1) = 2N \text{ (say)} = \text{Even qty.}$$

This implies that the sum of degrees of all the odd degree vertices is even.

Hence the number of odd degree vertices is even.

Given that G is a complete graph on n vertices. The degree of every vertex in G is n-1. By handshaking lemma the sum of degrees of vertices in G will be equal to twice the number of edges in G. Hence, n(n-1)=2(no. of edges in G). Therefore, no. of edges in G= n(n-1)/2.

5.

Complement of a graph. nvertices → no loop

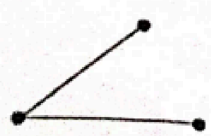
Let G be an undirected graph having n vertices without any loop. The complement of G denoted by $\overline{G}$ is the subgraph of K_n consisting of the n vertices in G and all the edges that are not in G.

If $G = K_n$ then $\overline{G}$ is a graph consisting of n vertices and no edges.

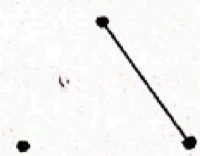
The same is called a *null graph*.

Examples

(1) G:



$\overline{G}$:



Ex – 12. *Let G be a simple graph of order n. If the size of G is 56 and size of $\overline{G}$ is 80, what is n?* **[July 2014, Jan 2015, 17]**

➤ By data, $|E(G)| = 56, \ |E(\overline{G})| = 80$

We have $|E(G)| + |E(\overline{G})| = |E(k_n)|$

Hence, $56 + 80 = \dfrac{n(n-1)}{2}$ or $n(n-1) = 272 \Rightarrow n = 17$

Thus the required, $\boldsymbol{n = 17}$.

6.

Let graph G be isomorphic to H, and let $\overline{G}$, $\overline{H}$ denote their complements.

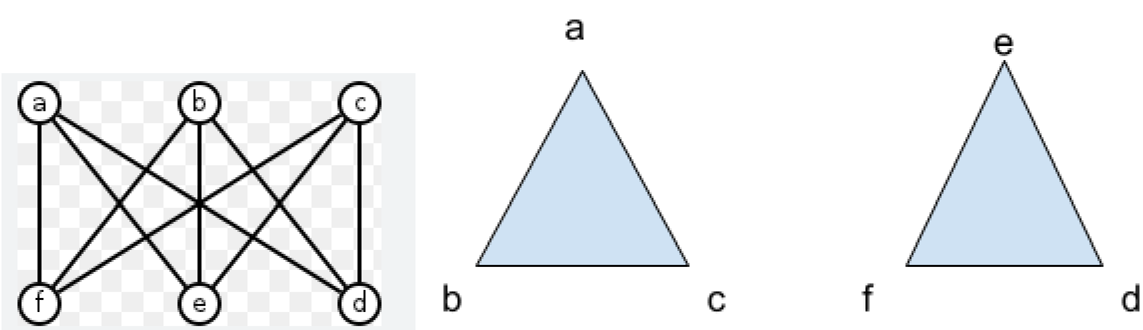
Since G is isomorphic to H, then there exists a bijection $f : V(G) \to V(H)$, such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. -> [this should be edge set]

Equivalently, there exists a bijection $f : V(G) \to V(H)$, such that $uv \notin E(G)$ if and only if $f(u)f(v) \notin E(H)$. -> [this should be edge set]

Since the vertex set of G and $\overline{G}$ are the same, therefore f is a bijection from $V(\overline{G})$ to $V(\overline{H})$. Then suppose $uv \notin E(G)$, by definition of a complement, $uv \in E(\overline{G})$. Likewise, if $f(u)f(v) \notin E(H)$, then $f(u)f(v) \in E(\overline{H})$.

Hence $\overline{G}$ and $\overline{H}$ are isomorphic.

7.



8.

Two graphs G_1 and G_2 are said to be ***isomorphic*** to each other if the following conditions are satisfied.

(i) There exists a one to one correspondence between the vertices of G_1 and G_2.

(ii) There exists a one to one correspondence between the edges of G_1 and the edges of G_2.

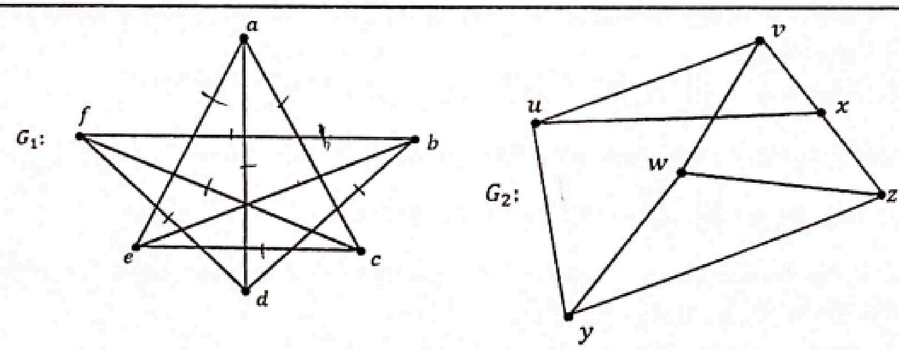
If $e_1 = (v_i, v_j)$ is an edge in G_1, then the corresponding edge $e_2 = (v_i', v_j')$ be such that $v_i'v_j'$ in G_2 corresponds to $v_i v_j$ in G_1.

Thus we can say that two graphs G_1 and G_2 are ***isomorphic*** to each other if there exists a one to one correspondence between their vertices and also between their edges such that adjacency of vertices is preserved.

The notation $G_1 \cong G_2$ symbolizes that G_1 is isomorphic to G_2. This means that G_1 and G_2 will have the same structure with only a difference in their geometric representation. Further we can also infer the following if $G_1 \cong G_2$.

- G_1 and G_2 will have the same number of vertices and the same number of edges. Symbolically we write, $|V(G_1)| = |V(G_2)|$ and $|E(G_1)| = |E(G_2)|$.

Ex – 19. **[Dec 2008]**



➢ The one to one correspondence between the vertices is as follows.

$$a \leftrightarrow u, b \leftrightarrow w, c \leftrightarrow x, d \leftrightarrow y, e \leftrightarrow v, f \leftrightarrow z$$

In this correspondence edges determined by the corresponding vertices correspond such that adjacency of vertices is maintained.

Thus G_1 is isomorphic to G_2 or $G_1 \cong G_2$