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Internal Assesment Test-II									
Sub:	Electromagnetic Theory							Code:	BEC401
Date:	08/07/2024	Duration:	90 mins	Max Marks:	50	Sem:	4th	Branch:	ECE(A,B,C,D)
Answer any **FIVE FULL** Questions									

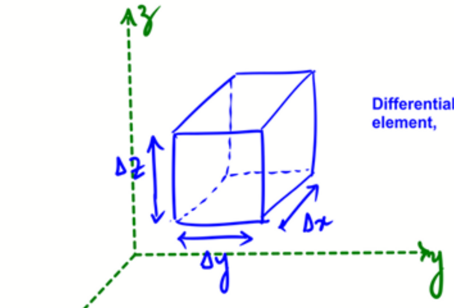
		Marks	OBE CO	OBE RBT
1.	Derive Maxwell's first equation of electrostatics. Also obtain the expression for Gauss's Divergence theorem.	[10]	CO2	L2
2.(a)	Derive an expression for the work done in moving a point charge Q in the presence of an electric field **E**.	[04]	CO3	L2
2.(b)	Define electric potential. Prove that electric field intensity is negative potential gradient for electrostatics.	[06]	CO3	L2
3.(a)	Define current and current density. Derive the expression for equation of continuity of current.	[06]	CO3	L2
3.(b)	Calculate volume charge density at the point $P(3, -45°, 5)$, given $\boldsymbol{D} = 5z^2\, \boldsymbol{a_\rho} + 10\rho z\, \boldsymbol{a_z}\, \frac{C}{m^2}$.	[04]	CO2	L3
4.	A cube is defined by $1 < x, y, z < 1.2$. If $\boldsymbol{D} = 2x^2 y \boldsymbol{a_x} + 3x^2\, y^2 \boldsymbol{a_y} \frac{C}{m^2}$. Evaluate both sides of divergence theorem.	[10]	CO2	L3
5.	Using Laplace's equation, derive an expression for capacitance of a coaxial cylindrical capacitor.	[10]	CO3	L3
6.(a)	Starting from the Gauss's law deduce Poisson's and Laplace's equations. Write the equations in Cartesian, Cylindrical and Spherical coordinate systems.	[07]	CO3	L2
6.(b)	Given potential function $V = x^2yz + Ay^3z$. Find A so that Laplace's equation is satisfied.	[03]	CO3	L3
7.(a)	State and prove the Uniqueness theorem.	[07]	CO3	L2
7.(b)	Determine whether or not the given potential field satisfy the Laplace equation: $V = \rho^2 + z^2$.	[03]	CO3	L3

CI CCI HoD

Application of Gauss's law to differential volume element:
Derive Maxwell's First Equation of electrostatics from Gauss's law.

1. Derive Maxwell's first equation of electrostatics. Also obtain the expression for Gauss's Divergence theorem. [10] CO2 L2

Gauss's Law: $\oint_S \vec{D}\cdot\vec{ds} = Q_{enclosed}$

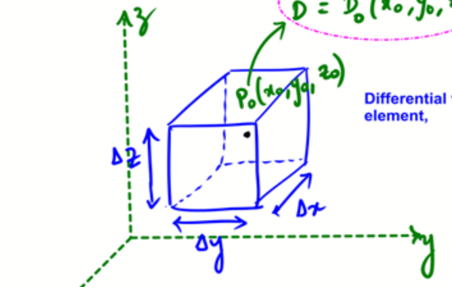


Differential volume element, $\Delta V = \Delta x\,\Delta y\,\Delta z$

Volume charge density: $\rho_v = \lim_{\Delta V \to 0} \frac{Q}{\Delta V}$ C/m^3

$Q_{enc} = \rho_v \Delta V = \rho_v \Delta x\,\Delta y\,\Delta z$

$\vec{D} = \vec{D}_0(x_0, y_0, z_0)$ is known at the point P_0 at the center.



Differential volume element, $\Delta V = \Delta x\,\Delta y\,\Delta z$

Gauss's Law: $\oint_S \vec{D}\cdot\vec{ds} = Q_{enclosed}$

$$\oint_S \vec{D}\cdot\vec{ds} = \iint_{\substack{\text{Front}\\\text{Surface}}} \vec{D}\cdot\vec{ds} + \iint_{\substack{\text{Back}\\\text{Surface}}} \vec{D}\cdot\vec{ds} + \iint_{\substack{\text{Left}\\\text{Surface}}} \vec{D}\cdot\vec{ds} + \iint_{\substack{\text{Right}\\\text{Surface}}} \vec{D}\cdot\vec{ds} + \iint_{\substack{\text{Top}\\\text{Surface}}} \vec{D}\cdot\vec{ds} + \iint_{\substack{\text{Bottom}\\\text{Surface}}} \vec{D}\cdot\vec{ds}$$

D is constant on each of the surfaces

$$\oint_S \vec{D}\cdot\vec{ds} = D_{front}\iint_{\substack{\text{Front}\\\text{Surface}}} \vec{ds} + D_{back}\iint_{\substack{\text{Back}\\\text{Surface}}} \vec{ds} + D_{left}\iint_{\substack{\text{Left}\\\text{Surface}}} \vec{ds} + D_{right}\iint_{\substack{\text{Right}\\\text{Surface}}} \vec{ds} + D_{top}\iint_{\substack{\text{Top}\\\text{Surface}}} \vec{ds} + D_{bottom}\iint_{\substack{\text{Bottom}\\\text{Surface}}} \vec{ds}$$

