CMR INSTITUTE OF TECHNOLOGY

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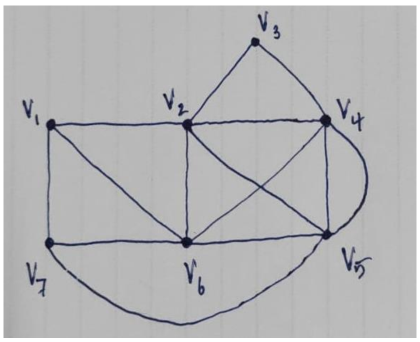
Internal Assessment Test II July 2024

Sub:	**Graph Theory**							Code:	BCS405B
Date:	08/07/2024	Duration:	90 mins	Max Marks:	50	Sem:	IV	Branch:	**CSE/IS**

	Question 1 is compulsory and Answer any 6 from the remaining questions.	Marks	OBE	
			CO	RBT
1	(a) Prove that a connected planar graph with 'n' vertices and 'e' edges has (e-n+2) regions. (b) Find the number of tree branches and chords in the following graph with 7 vertices and 14 edges.	[8]	CO4	L2,L3



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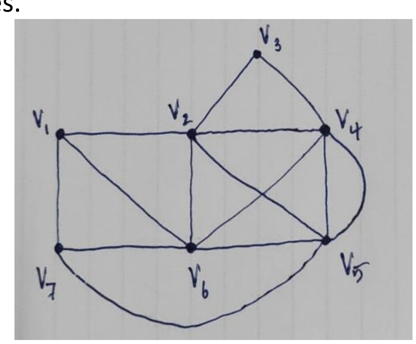

2.	Define the following (a) Binary tree (b) Spanning tree (c) Vertex connectivity (d) Fundamental circuits (e) Fundamental cut-sets.	[7]	CO3	L1
3	Prove that K_5 and $K_{3,3}$ are non-planar graphs.	[7]	CO4	L3
4	(a) Prove that every tree has one or two centres. (b) Show that the number of vertices in a binary tree is always odd.	[7]	CO3	L3
5	Show that for any graph G, $k(G) \leq \lambda(G) \leq \delta(G)$, where $k(G), \lambda(G),\ \delta(G)\ denote$ vertex connectivity, edge connectivity and minimum degree of a graph G respectively.	[7]	CO3	L3
6	If G is a simple planar graph with at least 3 vertices, then show that (a) $e \leq 3n - 6$, and (b) $e \leq 2n - 4$, for triangle -free graphs.	[7]	CO4	L3
7	Prove the following: (a) A tree with ‘n’ vertices has (n-1) edges. (b) There is one and only one path between every pair of vertices in a tree.	[7]	CO3	L3
8	(a) Prove that every tree is a planar graph. (b) Show that a complete graph K_n is not a tree, when $n > 2$..	[7]	CO4	L3

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1.

Euler's formula.

Theorem: If G is a connected plane graph with p-vertices, q-edges and r-regions, then $p-q+r=2$.

Proof: We apply the induction on q (number of edges), if G is a trivial graph then result is obvious ($\because p=1, q=0, r=1 \Rightarrow p-q+r=1-0+1=2$).

Assume that the result is true is all connected graphs with number of edges less than q (where $q>0$).

Suppose G is a graph with q-edges, then we have two cases.

Case 1 : G is a tree

In this case $p = q+1$ and $r = 1$.

$\therefore p - q + r = q+1-q+1 = 2$

$\therefore p - q + r = 2.$

Case 2 :- G is not a tree

In this case, G contains cycle (one or more)

Let e be an edge of a cycle.

Now G-e is a connected graph with p-vertices.

$(q-1)$ edges and $(r-1)$ regions.

By induction hypothesis, we have

$p-(q-1)+(r-1) = 2$

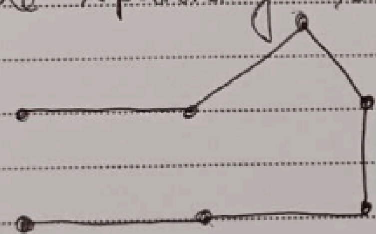
$\therefore p - q + r = 2.$

The formula holds for (p,q)-graph is r-regions.

Soln

We know that the connected graph with n vertices and e edges has $e-n+1$ ~~ch~~ chords and $n-1$ tree branches

The spanning tree is



No of vertices = 7; No of edges = 14

So No of chords = $14-7+1 = 8$

No of Tree branches = $7-1 = 6$

2. a)

A binary tree can also be defined as a tree in which there is exactly one vertex of degree 2 and each of the remaining vertices is of degree 1 or degree 3.

The vertex with degree 2 in a binary tree is a root.

