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Second Semester B.E./B.Tech. Degree Supplementary Examination, June/July 2024

Mathematics – II for EEE Stream

Time: 3 hrs. Max. Marks: 100

Note: 1. Answer any FIVE full questions, choosing ONE full question from each module.
2. VTU Formula Hand Book is permitted.
3. M : Marks , L: Bloom's level , C: Course outcomes.

			M	L	C
		Module – 1			
Q.1	a.	Find the angle between the surfaces $xy^2y = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at the point (1, -2, 1).	7	L2	CO1
	b.	If $\vec{F} = \nabla(x^3 + y^3 + z^3 - 3xyz)$, find div $\vec{F}$ and curl $\vec{F}$.	7	L2	CO1
	c.	Show that the vector $\vec{F} = \dfrac{x\hat{i} + y\hat{j}}{x^2 + y^2}$ is both solenoidal and irrotational.	6	L3	CO1
		OR			
Q.2	a.	Find the total work done by the force $\vec{F} = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$ along the curve $x = t^2 + 1$; $y = 2t^2$, $z = t^3$ from t = 1 to t = 2.	7	L2	CO1
	b.	Using Green's theorem, evaluate $\int_c (xy + y^2)dx + x^2dy$ where 'c' is the closed curve of the region bounded by $y = x$ and $y = x^2$.	7	L3	CO1
	c.	Using modern mathematical tools, write the code to find the find the gradient of $\phi = x^2y + 2xz - 4$.	6	L2	CO5
		Module – 2			
Q.3	a.	Define a Subspace. Show that the intersection of two subspaces of a vector V is also a subspace of V.	7	L2	CO2
	b.	Show that T : $R^3 \rightarrow R^3$ defined by T(x, y, z) = (x, y, –z) is linear transformation.	7	L3	CO2
	c.	If $u = [2, -5, -1]^T$, $V = [-7, -4, 6]^T$, compute : i) $<u, v>$ ii) $\|\|u\|\|^2$ iii) $\|\|v\|\|^2$ iv) $\|\|u + v\|\|^2$.	6	L2	CO2
		OR			
Q.4	a.	Define linearly independent and linearly dependent set of vectors. Test the vectors $v_1 = [3, 0, -6]^T$, $v_2 = [-4, 1, 7]^T$ and $v_3 = [-2, 1, 5]^T$ forms a basis.	7	L2	CO2
	b.	State Rank – Nullity Theorem. For the matrix $A = \begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix}$, Find : i) Rank of A ii) Dim (Nul A) iii) Bases	7	L3	CO2

	c.	Using the modern mathematical tool, write the code to represent the reflection transformation $T : R^2 \to R^2$ and to find the image of vector (10, 0) when it is reflected about y – axis.	6	L2	CO5

Module – 3

Q.5	a.	Find the Laplace Transform of (i) $e^{-3t}\cos 2t$ ii) $\dfrac{\cos at - \cos bt}{t}$.	7	L2	CO3
	b.	Find the Laplace Transform of the square wave function of period Za, defined by $f(t) = \begin{cases} k & 0 < t < a \\ -k & a < t < 2a \end{cases}$	7	L2	CO3
	c.	Explain $f(t) = \begin{cases} \cos t & 0 < t < \pi \\ \cos 2t & \pi < t < 2\pi \\ \cos 3t & t > 2\pi \end{cases}$ in term of the unit step function and hence find L[f(t)].	6	L3	CO3

OR

Q.6	a.	Find the inverse Laplace transformer of i) $\dfrac{2s-1}{s^2+4s+29}$ ii) $\dfrac{1}{(s-4)^2}$.	7	L2	CO3
	b.	Using the convolution theorem, find the inverse Laplace transform of $\dfrac{1}{(s-1)(s^2+1)}$.	7	L3	CO3
	c.	Solve by the Laplace transforms $y'' + k^2y = 0$, given that $y(0) = 2$, $y'(0) = 0$.	6	L2	CO3

Module – 4

Q.7	a.	Find the real root of $x\log_{10} x = 1.2$ by Regula – Falsi method correct to 2 decimal places the root lies between (2, 3).	7	L2	CO4
	b.	Find interpolating polynomical by Newton's divided difference formula for the data f(1) = 4, f(3) = 32, f(4) = 55 and f(6) = 119.	7	L2	CO4
	c.	Evaluate using Simpson's $\frac{1}{3}^{rd}$ rule $\int_0^6 \frac{e^x}{1+x}dx$ by taking six equal parts.	6	L2	CO4