If f(x=x0) is known, we can find f(x) using Taylor Series Expansion, as given by the expression:

Taylor Series (One Dimensional): At $x = x_0$, $f(x = x_0)$ is known

$$f(x) = f(x_0) + \frac{(x-x_0)}{1!}f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \frac{(x-x_0)^3}{3!}f'''(x_0) + \text{higher order terms} \ldots$$

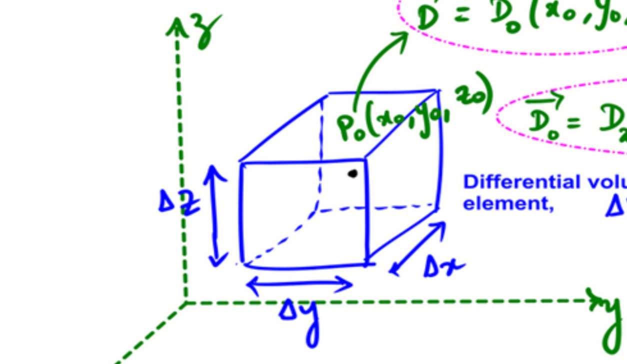
If f(x0,y0,z0) is known, we can find f(x,y,z) using Taylor Series Expansion, as given by the expression:

Taylor Series (Three Dimensional):

$$f(x,y,z) = f(x_0,y_0,z_0) + \frac{(x-x_0)}{1!}\frac{\partial f}{\partial x} + \frac{(y-y_0)}{1!}\frac{\partial f}{\partial y} + \frac{(z-z_0)}{1!}\frac{\partial f}{\partial z} + \text{higher order terms}$$

$\vec{D} = \vec{D}_0(x_0, y_0, z_0)$ is known at the point P_0 at the center.

$\vec{D}_0 = D_{x_0}\hat{a}_x + D_{y_0}\hat{a}_y + D_{z_0}\hat{a}_z$



Differential volume element, $\Delta V = \Delta x\,\Delta y\,\Delta z$

The front face is at a distance of Δx/2 from P0

$\therefore (x - x_0) = \frac{\Delta x}{2}$

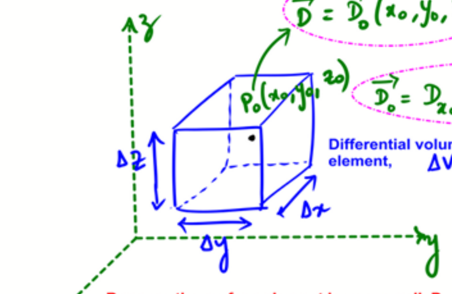
Because the surface element is very small, D is essentially constant (over this portion of the entire closed surface) and we have only to approximate the value of Dx at this front face

$$D_{front} = D_{x_0} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x} + \text{higher order terms}$$

negligible

Taylor Series (Three Dimensional):

$$f(x,y,z) = f(x_0,y_0,z_0) + \frac{(x-x_0)}{1!}\frac{\partial f}{\partial x} + \frac{(y-y_0)}{1!}\frac{\partial f}{\partial y} + \frac{(z-z_0)}{1!}\frac{\partial f}{\partial z} + \text{higher order terms}$$



$\vec{D} = \vec{D}_0(x_0, y_0, z_0)$ is known at the point P_0 at the center.

$\vec{D}_0 = D_{x_0}\hat{a}_x + D_{y_0}\hat{a}_y + D_{z_0}\hat{a}_z$

Differential volume element, $\Delta V = \Delta x \, \Delta y \, \Delta z$

The back face is at a distance of $-\Delta x/2$ from P0

$(x - x_0) = -\frac{\Delta x}{2}$

Because the surface element is very small, D is essentially constant (over this portion of the entire closed surface) and we have only to approximate the value of Dx at this back face

$$\boxed{D_{back} = D_{x_0} - \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}}$$

Taylor Series (Three Dimensional):

$$f(x,y,z) = f(x_0,y_0,z_0) + \frac{(x-x_0)}{1!}\frac{\partial f}{\partial x} + \frac{(y-y_0)}{1!}\frac{\partial f}{\partial y} + \frac{(z-z_0)}{1!}\frac{\partial f}{\partial z} + \underbrace{\text{higher order terms}}_{\text{negligible}}$$

If we combine these two integrals, we have

$$D_{front}\iint_{\substack{\text{Front}\\\text{Surface}}} d\vec{S} + D_{back}\iint_{\substack{\text{Back}\\\text{Surface}}} d\vec{S} = \left[D_{x_0} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}\right](\Delta y \Delta z) + \left[D_{x_0} - \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}\right](-\Delta y \Delta z)$$

$$\boxed{D_{front}\iint_{\substack{\text{Front}\\\text{Surface}}} d\vec{S} + D_{back}\iint_{\substack{\text{Back}\\\text{Surface}}} d\vec{S} = \frac{\partial D_x}{\partial x}\Delta x \, \Delta y \, \Delta z}$$