$\Rightarrow$ Every binary tree is a rooted tree.

b) A spanning tree T of an undirected graph G is a subgraph that is a tree which includes all of the vertices of G.

c)

Vertex-connectivity

Let G a connected graph. The minimum number of vertices whose removal results in a disconnected or trivial graph is called the *vertex-connectivity* (or *point-connectivity*) of G.

d) Fundamental cut set or f-cut set is the minimum number of branches that are removed from a graph in such a way that the original graph will become two isolated subgraphs.

e) A circuit formed by adding a chord of G to a spanning tree T such that it creates only one circuit is called a fundamental circuit.

3.

2. The graph K_5 and $K_{3,3}$ are non-planar

G is planar.

$q \leq 3p-6$

$q > 3p-6$

G is non-planar.

$G : K_5$.

$p = 5,\ q = 10 \Rightarrow 3p-6 = 3(5)-6 = 15-6 = 9$.

$\Rightarrow 10 > 9$

$\therefore$ G is non-planar.

$G : K_{3,3}$.

Here, $p = 6,\ q = 9$.

$3p-6 = 3(6)-6 = 18-6 = 12$

$K_{3,3}$ is a bipartite graph which have no triangle. if such graph is to be planar, then $q \leq 2p-4$.

$\therefore q \geq 2p-4$.

$\Rightarrow$ G is non-planar.

4. a)

a_5 is also a centre

Every tree has either one or two centers.

Proof: The max. distance, denoted by $d(v, v_i)$ from a vertex v to a vertex v_i occurs only when v_i is a pendant vertex.

Let T be a tree with more than two vertices

When pendant vertices are deleted, we get •—• . This tree, denoted by T' is also a tree

Further Eccentricity $E(v)$ in T' is just one less than $E(v)$ in T for every v in T'.

continue the process of deleting pendant vertices from T' to get another tree T''.

T'' is still a tree with same centers.

In this process all vertices that T had as centres will still remain to be centres in T', in T'', T''' so on.

Continue the process ~~still~~ until the given tree reduces to a tree T^* that has either 1 vertex or 1 edge (2 vertices)

If the tree T^* has 1 vertex, it implies T has one center.

If T^* has 1 edge (2 vertices), it implies that T has two centres.

If T^* has 1 edge (2 vertices), it implies that T has two centres.

This proves the Theorem.

b) The number of vertices n in a binary tree is always odd. This is because there is exactly one vertex of even degree, and the remaining n − 1 vertices are of odd degrees. Since from Theorem 1-1 the number of vertices of odd degrees are even, n − 1 is even. Hence n is odd.

5.

Proof. First, we prove $\lambda(G) \leq \delta(G)$, if G has no edges then $\lambda(G) = \delta(G) = 0$. Otherwise, removal of all edges incident with a vertex of minimum degree result in a disconnected graph. Hence $\lambda(G) \leq \delta(G)$.

Now we prove $\kappa(G) \leq \lambda(G)$. We consider the following cases:

Case (i) : If G is disconnected or trivial then $\kappa(G) = \lambda(G) = 0$

Case (ii) : G is connected graph with a bridge x then $\lambda(G) = 1$ becomes 1. Further in this case $G = K_2$ are one of the vertex incident with x is a cut vertex. Hence $\kappa(G) = 1$, thus $\kappa(G) = \lambda(G) = 1$.

Case (iii) : Let $\lambda(G) > 2$ then there exist $\lambda(G)$. There removal its disconnect the graph. Hence the removal of $\lambda(G) - 1$ of each edges result in G with a bridge $x = uv$. For each of the $\lambda(G) - 1$ edges select and incident vertex different from u and v. The removal of the $\lambda(G) - 1$ vertices remove all the $\lambda(G) - 1$ edges. If the result in graph is disconnected. Then $\kappa(G) \leq \lambda(G) - 1$.

If not x is not a bridge of this subgraph and hence the removal of u of v reuslt in a disconnected or trivial graph. Hence $\kappa(G) \leq \lambda(G)$. □

6. a)

Proof: Let G be a connected plane graph with $p \geq q$ vertices then we have $\leq q$

$1 \leq p - q$

Case 1: Let G be a tree $r=1$, $q=p-1$, $p\geq 3$.

Hence $q \geq \frac{3r}{2}$ and $q \leq 3p-6$

Since $p-1 \leq 3p-6$ as $p\geq 3$.

Case 2: Let G be a cycle (Let G be a not tree) and f_i ($i=1,2,\ldots,r$) be the faces of G.