OR

Q.8	a.	Find the real root of the equation $\cos x = xe^x$, using Newton's – Raphson method, correct to 3 decimal places taking $x_0 = 0.5$.	7	L2	CO4
	b.	Use Newton's backward interpolation formula to compute the value of y when x = 6, given that	7	L3	CO4

x	1	2	3	4	5
y	1	–1	1	–1	1

	c.	Evaluate $\int_0^5 \frac{dx}{4x+5}$, by Trapezoidal rule, taking 6 ordinates.	6	L2	CO4
		Module – 5			
Q.9	a.	Employ Taylors series method to find y(0.2), given that $\frac{dy}{dx} = 2y + 3e^x$, y(0) = 0.	7	L3	CO4
	b.	Using Modified Euler's method, find y(0.1) correct to 4 decimal places, given that $y' = x - y^2$, y(0) = 1, h = 0.1, perform 2 iterations.	7	L2	CO4
	c.	Employ Milne's predictor – corrector method given that $y' = x^2(1 + y)$ y(1) = 1, y(1.1) = 1.233, y(1.2) = 1.548, y(1.3) = 1.979 to find y(1.4).	6	L3	CO4
		OR			
Q.10	a.	Solve $y' = \log_{10}(x + y)$, by modified Euler's method at x = 0.2 and x = 0.4 with h = 0.2, perform 2 iterations at each stage.	7	L2	CO4
	b.	Use 4th order Runge – Kutta method to solve (x + y) y′ = 1 with y(0.4) = 1, at x = 0.5 correct to 4 decimal places.	7	L2	CO4
	c.	Using modern mathematical tools, write a code to find y(0.1), given y′ = x – y, y(0) = 1 by Taylors series.	6	L3	CO5

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Solutions

Q1.

(a) To find the angle between the surfaces, we first determine the normal vectors of each surface at the point $(1, -2, 1)$.

For the first surface $xy^2z = 3x + z^2$, let $f(x, y, z) = xy^2z - 3x - z^2$. Then, the normal vector is given by ∇f.

$$\begin{aligned}\nabla f &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \\ &= \left(y^2z - 3, 2xyz, xy^2 - 2z\right)\end{aligned}$$

Substituting $(x, y, z) = (1, -2, 1)$:

$$\nabla f(1, -2, 1) = (-2 - 3, -4, -2 - 2) = (-5, -4, -4)$$

For the second surface $3x^2 - y^2 + 2z = 1$, let $g(x, y, z) = 3x^2 - y^2 + 2z - 1$. Then, the normal vector is given by ∇g.

$$\begin{aligned}\nabla g &= \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right) \\ &= (6x, -2y, 2)\end{aligned}$$

Substituting $(x, y, z) = (1, -2, 1)$:

$$\nabla g(1, -2, 1) = (6, 4, 2)$$

The angle θ between the surfaces is the angle between ∇f and ∇g, given by

$$\cos\theta = \frac{\nabla f \cdot \nabla g}{|\nabla f||\nabla g|}$$

where

$$\nabla f \cdot \nabla g = (-5)(6) + (-4)(4) + (-4)(2) = -30 - 16 - 8 = -54$$

and

$$\begin{aligned}|\nabla f| &= \sqrt{(-5)^2 + (-4)^2 + (-4)^2} = \sqrt{25 + 16 + 16} = \sqrt{57} \\ |\nabla g| &= \sqrt{(6)^2 + (4)^2 + (2)^2} = \sqrt{36 + 16 + 4} = \sqrt{56}\end{aligned}$$

Thus,

$$\cos\theta = \frac{-54}{\sqrt{57}\sqrt{56}}$$

(b) To find $\operatorname{div}(\vec{F})$ and $\operatorname{curl}(\vec{F})$ for $\vec{F} = \nabla(x^3 + y^3 + z^3 - 3xyz)$:

$$f(x, y, z) = x^3 + y^3 + z^3 - 3xyz$$

Then,

$$\vec{F} = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \left(3x^2 - 3yz, 3y^2 - 3xz, 3z^2 - 3xy\right)$$

For the divergence,

$$\begin{aligned}\operatorname{div}(\vec{F}) &= \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3xz) + \frac{\partial}{\partial z}(3z^2 - 3xy) \\ &= 6x + 6y + 6z = 6(x + y + z)\end{aligned}$$

For the curl,

$$\operatorname{curl}(\vec{F}) = \nabla \times \vec{F} = 0$$

since $\vec{F}$ is a gradient field and the curl of a gradient is zero.