By exactly the same process we find that,

$$D_{left}\iint_{\substack{\text{Left}\\\text{Surface}}} d\vec{S} + D_{right}\iint_{\substack{\text{Right}\\\text{Surface}}} d\vec{S} = \left[D_{y_0} + \frac{\Delta y}{2}\frac{\partial D_y}{\partial y}\right](\Delta x \Delta z) + \left[D_{y_0} - \frac{\Delta y}{2}\frac{\partial D_y}{\partial y}\right](-\Delta x \Delta z)$$

$$\boxed{D_{left}\iint_{\substack{\text{Left}\\\text{Surface}}} d\vec{S} + D_{right}\iint_{\substack{\text{Right}\\\text{Surface}}} d\vec{S} = \frac{\partial D_y}{\partial y}\Delta x \, \Delta y \, \Delta z}$$

By exactly the same process we find that,

$$D_{top}\iint_{\substack{\text{Top}\\\text{Surface}}} d\vec{S} + D_{bottom}\iint_{\substack{\text{Bottom}\\\text{Surface}}} d\vec{S} = \left[D_{z_0} + \frac{\Delta z}{2}\frac{\partial D_z}{\partial z}\right](\Delta x \Delta y) + \left[D_{z_0} - \frac{\Delta z}{2}\frac{\partial D_z}{\partial z}\right](-\Delta x \Delta y)$$

$$\boxed{D_{top}\iint_{\substack{\text{Top}\\\text{Surface}}} d\vec{S} + D_{bottom}\iint_{\substack{\text{Bottom}\\\text{Surface}}} d\vec{S} = \frac{\partial D_z}{\partial z}\Delta x \, \Delta y \, \Delta z}$$

$$\boxed{\iint_{\substack{\text{Front}\\\text{Surface}}} \vec{D}\cdot d\vec{S} + \iint_{\substack{\text{Back}\\\text{Surface}}} \vec{D}\cdot d\vec{S} = D_{front}\iint_{\substack{\text{Front}\\\text{Surface}}} d\vec{S} + D_{back}\iint_{\substack{\text{Back}\\\text{Surface}}} d\vec{S} = \frac{\partial D_x}{\partial x}\Delta x \, \Delta y \, \Delta z}$$

$$\boxed{\iint_{\substack{\text{Left}\\\text{Surface}}} \vec{D}\cdot d\vec{S} + \iint_{\substack{\text{Right}\\\text{Surface}}} \vec{D}\cdot d\vec{S} = D_{left}\iint_{\substack{\text{Left}\\\text{Surface}}} d\vec{S} + D_{right}\iint_{\substack{\text{Right}\\\text{Surface}}} d\vec{S} = \frac{\partial D_y}{\partial y}\Delta x \, \Delta y \, \Delta z}$$

$$\boxed{\iint_{\substack{\text{Top}\\\text{Surface}}} \vec{D}\cdot d\vec{S} + \iint_{\substack{\text{Bottom}\\\text{Surface}}} \vec{D}\cdot d\vec{S} = D_{top}\iint_{\substack{\text{Top}\\\text{Surface}}} d\vec{S} + D_{bottom}\iint_{\substack{\text{Bottom}\\\text{Surface}}} d\vec{S} = \frac{\partial D_z}{\partial z}\Delta x \, \Delta y \, \Delta z}$$

$$\oint_S \vec{D}\cdot d\vec{S} = \iint_{\substack{\text{Front}\\\text{Surface}}} \vec{D}\cdot d\vec{S} + \iint_{\substack{\text{Back}\\\text{Surface}}} \vec{D}\cdot d\vec{S} + \iint_{\substack{\text{Left}\\\text{Surface}}} \vec{D}\cdot d\vec{S} + \iint_{\substack{\text{Right}\\\text{Surface}}} \vec{D}\cdot d\vec{S} + \iint_{\substack{\text{Top}\\\text{Surface}}} \vec{D}\cdot d\vec{S} + \iint_{\substack{\text{Bottom}\\\text{Surface}}} \vec{D}\cdot d\vec{S}$$

$$\boxed{\oint_S \vec{D}\cdot d\vec{S} = \left[\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right]\Delta x \, \Delta y \, \Delta z}$$

Gauss's Law: $\oint\!\!\oint_S \vec{D}\cdot d\vec{S} = Q_{enclosed}$

$$\oint\!\!\oint_S \vec{D}\cdot d\vec{S} = \left[\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right]\Delta x\,\Delta y\,\Delta z$$

$$Q_{enc} = \rho_v\,\Delta V = \rho_v\,\Delta x\,\Delta y\,\Delta z$$

Equating the two sides of Gauss's Law:

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho_v$$

$$\text{div}\,\vec{D} = \vec{\nabla}\cdot\vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho_v$$

$$\text{div}\,\vec{D} = \vec{\nabla}\cdot\vec{D} = \rho_v$$

Maxwell's First Equation of Electrostatics

This equation is also called Point (Differential) form of Gauss's law

The divergence of the vector flux density A is the outflow of flux from a small closed surface per unit volume as the volume shrinks to zero.

$$\text{Divergence of } \mathbf{A} = \text{div}\,\mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A}\cdot d\mathbf{S}}{\Delta v}$$

$$\text{div}\,\mathbf{D} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) \quad \text{(rectangular)}$$

$$\text{div}\,\mathbf{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z} \quad \text{(cylindrical)}$$

$$\text{div}\,\mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial\phi} \quad \text{(spherical)}$$

Gauss's Law: $\oint\!\!\oint_S \vec{D}\cdot d\vec{S} = Q_{enclosed} = \iiint_V \rho_v\, dv$

Integral Form of Gauss's Law

Gauss's Law: $\text{div}\,\vec{D} = \vec{\nabla}\cdot\vec{D} = \rho_v$

Maxwell's First Equation of Electrostatics

This equation is also called Point (Differential) form of Gauss's law

Divergence Theorem:

This theorem applies to any vector field for which the appropriate partial derivatives exist.