Since each edges lies on boundary of atmost two faces.

i.e., $2q \geq \sum_{i=1}^{r}$ (Number of edges in the boundary of face f_i).

i.e, $2q \geq 3r$ (Since each faces is bounded by atleast 3 edges)

$q \geq \frac{3r}{2} \rightarrow ①$

By Euler formula, $p-q+r=2 \rightarrow ②$

$r = 2-p+q$

Substituting ② in ①

We have $q \geq \frac{3}{2}(2-p+q)$

$\Rightarrow q \leq 3p-6$

b)

Proof:) Let G be a planar connected graph without triangles following cases arise:

Case 1: If G is a tree, then $q = p-1$

$\therefore p \geq 3 \Rightarrow p \geq 4-1$

$p-4 \geq 1$

$2p-4 \geq p-1$

ie, $q \leq 2p-4$.

Case 2: If G is not tree, Since G has no triangle it follows that the boundary of each region $f_1, f_2, \dots, f_r$ as atleast four edges.

$\therefore 4 \leq$ number of edges in faces $f_i \; \forall i = 1, 2, \dots r$.

ie, $\sum_{i=1}^{r} 4 \leq \sum_{i=1}^{r}$ (Number of edges in each faces f_i) $= 2q$

$\therefore \sum_{i=1}^{r} 4 \leq 2q$ or $4r \leq 2q \longrightarrow$ ①

From Euler formula, we have $p-q+r=2$.

$r = 2+q-p \longrightarrow$ ②

② in ①

$$4(2+q-p) \leq 2q$$

$$8+4q-4p \leq 2q$$

$$q \leq 2p-4.$$

7. a)

Proof: We prove the theorem by induction on n.

The theorem is obvious for $n = 1, n = 2$ and $n = 3$; see the trees in Figure 3.1.

Assume that the theorem holds for all trees with n vertices where $n \leq k$, for a specified positive integer k.

Consider a tree T with $k+1$ vertices. In T, let e be an edge with end vertices u and v. Since T is a tree, it has no cycles and therefore there exists no other edge or path between u and v. Hence, deletion of e from T will disconnect the graph and $T - e$ consists of exactly two components, say T_1 and T_2. Since T does not contain any cycle, the components T_1 and T_2 too do not contain any cycles. Hence, T_1 and T_2 are trees in their own right. Both of these trees have less than $k + 1$ vertices each, and therefore, according to the assumption made, the theorem holds for these trees; that is, each of T_1 and T_2 contains one less edge than the number of vertices in it. Therefore, since the total number of vertices in T_1 and T_2 (taken together) is $k+1$, the total number of edges in T_1 and T_2 (taken together) is $(k + 1) - 2 = k-1$. But T_1 and T_2 taken together is $T - e$. Thus, $T - e$ contains $k - 1$ edges. Consequently, T has exactly k edges.

Thus, if the theorem is true for a tree with $n \leq k$ vertices, it is true for a tree with $n = k + 1$ vertices. Hence, by induction, the theorem is true for all positive integers n. •

b)

Theorem 2 *If in a graph G there is one and only one path between every pair of vertices, then G is a tree.*

Proof: Since there is a path between every pair of vertices in G, it is obvious that G is connected. Since there is only one path between every pair of vertices, G cannot have a cycle. Because, if there is a cycle, then there exist two paths between two vertices on the cycle. Thus, G is a connected graph containing no cycles. This means that G is a tree. •

8. a)

Let T be a tree. When T is a connected graph that does not contain any circuits.

Since T does not contain any circuit and since □5 contains circuits, T does not contain a subgraph that is homeomorphicto □5.

Let us assume, for the sake of contradiction, that T contains a subgraph G that is homeomorphicto □3,3.

This would then imply that there are distinct verticesV1,V2,V3,V4,V5,V6 in T such that there is a unique path from every vertex in {V1,V2,V3} to every vertex in {V4,V5,V6} where none of these paths have edges in common.

Let P be the unique path from V1to V4, let Q be the unique path from V4 to V2, let R be the unique path from V2 to V5 and let S be the unique path from V5 to V1.

We assume that none of the path's P, Q, R, and S have any edges in common.

However, the paths P, Q, R, S then form a circuit within T.

Since a tree T does not contain any circuit, we have then found a contradiction.

So, our assumption "T contains a subgraph G that is homeomorphicto K3,3" is incorrect. That is, T does not contain a subgraph G that is homeomorphic to K3,3.

We obtained that T does not contain a subgraph homeomorphic to K5 nor K3,3.

Conclusion: By Kuratowski's theorem, T is planar.

Hence proved.

b) If v_1, v_2, v_3 are any three vertices of K_n , where $n > 2$, then the closed walk $v_1 v_2 v_3 v_1$ is a cycle in K_n. Since K_n has a cycle, it cannot be a tree.