(c) For the vector $\vec{F} = \frac{x\vec{i}+y\vec{j}}{x^2+y^2}$:

To check if $\vec{F}$ is solenoidal (divergence-free),

$$\text{div}(\vec{F}) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$$

Using the quotient rule,

$$= \frac{(x^2+y^2) - x \cdot 2x}{(x^2+y^2)^2} + \frac{(x^2+y^2) - y \cdot 2y}{(x^2+y^2)^2} = 0$$

so $\vec{F}$ is solenoidal.

To check if $\vec{F}$ is irrotational (curl-free),

$$\text{curl}(\vec{F}) = \nabla \times \vec{F} = 0$$

since $\vec{F}$ has zero curl in 2D.

Q2.

(a) To find the work done by $\vec{F}$ along the curve, we need to compute the line integral $W = \int_C \vec{F} \cdot d\vec{r}$.

Parameterize the curve with $x = t^2+1$, $y = 2t^2$, $z = t^3$, and let $t \in [1, 2]$.

$\vec{r}(t) = (x(t), y(t), z(t)) = \left(t^2+1\right)\vec{i} + \left(2t^2\right)\vec{j} + \left(t^3\right)\vec{k}$.

$\frac{d\vec{r}}{dt} = \frac{d}{dt}\left(t^2+1, 2t^2, t^3\right) = (2t)\vec{i} + (4t)\vec{j} + (3t^2)\vec{k}$.

Substitute $x = t^2+1$, $y = 2t^2$, and $z = t^3$ into $\vec{F}$:

$$\vec{F}(t) = 3(t^2+1)(2t^2)\,\vec{i} - 5(t^3)\,\vec{j} + 10(t^2+1)\,\vec{k}$$
$$= (6t^4 + 6t^2)\,\vec{i} - 5t^3\,\vec{j} + (10t^2+10)\,\vec{k}$$

$$\vec{F}(t) \cdot \frac{d\vec{r}}{dt} = (6t^4+6t^2)(2t) + (-5t^3)(4t) + (10t^2+10)(3t^2)$$
$$= 12t^5 + 12t^3 - 20t^4 + 30t^4 + 30t^2 = 12t^5 + 10t^4 + 12t^3 + 30t^2$$

Integrate from $t = 1$ to $t = 2$:

$$W = \int_1^2 (12t^5 + 10t^4 + 12t^3 + 30t^2)\,dt$$

Integrate each term separately:

$$W = \left[\frac{12t^6}{6} + \frac{10t^5}{5} + \frac{12t^4}{4} + \frac{30t^3}{3}\right]_1^2$$
$$= \left[2t^6 + 2t^5 + 3t^4 + 10t^3\right]_1^2$$
$$= (2 \cdot 64 + 2 \cdot 32 + 3 \cdot 16 + 10 \cdot 8) - (2 \cdot 1 + 2 \cdot 1 + 3 \cdot 1 + 10 \cdot 1)$$
$$= (128 + 64 + 48 + 80) - (2 + 2 + 3 + 10) = 320 - 17 = 303$$

Thus, the total work done is $W = 303$.

(b) Using Green's theorem to evaluate $\oint_C (xy + y^2)\,dx + x^2\,dy$:

According to Green's theorem,

$$\oint_C M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

where $M = xy + y^2$ and $N = x^2$.

Calculate $\frac{\partial N}{\partial x}$ and $\frac{\partial M}{\partial y}$:

$$\frac{\partial N}{\partial x} = 2x, \quad \frac{\partial M}{\partial y} = x + 2y$$

Then,

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - (x + 2y) = x - 2y$$

The region D is bounded by $y = x$ and $y = x^2$. Set up the double integral:

$$\iint_D (x - 2y)\,dA = \int_0^1 \int_{x^2}^{x} (x - 2y)\,dy\,dx$$

Evaluate the inner integral with respect to y:

$$= \int_0^1 \left[xy - y^2 \right]_{y=x^2}^{y=x} \, dx$$

$$= \int_0^1 \left(x \cdot x - x^2 - (x \cdot x^2 - (x^2)^2) \right) \, dx$$

$$= \int_0^1 \left(x^2 - x^2 - x^3 + x^4 \right) \, dx$$

Simplify and integrate term by term:

$$= \int_0^1 (x^2 - x^3)\,dx = \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1$$

$$= \frac{1}{3} - \frac{1}{4} = \frac{4-3}{12} = \frac{1}{12}$$

Thus, the value of the integral is $\frac{1}{12}$.