This theorem can be derived from Gauss's law.

Gauss's Law: $\oint\!\!\oint_S \vec{D}\cdot d\vec{S} = Q_{enclosed}$

$$\oint\!\!\oint_S \vec{D}\cdot d\vec{S} = Q_{enclosed}$$

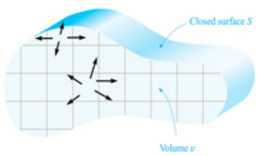
Maxwell's 1st Equation of electrostatics: $\rho_v = \vec{\nabla}\cdot\vec{D}$

$$\oint\!\!\oint_S \vec{D}\cdot d\vec{S} = Q_{enclosed} = \iiint_V \rho_v\, dv = \iiint_V \vec{\nabla}\cdot\vec{D}\, dv$$

$$\oint\!\!\oint_S \vec{D}\cdot d\vec{S} = \iiint_V \vec{\nabla}\cdot\vec{D}\, dv$$

Gauss's Divergence Theorem

The divergence theorem states that the total flux crossing the closed surface is equal to the integral of the divergence of the flux density throughout the enclosed volume.

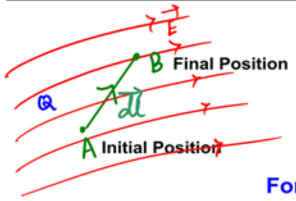


$$\oint\!\!\oint_S \vec{D}\cdot d\vec{S} = \iiint_V \vec{\nabla}\cdot\vec{D}\, dv$$

Gauss's Divergence Theorem

2.(a) Derive an expression for the work done in moving a point charge Q in the presence of an electric field **E**. [04] CO3 L2

Energy expended in moving a point charge in an electric field - (Work Done)



If we attempt to move the test charge against the electric field, we have to exert a force equal and opposite to that exerted by the field, and this requires us to expend energy or do work.

Work Done, $dW = \vec{F}\cdot\vec{dl}$ J

$$W = \int_{Initial}^{Final} \vec{F}\cdot\vec{dl}$$

Force on the point charge in the electric field,

$$\vec{F} = Q\vec{E}$$

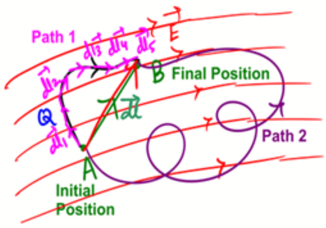
Force applied to move the charge against electric field,

$$\vec{F}_{applied} = -Q\vec{E}$$

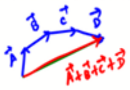
Work Done, $dW = -Q\vec{E}\cdot\vec{dl}$

$$\boxed{W = -\int_{Initial}^{Final} Q\vec{E}\cdot\vec{dl} = -Q\int_A^B \vec{E}\cdot\vec{dl}} \text{ J}$$

The line integral :



Addition of vectors



$$dW = -Q\vec{E}\cdot\vec{dl}$$

$$dW = -Q\vec{E}\cdot\vec{dl_1} - Q\vec{E}\cdot\vec{dl_2} - Q\vec{E}\cdot\vec{dl_3} - Q\vec{E}\cdot\vec{dl_4} - Q\vec{E}\cdot\vec{dl_5}$$

$$dW = -Q\vec{E}\cdot\underbrace{(\vec{dl_1}+\vec{dl_2}+\vec{dl_3}+\vec{dl_4}+\vec{dl_5})}_{\vec{dl}}$$

$$dW = -Q\vec{E}\cdot\vec{dl}$$

$$\boxed{W = -Q\int_A^B \vec{E}\cdot\vec{dl}}$$

*) Work done remains same irrespective of the path chosen in moving the charge from A to B

*) Work done around a closed path is zero

$$\boxed{W = -Q\oint_L \vec{E}\cdot\vec{dl} = 0}$$

*) In general, vectors whose line integral does not depend on the path of integration are called conservative. Thus, E is conservative field.