(c) Python code to find the gradient of $\Phi = x^2 y + 2xz - 4$:

```
import sympy as sp

# Define variables
x, y, z = sp.symbols('x y z')

# Define the scalar field Phi
Phi = x**2 * y + 2 * x * z - 4

# Calculate the gradient
gradient_Phi = sp.Matrix([sp.diff(Phi, var) for var in (x, y, z)])

# Display the gradient
gradient_Phi
```

Q3.

(a) **Definition of a Subspace:** A subset $W \subseteq V$ of a vector space V is called a subspace of V if W itself forms a vector space under the same operations of addition and scalar multiplication defined in V. For W to be a subspace, it must satisfy: 1. The zero vector of V is in W. 2. W is closed under vector addition: if $\vec{u}, \vec{v} \in W$, then $\vec{u} + \vec{v} \in W$. 3. W is closed under scalar multiplication: if $\vec{u} \in W$ and $c \in \mathbb{R}$, then $c\vec{u} \in W$.

Intersection of Two Subspaces: Let U and W be two subspaces of V. The intersection $U \cap W = \{\vec{v} \in V : \vec{v} \in U \text{ and } \vec{v} \in W\}$ is also a subspace of V.

Proof: To show that $U \cap W$ is a subspace, we check the three conditions: 1. The zero vector is in both U and W, so it is in $U \cap W$. 2. If $\vec{u}, \vec{v} \in U \cap W$, then $\vec{u} + \vec{v} \in U$ (since U is closed under addition) and $\vec{u} + \vec{v} \in W$ (since W is closed under addition), so $\vec{u} + \vec{v} \in U \cap W$. 3. If $c \in \mathbb{R}$ and $\vec{u} \in U \cap W$, then $c\vec{u} \in U$ (since U is closed under scalar multiplication) and $c\vec{u} \in W$ (since W is closed under scalar multiplication), so $c\vec{u} \in U \cap W$.

Thus, $U \cap W$ is a subspace of V.

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x, y, z) = (x, y, -z)$. To show that T is a linear transformation, we need to check that T satisfies the properties of additivity and scalar multiplication.

Additivity: Let (x_1, y_1, z_1) and $(x_2, y_2, z_2) \in \mathbb{R}^3$. Then,

$$T((x_1, y_1, z_1) + (x_2, y_2, z_2)) = T(x_1 + x_2, y_1 + y_2, z_1 + z_2) = (x_1 + x_2, y_1 + y_2, -(z_1 + z_2))$$

and

$$T(x_1, y_1, z_1) + T(x_2, y_2, z_2) = (x_1, y_1, -z_1) + (x_2, y_2, -z_2) = (x_1 + x_2, y_1 + y_2, -(z_1 + z_2))$$

so $T((x_1, y_1, z_1) + (x_2, y_2, z_2)) = T(x_1, y_1, z_1) + T(x_2, y_2, z_2)$.

Scalar Multiplication: Let $c \in \mathbb{R}$ and $(x, y, z) \in \mathbb{R}^3$. Then,

$$T(c \cdot (x, y, z)) = T(cx, cy, cz) = (cx, cy, -cz)$$

and

$$c \cdot T(x, y, z) = c \cdot (x, y, -z) = (cx, cy, -cz)$$

Thus, $T(c \cdot (x, y, z)) = c \cdot T(x, y, z)$.

Since T satisfies both properties, T is a linear transformation.

(c) Given $\vec{u} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -7 \\ -4 \\ 6 \end{bmatrix}$, we compute:

(i) $\langle \vec{u}, \vec{v} \rangle = 2(-7) + (-5)(-4) + (-1)(6) = -14 + 20 - 6 = 0$.

(ii) $\|\vec{u}\|^2 = \langle \vec{u}, \vec{u} \rangle = 2^2 + (-5)^2 + (-1)^2 = 4 + 25 + 1 = 30$.

(iii) $\|\vec{v}\|^2 = \langle \vec{v}, \vec{v} \rangle = (-7)^2 + (-4)^2 + 6^2 = 49 + 16 + 36 = 101$.

(iv) To find $\|\vec{u} + \vec{v}\|^2$, first calculate $\vec{u} + \vec{v}$:

$$\vec{u} + \vec{v} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix} + \begin{bmatrix} -7 \\ -4 \\ 6 \end{bmatrix} = \begin{bmatrix} -5 \\ -9 \\ 5 \end{bmatrix}$$

Then,

$$\|\vec{u} + \vec{v}\|^2 = (-5)^2 + (-9)^2 + 5^2 = 25 + 81 + 25 = 131$$

Q4.

(a) **Definition of Linearly Independent and Linearly Dependent Sets:** A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ in a vector space V is said to be *linearly independent* if the only solution to the equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = 0$$

is $c_1 = c_2 = \cdots = c_n = 0$. If there exists a nontrivial solution (some $c_i \neq 0$), then the vectors are said to be *linearly dependent.*

Testing for Basis: To determine if $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ forms a basis in $\mathbb{R}^3$, we check if these vectors are linearly independent. This can be done by setting up the matrix $B = \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 5 \end{bmatrix}$ and row reducing it to check for pivot columns.

$$\begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 5 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since we have two pivot columns, $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dependent and do not form a basis in $\mathbb{R}^3$.

(b) **Rank-Nullity Theorem:** For a matrix A with dimensions $m \times n$, the Rank-Nullity Theorem states that

$$\text{rank}(A) + \text{nullity}(A) = n$$

where rank(A) is the dimension of the column space of A, and nullity(A) is the dimension of the null space of A.

Given the matrix $A = \begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix}$:

(i) **Rank of** A: Row reduce A to determine the number of pivot columns.

$$\begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -4 & 0 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of A is 2 (two pivot columns).

(ii) **Dimension of Null Space (Nullity of** A**)**: Since A has 4 columns, by the Rank-Nullity Theorem:

$$\text{nullity}(A) = 4 - 2 = 2$$

(iii) **Bases for Column Space and Null Space**: - The basis for the column space can be formed by the pivot columns of the original matrix A, so a basis for the column space of A is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix} \right\}$$

- To find the basis for the null space, we solve $A\vec{x} = 0$ and express the solutions in terms of free variables:

Let $x_3 = t$ and $x_4 = s$ be free variables, then

$$\vec{x} = \begin{bmatrix} 4s+t \\ -2s \\ t \\ s \end{bmatrix} = s \begin{bmatrix} 4 \\ -2 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Therefore, a basis for the null space is

$$\left\{ \begin{bmatrix} 4 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

(c) **Python Code for Reflection Transformation:**

The reflection transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ about the y-axis can be represented by the matrix

$$T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

To find the image of the vector $\vec{v} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$ under T, we can use the following Python code:

```
import numpy as np

# Define the reflection matrix for reflection about the y-axis
T = np.array([[-1, 0],
              [0, 1]])

# Define the vector to be reflected
v = np.array([10, 0])

# Compute the image of v under T
image_v = T @ v

print("The image of the vector (10, 0) under the reflection is:", image_v)
```

The result will give the image of $(10, 0)$ after reflection about the y-axis, which is $(-10, 0)$.

Q5.

(a) (i) The Laplace transform of $f(t) = e^{-3t}\cos(2t)$ is given by

$$\mathcal{L}\{e^{-3t}\cos(2t)\} = \int_0^\infty e^{-st}e^{-3t}\cos(2t)\,dt = \int_0^\infty e^{-(s+3)t}\cos(2t)\,dt$$

Using the Laplace transform property for $e^{at}\cos(bt)$,

$$\mathcal{L}\{e^{at}\cos(bt)\} = \frac{s-a}{(s-a)^2+b^2}$$

with $a = -3$ and $b = 2$, we get

$$\mathcal{L}\{e^{-3t}\cos(2t)\} = \frac{s+3}{(s+3)^2+4}$$

(ii) The Laplace transform of $f(t) = \frac{\cos(at)-\cos(bt)}{t}$ can be computed using the result that

$$\mathcal{L}\left\{\frac{\cos(at)-\cos(bt)}{t}\right\} = \ln\left(\frac{s+a}{s+b}\right)$$

Thus,

$$\mathcal{L}\left\{\frac{\cos(at)-\cos(bt)}{t}\right\} = \ln\left(\frac{s+a}{s+b}\right)$$