2.(b) Define electric potential. Prove that electric field intensity is negative potential gradient for electrostatics. [06] CO3 L2

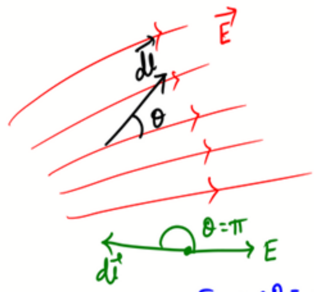
POTENTIAL GRADIENT

Show that electric field is the negative gradient of potential for electrostatics
(or) Derive the relation between electric field intensity and electric potential

$$\boxed{\vec{E} = -\,\text{grad}\,V = -\vec{\nabla}V}$$

Potential:

$$V = -\int_L \vec{E}\cdot\vec{dl}$$



$$dV = -\vec{E}\cdot\vec{dl}$$

$$dV = -|\vec{E}||\vec{dl}|\cos\theta$$

$$\frac{dV}{dl} = -E\cos\theta$$

If $\theta = \pi$, $\cos\theta = -1$

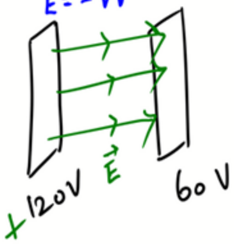
$$\left.\frac{dV}{dl}\right|_{max} = E$$

$$E_{max} = -\frac{dV}{dl}$$

In vector representation, $\vec{E} = -\frac{dV}{dl}\hat{a}_L$ Electric field is in the direction of decreasing potential.

$$\boxed{\vec{E} = -\,\text{grad}\,V = -\vec{\nabla}V}$$ Electric field is negative potential gradient

For example, $\vec{E} = -\vec{\nabla}V$



Electric field is in the direction of decreasing potential.

In Rectangular system, $V(x,y,z)$

$$\vec{E} = -\left[\frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z\right] = -\vec{\nabla}V = -\text{grad}\,V$$

In Cylindrical system, $V(\rho,\phi,z)$

$$\vec{E} = -\left[\frac{\partial V}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{a}_\phi + \frac{\partial V}{\partial z}\hat{a}_z\right] = -\vec{\nabla}V = -\text{grad}\,V$$

In Spherical system, $V(r,\theta,\phi)$

$$\vec{E} = -\left[\frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{a}_\phi\right] = -\vec{\nabla}V = -\text{grad}\,V$$

3.(a) Define current and current density. Derive the expression for equation of continuity of current. [06] CO3 L2

Current and Current Density

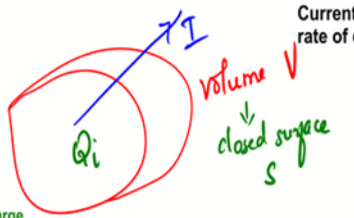
The current is defined as a rate of movement of charge passing a given reference point (or crossing a given reference plane) of one Coulomb per second.
Current is symbolized by I.

$$\boxed{I = \frac{dQ}{dt}} \; A$$

The current density, measured in Amperes per square meter, is a vector flux density represented by J. It is defined as the current per unit cross sectional area.

$$\boxed{|\vec{J}| = \frac{dI}{ds}} \; A/m^2 \qquad dI = \vec{J}\cdot\vec{ds} \qquad \boxed{I = \iint_S \vec{J}\cdot\vec{ds}}$$

Continuity Equation of current (or) Equation of continuity of current



Current flowing out of the given volume V is equal to the rate of decrease in charge within the volume

$$I = -\frac{dQ_i}{dt}$$

Charge within the volume V,

$$Q_i = \iiint_V \rho_v \, dv$$

$$I = -\frac{d}{dt}\left[\iiint_V \rho_v \, dv\right]$$

Volume charge density, $\rho_v(x,y,z,t)$

$$I = -\iiint_V \frac{\partial \rho_v}{\partial t} dv$$

Current Density, $|\vec{J}| = \frac{dI}{ds}$

$$I = \oint_S \vec{J}\cdot\vec{ds}$$

Equating the above two equaitions,

$$\oint_S \vec{J}\cdot\vec{ds} = -\iiint_V \frac{\partial \rho_v}{\partial t} dv$$

Using Gauss's Divergence theorem:

$$\oint_S \vec{J}\cdot\vec{ds} = \iiint_V \vec{\nabla}\cdot\vec{J} \, dv$$

$$\iiint_V \vec{\nabla}\cdot\vec{J} \, dv = -\iiint_V \frac{\partial \rho_v}{\partial t} dv$$

$$\boxed{\vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho_v}{\partial t}}$$

Equation of continuity of current

3.(b) Calculate volume charge density at the point $P(3, -45°, 5)$, given $\boldsymbol{D} = 5z^2\,\boldsymbol{a}_\rho + 10\rho z\,\boldsymbol{a}_z \frac{C}{m^2}$. [04] CO2 L3

b) $\mathbf{D} = 5z^2\mathbf{a}_\rho + 10\rho z\,\mathbf{a}_z$ at $P(3, -45°, 5)$: In cylindrical coordinates, we have

$$\nabla\cdot\mathbf{D} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial\phi} + \frac{\partial D_z}{\partial z} = \left[\frac{5z^2}{\rho} + 10\rho\right]_{(3,-45°,5)} = \underline{71.67}$$

4. A cube is defined by $1 < x, y, z < 1.2$. If $\boldsymbol{D} = 2x^2y\boldsymbol{a}_x + 3x^2y^2\boldsymbol{a}_y \frac{C}{m^2}$. Evaluate both sides of divergence theorem. [10] CO2 L3