(b) The square wave function $f(t)$ with period $2a$ is defined by

$$f(t) = \begin{cases} k & 0 < t < a \\ -k & a < t < 2a \end{cases}$$

We can represent this function using the unit step function as:

$$f(t) = k\left[u(t) - 2u(t-a) + u(t-2a)\right]$$

The Laplace transform of $f(t)$ is then

$$\mathcal{L}\{f(t)\} = k\left[\frac{1}{s} - \frac{2e^{-as}}{s} + \frac{e^{-2as}}{s}\right] = \frac{k}{s}\left(1 - 2e^{-as} + e^{-2as}\right)$$

(c) To express $f(t) = \begin{cases} \cos(t) & 0 < t < \pi \\ \cos(2t) & \pi < t < 2\pi \\ \cos(3t) & t > 2\pi \end{cases}$ in terms of the unit step function, we write:

$$f(t) = \cos(t) + (\cos(2t) - \cos(t))u(t-\pi) + (\cos(3t) - \cos(2t))u(t-2\pi)$$

Thus,

$$f(t) = \cos(t) + (\cos(2t) - \cos(t))u(t-\pi) + (\cos(3t) - \cos(2t))u(t-2\pi)$$

To find $\mathcal{L}[f(t)]$, we use the linearity of the Laplace transform and the shifting property:

$$\mathcal{L}[f(t)] = \mathcal{L}[\cos(t)] + \mathcal{L}[(\cos(2t) - \cos(t))u(t-\pi)] + \mathcal{L}[(\cos(3t) - \cos(2t))u(t-2\pi)]$$

Using the shifting property, we have:

$$\mathcal{L}[\cos(t)] = \frac{s}{s^2+1}$$

$$\mathcal{L}[(\cos(2t) - \cos(t))u(t-\pi)] = e^{-\pi s}\left(\frac{s}{s^2+4} - \frac{s}{s^2+1}\right)$$

$$\mathcal{L}[(\cos(3t) - \cos(2t))u(t-2\pi)] = e^{-2\pi s}\left(\frac{s}{s^2+9} - \frac{s}{s^2+4}\right)$$

Thus,

$$\mathcal{L}[f(t)] = \frac{s}{s^2+1} + e^{-\pi s}\left(\frac{s}{s^2+4} - \frac{s}{s^2+1}\right) + e^{-2\pi s}\left(\frac{s}{s^2+9} - \frac{s}{s^2+4}\right)$$

Q6.

(a) (i) To find the inverse Laplace transform of $\frac{2s-1}{s^2+4s+29}$:

$$\mathcal{L}^{-1}\left\{\frac{2s-1}{s^2+4s+29}\right\}$$

First, we complete the square for the denominator:

$$s^2 + 4s + 29 = (s+2)^2 + 25$$

So we rewrite the fraction as:

$$\frac{2s-1}{(s+2)^2+25}$$

We decompose $2s-1$ to express it in terms of $s+2$:

$$2s-1 = 2(s+2)-5$$

Thus,

$$\frac{2s-1}{(s+2)^2+25} = \frac{2(s+2)}{(s+2)^2+25} - \frac{5}{(s+2)^2+25}$$

Using the Laplace transform properties:

$$\mathcal{L}^{-1}\left\{\frac{2(s+2)}{(s+2)^2+25}\right\} = 2e^{-2t}\cos(5t)$$

and

$$\mathcal{L}^{-1}\left\{\frac{5}{(s+2)^2+25}\right\} = e^{-2t}\sin(5t)$$

Therefore,

$$\mathcal{L}^{-1}\left\{\frac{2s-1}{s^2+4s+29}\right\} = 2e^{-2t}\cos(5t) - e^{-2t}\sin(5t)$$

(ii) To find the inverse Laplace transform of $\frac{1}{(s-4)^2}$:

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-4)^2}\right\}$$

Using the property $\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^2}\right\} = te^{at}$, with $a = 4$, we get:

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-4)^2}\right\} = te^{4t}$$

(b) Using the convolution theorem to find the inverse Laplace transform of $\frac{1}{(s-1)(s^2+1)}$:

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s^2+1)}\right\} = f(t) * g(t)$$

where $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^t$ and $g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin(t)$.

The convolution of $f(t)$ and $g(t)$ is:

$$f(t) * g(t) = \int_0^t e^{\tau}\sin(t-\tau)\,d\tau$$

Using the identity $\int e^{a\tau}\sin(b\tau)\,d\tau = \frac{e^{a\tau}(a\sin(b\tau)-b\cos(b\tau))}{a^2+b^2}$, we get:

$$f(t) * g(t) = \frac{1}{2}\left(e^t - e^{-t}\right) = \sinh(t)$$

Thus,

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s^2+1)}\right\} = \sinh(t)$$

(c) To solve the differential equation $y'' + k^2 y = 0$ with initial conditions $y(0) = 2$ and $y'(0) = 0$ using Laplace transforms:

Take the Laplace transform of both sides:

$$\mathcal{L}\{y''\} + k^2\mathcal{L}\{y\} = 0$$

Using $\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0)$ and $\mathcal{L}\{y\} = Y(s)$, we get:

$$s^2Y(s) - 2s + k^2Y(s) = 0$$

Rearranging terms:

$$(s^2 + k^2)Y(s) = 2s$$

So,

$$Y(s) = \frac{2s}{s^2 + k^2}$$

Using the inverse Laplace transform, we find:

$$y(t) = 2\cos(kt)$$

Thus, the solution to the differential equation is:

$$y(t) = 2\cos(kt)$$

Q7.

(a) **Regula-Falsi Method**

We aim to find the root of $f(x) = x\log_{10} x - 1.2$ in the interval $x = [2, 3]$.

$f(x) = x\log_{10} x - 1.2$. $f(2)$ and $f(3)$:

$$f(2) = 2\log_{10} 2 - 1.2 \approx -0.90, \quad f(3) = 3\log_{10} 3 - 1.2 \approx 0.43$$

Using the Regula-Falsi method, apply the formula:

$$x = x_1 - \frac{f(x_1)(x_2 - x_1)}{f(x_2) - f(x_1)}$$

Iteration 1:

$$x_1 = 2, \quad x_2 = 3$$

$$x = 2 - \frac{-0.90 \times (3 - 2)}{0.43 - (-0.90)} \approx 2.41$$

$f(2.41) \approx -0.07$. Since $f(2.41) < 0$, we update $x_1 = 2.41$.

Iteration 2:

$$x_1 = 2.41, \quad x_2 = 3$$

$$x = 2.41 - \frac{-0.07 \times (3 - 2.41)}{0.43 - (-0.07)} \approx 2.46$$

$f(2.46) \approx 0.01$. Since $f(2.46) > 0$, update $x_2 = 2.46$.

Iteration 3:

$$x_1 = 2.41, \quad x_2 = 2.46$$

$$x = 2.41 - \frac{-0.07 \times (2.46 - 2.41)}{0.01 - (-0.07)} \approx 2.45$$

$f(2.45) \approx -0.002$.

The root is accurate to two decimal places, yielding:

$$x \approx 2.45$$

(b) **Newton's Divided Difference Interpolating Polynomial**

Given data points: $(x_0, f(x_0)) = (1, 4)$, $(x_1, f(x_1)) = (3, 32)$, $(x_2, f(x_2)) = (4, 55)$, $(x_3, f(x_3)) = (6, 119)$.

Create the divided difference table:

$$f[x_0] = 4, \quad f[x_1] = 32, \quad f[x_2] = 55, \quad f[x_3] = 119$$
$$f[x_0, x_1] = \frac{32 - 4}{3 - 1} = 14, \quad f[x_1, x_2] = \frac{55 - 32}{4 - 3} = 23, \quad f[x_2, x_3] = \frac{119 - 55}{6 - 4} = 32$$
$$f[x_0, x_1, x_2] = \frac{23 - 14}{4 - 1} = 3, \quad f[x_1, x_2, x_3] = \frac{32 - 23}{6 - 3} = 3$$
$$f[x_0, x_1, x_2, x_3] = \frac{3 - 3}{6 - 1} = 0$$

$P(x)$:

$P(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)$

Substituting values:

$$P(x) = 4 + 14(x - 1) + 3(x - 1)(x - 3)$$

$P(x)$:

$$P(x) = 4 + 14(x - 1) + 3(x^2 - 4x + 3) = 3x^2 + 2x - 1$$

Expanding and simplifying further, we obtain the interpolating polynomial.

(c) **Simpson's $\frac{1}{3}$-Rule**

To evaluate $\int_0^6 \frac{e^x}{1+x}\, dx$ with $n = 6$, we apply Simpson's $\frac{1}{3}$-rule with $h = 1$, where $f(x) = \frac{e^x}{1+x}$ and $e = 2.7182$.