Divergence Theorem:

$$\oint_S \vec{D}\cdot\vec{ds} = \iiint_V \vec{\nabla}\cdot\vec{D}\, dv$$

LHS

$$\Phi = Q = \oint \mathbf{D}\cdot\mathbf{n}\,da = \underbrace{\int_1^{1.2}\int_1^{1.2} 2(1.2)^2 y\,dy\,dz}_{\text{front}} + \underbrace{\int_1^{1.2}\int_1^{1.2} -2(1)^2 y\,dy\,dz}_{\text{back}}$$

$$+ \underbrace{\int_1^{1.2}\int_1^{1.2} -3x^2(1)^2\,dx\,dz}_{\text{left}} + \underbrace{\int_1^{1.2}\int_1^{1.2} 3x^2(1.2)^2\,dx\,dz}_{\text{right}} = \underline{0.1028\,\mathrm{C}}$$

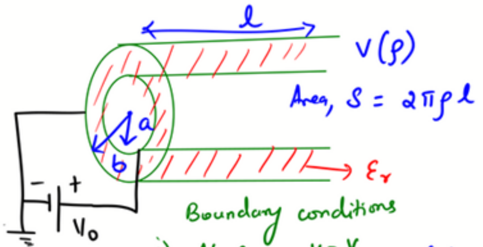
RHS

$$\nabla\cdot\mathbf{D} = \left[4xy + 6x^2y\right]$$

$$Q \doteq \nabla\cdot\mathbf{D}\big|_{\text{center}} \times \Delta v = 12.83 \times (0.2)^3 = \underline{0.1026}$$

5. Using Laplace's equation, derive an expression for capacitance of a coaxial cylindrical capacitor. [10] CO3 L3

2) Coaxial Cylindrical Capacitor:



Boundary conditions

i) At $\rho = a$, $V = V_0$ $(b > a)$

ii) At $\rho = b$, $V = 0$

Laplace's Equation:

$$\nabla^2 V = 0$$

$$\boxed{\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2} = 0}$$

Since Potential V is a function of only one dimension 'ρ', $V(\rho)$

$$\boxed{\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \cancel{\frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2}}^{0} + \cancel{\frac{\partial^2 V}{\partial z^2}}^{0} = 0}$$

$$\nabla^2 V = \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dV}{d\rho}\right) = 0$$

$$\frac{d}{d\rho}\left(\rho\frac{dV}{d\rho}\right) = 0$$

Integrating w.r.t. $\rho \Rightarrow \int \frac{d}{d\rho}\left(\rho\frac{dV}{d\rho}\right)d\rho = \int 0\, d\rho$

$$\rho\frac{dV}{d\rho} = 0 + C_1$$

$$\frac{dV}{d\rho} = \frac{C_1}{\rho}$$

$$\frac{dV}{d\rho} = \frac{c_1}{\rho}$$

Integrating w.r.t. ρ $\Rightarrow$ $\int \frac{dV}{d\rho}\, d\rho = \int \frac{c_1}{\rho}\, d\rho$

$$\boxed{V = c_1 \ln\rho + c_2}$$

Applying Boundary Conditions to get the constants C1 and C2,

i) At $\rho = a$, $V = V_0$ $\Rightarrow$ $V_0 = c_1 \ln a + c_2$ $\quad (b > a)$

ii) At $\rho = b$, $V = 0$ $\Rightarrow$ $0 = c_1 \ln b + c_2$

Solving the above two equations, we get constants C1 and C2,

$$V_0 - 0 = c_1 \ln a - c_1 \ln b$$

$$-V_0 = c_1 \ln(b/a)$$

$$\boxed{c_1 = \frac{-V_0}{\ln(b/a)}}$$

$$c_2 = -c_1 \ln b$$

$$\boxed{c_2 = \frac{V_0 \ln b}{\ln(b/a)}}$$

1) $$\boxed{V(\rho) = \frac{-V_0}{\ln(b/a)} \ln\rho + \frac{V_0}{\ln(b/a)} \ln b} \ \text{V}$$

2) $$\vec{E} = -\vec{\nabla} V = -\left[\frac{\partial V}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\cancel{\frac{\partial V}{\partial \phi}}\hat{a}_\phi + \cancel{\frac{\partial V}{\partial z}}\hat{a}_z\right]$$

$$= -\left[\frac{-V_0}{\ln(b/a)}\frac{1}{\rho}\hat{a}_\rho\right]$$

$$\boxed{\vec{E} = \frac{V_0}{\rho \ln(b/a)}\hat{a}_\rho} \ \text{V/m}$$

3) $$\boxed{\vec{D} = \varepsilon_0 \varepsilon_r \vec{E} = \frac{\varepsilon_0 \varepsilon_r V_0}{\rho \ln(b/a)}\hat{a}_\rho} \ \text{C/m}^2$$

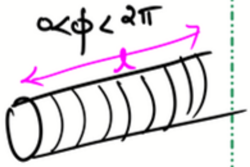
4) $$Q = \oint\!\!\oint_s \vec{D}\cdot\vec{ds} = \iint_s \frac{\varepsilon_0\varepsilon_r V_0}{\rho \ln(b/a)}\hat{a}_\rho \cdot \vec{ds}$$

$$Q = \frac{\varepsilon_0\varepsilon_r V_0}{\rho \ln(b/a)} \cdot 2\pi\rho l$$

$$\boxed{Q = \frac{2\pi\varepsilon_0\varepsilon_r l V_0}{\ln(b/a)}} \ \text{C}$$

$0 < z < l$

$0 < \phi < 2\pi$



Surface area of the cylinder = $\iint_s ds = \iint \rho\, d\phi\, dz = 2\pi\rho l$

6.(a) Starting from the Gauss's law deduce Poisson's and Laplace's equations. Write the equations in Cartesian, Cylindrical and Spherical coordinate systems. [07] CO3 L2

Poisson's and Laplace's Equations:

Starting from Maxwell's first equation of electrostatics (Point form of Gauss's Law):

$$\vec{\nabla}\cdot\vec{D} = \rho_V$$

We can make the substitutions,

$$\boxed{\begin{aligned}\vec{D} &= \varepsilon\vec{E}\\ \vec{E} &= -\vec{\nabla}V\end{aligned}}$$

$$\vec{\nabla}\cdot\varepsilon\vec{E} = \rho_V$$

For a Linear , Isotropic, Homogeneous Medium, $\mathcal{E}$ is constant

$$\varepsilon\,\vec{\nabla}\cdot\vec{E} = \rho_V$$

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho_V}{\varepsilon}$$

$$\vec{\nabla}\cdot(-\vec{\nabla}V) = \frac{\rho_V}{\varepsilon}$$

$$\boxed{\vec{\nabla}\cdot\vec{\nabla}V = -\frac{\rho_V}{\varepsilon}}$$

Divergence of Gradient of potential function 'V' => $\vec{\nabla}\cdot\vec{\nabla} = \nabla^2$ (Laplacian operator)

Poisson's Equation: $\boxed{\nabla^2 V = -\frac{\rho_V}{\varepsilon}}$

Laplace's Equation is a special case of Poisson's Equation, when the region is free of charges. $\rho_V = 0 \longrightarrow \boxed{\nabla^2 V = 0}$ **Laplace's Equation**

Poisson's and Laplace's Equations in rectangular system:

$$\boxed{\begin{aligned}\nabla^2 V &= -\frac{\rho_V}{\varepsilon}\\ \nabla^2 V &= 0\end{aligned}}$$

Divergence of Gradient of potential function 'V' => $\nabla^2 V = \vec{\nabla}\cdot\vec{\nabla}V$

$$\text{grad } V = \vec{\nabla}V = \frac{\partial V}{\partial x}\vec{a_x} + \frac{\partial V}{\partial y}\vec{a_y} + \frac{\partial V}{\partial z}\vec{a_z}$$

$$\text{div } \vec{A} = \vec{\nabla}\cdot\vec{A} = \frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y + \frac{\partial}{\partial z}A_z$$

$$\nabla^2 V = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial z}\right)$$

$$\boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_V}{\varepsilon}} \leftarrow \text{Poisson's Equation}$$

$$\boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0} \leftarrow \text{Laplace's Equation}$$

Poisson's and Laplace's Equations in cylindrical system:

$$\boxed{\begin{aligned}\nabla^2 V &= -\frac{\rho_V}{\varepsilon}\\ \nabla^2 V &= 0\end{aligned}}$$

Divergence of Gradient of potential function 'V' => $\nabla^2 V = \vec{\nabla}\cdot\vec{\nabla}V$

$$\text{grad } V = \frac{\partial V}{\partial \rho}\vec{a_\rho} + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\vec{a_\phi} + \frac{\partial V}{\partial z}\vec{a_z}$$

$$\text{div } \vec{A} = \vec{\nabla}\cdot\vec{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial}{\partial \phi}(A_\phi) + \frac{\partial}{\partial z}(A_z)$$

$$\nabla^2 V = \vec{\nabla}\cdot\vec{\nabla}V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho}\frac{\partial}{\partial \phi}\left(\frac{1}{\rho}\frac{\partial V}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial z}\right)$$

$$\boxed{\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_V}{\varepsilon}} \leftarrow \text{Poisson's Equation}$$

$$\boxed{\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0} \leftarrow \text{Laplace's Equation}$$

Poisson's and Laplace's Equations in spherical system:

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

$$\nabla^2 V = 0$$

Divergence of Gradient of potential function 'V' => $\nabla^2 V = \vec{\nabla} \cdot \vec{\nabla} V$

$$\text{grad } V = \frac{\partial V}{\partial r}\vec{a_r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\vec{a_\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\vec{a_\phi}$$

$$\text{div } \vec{A} = \vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(A_\phi)$$

$$\nabla^2 V = \vec{\nabla}\cdot\vec{\nabla} V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{1}{r}\frac{\partial V}{\partial\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\right)$$

$$\boxed{\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = -\frac{\rho_v}{\varepsilon}}$$ ← Poisson's Equation

$$\boxed{\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0}$$ ← Laplace's Equation

6.(b) Given potential function $V = x^2yz + Ay^3z$. Find A so that Laplace's equation is satisfied. [03] CO3 L3

$$\nabla^2 V = 0$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$\frac{\partial}{\partial x}(2xyz) + \frac{\partial}{\partial y}(x^2z + 3y^2Az) + \frac{\partial}{\partial z}(x^2y + Ay^3) = 0$$

$$2yz + 6yzA = 0$$

$$2yz(1 + 3A) = 0$$

$$3A = -1$$

$$\boxed{A = -1/3}$$

7.(a) State and prove the Uniqueness theorem. [07] CO3 L2

Uniqueness Theorem:

If a solution to Laplace's equation can be found that satisfies the boundary conditions, then the solution is Unique.

The theorem applies to any solution of Poisson's or Laplace's equation in a given region or closed surface.

Proof:

The theorem is proved by contradiction.
We assume that there are two solutions V1 and V2 of Laplace's equation (or Poisson's equation) both of which satisfy the prescribed boundary conditions.

Proof:

If V1 and V2 are two different solutions:

case (i)

Poisson's Equation: $\nabla^2 V = -\frac{\rho_v}{\varepsilon}$

$\nabla^2 V_1 = -\frac{\rho_v}{\varepsilon} \rightarrow (1)$

$\nabla^2 V_2 = -\frac{\rho_v}{\varepsilon} \rightarrow (2)$

$(1)-(2) \Rightarrow \boxed{\nabla^2 V_1 - \nabla^2 V_2 = 0}$

case (ii)

Laplace's Equation: $\nabla^2 V = 0$

$\nabla^2 V_1 = 0 \rightarrow (3)$

$\nabla^2 V_2 = 0 \rightarrow (4)$

$(3)-(4) \Rightarrow \boxed{\nabla^2 V_1 - \nabla^2 V_2 = 0}$

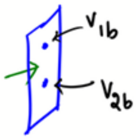
$V_1(x,y,z) - V_2(x,y,z) = V_d(x,y,z)$

Differential operator is distributive and hence we can write the above equation as follows:

$\boxed{\nabla^2 (V_1 - V_2) = 0}$ (or) $\boxed{\nabla^2 V_d = 0} \rightarrow (5)$

Considering the boundary conditions with atleast one conducting boundary,

Conductor surface is equipotential



$V_{1b} = V_{2b}$

$V_{1b} - V_{2b} = 0$

$\boxed{V_{db} = 0} \rightarrow (6)$

Any Scalar $\rightarrow \alpha(x,y,z)$

Any vector $\rightarrow \vec{A}(x,y,z)$

Vector Identity:

$\boxed{\vec{\nabla}\cdot(\alpha \vec{A}) = \alpha\, \vec{\nabla}\cdot\vec{A} + \vec{A}\cdot\vec{\nabla}\alpha}$

In our case,

$\alpha = V_d(x,y,z)$

$\vec{A} = \vec{\nabla} V_d$

$\boxed{\vec{\nabla}\cdot(V_d \vec{\nabla} V_d) = V_d\, \vec{\nabla}\cdot\vec{\nabla} V_d + \vec{\nabla} V_d \cdot \vec{\nabla} V_d}$

$\vec{A}\cdot\vec{A} = |\vec{A}|^2$

Integrating the above expression over the volume 'V' enclosed by the closed surface 'S',

$$\iiint_V \vec{\nabla}\cdot(V_d \vec{\nabla} V_d)\, dv = \iiint_V V_d \underbrace{\nabla^2 V_d}_{0}\, dv + \iiint_V |\vec{\nabla} V_d|^2 dv$$

From Eqn (5)

Using Divergence theorem, $\oint_S \vec{D}\cdot\vec{ds} = \iiint_V \vec{\nabla}\cdot\vec{D}\, dv$

$$\oint_S \underbrace{V_d}_{0} \vec{\nabla} V_d \cdot \vec{ds} = \iiint_V |\vec{\nabla} V_d|^2 dv$$

From eqn (6) as Integration is around the closed surface, which is the boundary itself.

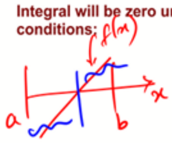
$$0 = \iiint_V |\vec{\nabla} V_d|^2 dv$$

$$0 = \iiint_V |\vec{\nabla} V_d|^2 dv$$

$|\vec{\nabla} V_d|^2 = 0$

$\vec{\nabla} V_d = 0$

Integral will be zero under any one of the below conditions;



$\int_a^b f(x)\, dx = 0$

i) $f(x) = 0$

ii) $f(x)$ is odd function

But the integrand is a square function, which cannot be odd.
Hence Integrand must be zero.

$\vec{\nabla} V_d = 0$ only if $\boxed{V_d = \text{constant}}$

For equipotential surface (i.e. at the boundary),

$V_d = 0$

$V_1 - V_2 = 0$

$\boxed{V_1 = V_2}$ Thus V1 and V2 are not two different solutions. They are both same.
Hence Uniqueness Theorem is proved.

7.(b) Determine whether or not the given potential field satisfy the Laplace equation: [03] CO3 L3
$V = \rho^2 + z^2$.

$V = \rho + z$

Solution
Cylindrical system:
Laplace's equation

$\nabla^2 V = 0$

$$\nabla^2 V = \frac{1}{\rho}\left(\frac{\partial}{\partial \rho}\rho\frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

$$= \frac{1}{\rho}\left(\frac{\partial}{\partial \rho}\left(\rho \cdot \frac{\partial}{\partial \rho}(\rho^2+z^2)\right)\right) + 0 + \frac{\partial^2}{\partial z^2}(\rho^2+z^2)$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho \cdot 2\rho) + \frac{\partial}{\partial z}(2z)$$

$$= \frac{1}{\rho}\cdot(2\times 2\rho) + 2$$

$$\nabla^2 V = 4+2$$

$$\boxed{\nabla^2 V = 6 \neq 0}$$

V does not satisfy Laplace's equation