$f(x)$ at the required points $x = 0, 1, 2, 3, 4, 5, 6$:

$$f(0) = \frac{e^0}{1 + 0} = 1$$
$$f(1) = \frac{e}{2} = \frac{2.7182}{2} = 1.3591$$
$$f(2) = \frac{e^2}{3} = \frac{2.7182^2}{3} = \frac{7.3891}{3} \approx 2.4630$$
$$f(3) = \frac{e^3}{4} = \frac{2.7182^3}{4} = \frac{20.0855}{4} \approx 5.0214$$
$$f(4) = \frac{e^4}{5} = \frac{2.7182^4}{5} = \frac{54.5982}{5} \approx 10.9196$$
$$f(5) = \frac{e^5}{6} = \frac{2.7182^5}{6} = \frac{148.4132}{6} \approx 24.7355$$
$$f(6) = \frac{e^6}{7} = \frac{2.7182^6}{7} = \frac{403.4288}{7} \approx 57.6327$$

Applying Simpson's $\frac{1}{3}$-rule:

$$\int_0^6 \frac{e^x}{1 + x}\, dx \approx \frac{1}{3}\left[f(0) + 4(f(1) + f(3) + f(5)) + 2(f(2) + f(4)) + f(6)\right]$$

Substitute the values:

$$\approx \frac{1}{3}\left[1 + 4(1.3591 + 5.0214 + 24.7355) + 2(2.4630 + 10.9196) + 57.6327\right]$$

$$= \frac{1}{3}\left[1 + 4(31.1160) + 2(13.3826) + 57.6327\right]$$

$$= \frac{1}{3}\left[1 + 124.464 + 26.7652 + 57.6327\right]$$

$$= \frac{1}{3} \times 209.8619 \approx 69.9540$$

Thus, the approximate value of the integral is:

$$\int_0^6 \frac{e^x}{1+x}\,dx \approx 69.95$$

Q8

(a)

The Newton-Raphson formula is given by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

For the equation $\cos(x) = xe^x$, we define:

$$f(x) = \cos(x) - xe^x$$

and

$$f'(x) = -\sin(x) - e^x - xe^x$$

Starting with the initial approximation $x_0 = 0.5$, we compute the iterations until the solution converges to 3 decimal places.

Iteration 1:

$$f(0.5) = \cos(0.5) - 0.5 \cdot e^{0.5} = 0.8776 - 0.8244 = 0.0532$$

$$f'(0.5) = -\sin(0.5) - e^{0.5} - 0.5 \cdot e^{0.5} = -0.4794 - 2.473 = -2.9524$$

$$x_1 = 0.5 - \frac{0.0532}{-2.9524} = 0.517$$

Iteration 2:

$$f(0.517) = \cos(0.517) - 0.517 \cdot e^{0.517} = 0.8562 - 0.8605 = -0.0043$$

$$f'(0.517) = -\sin(0.517) - e^{0.517} - 0.517 \cdot e^{0.517} = -0.4947 - 2.4811 = -2.9758$$

$$x_2 = 0.517 - \frac{-0.0043}{-2.9758} = 0.516$$

Thus, the root correct to 3 decimal places is:

$$x = 0.516$$

(b)

The general Newton's backward interpolation formula is:

$$y = y_n + u\Delta y_{n-1} + \frac{u(u+1)}{2!}\Delta^2 y_{n-2} + \frac{u(u+1)(u+2)}{3!}\Delta^3 y_{n-3} + \dots$$

where $u = \frac{x-x_n}{h}$. Here, $x_n = 5$, $h = 1$, and $x = 6$, so:

$$u = \frac{6-5}{1} = 1$$

Now, we compute the backward differences:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	1			
2	-1	-2		
3	1	2	4	
4	-1	-2	-4	-8
5	1	2	4	8

Using the formula:

$$y = y_5 + u\Delta y_4 + \frac{u(u+1)}{2!}\Delta^2 y_3 + \frac{u(u+1)(u+2)}{3!}\Delta^3 y_2$$

$$y = 1 + 1 \cdot 2 + \frac{1(1+1)}{2} \cdot 4 + \frac{1(1+1)(1+2)}{6} \cdot (-8)$$

$$y = 1 + 2 + 4 - 4 = 3$$

Thus, the value of y when $x = 6$ is:

$$y = 3$$

(c)

The formula for the Trapezoidal rule is:

$$I \approx \frac{h}{2}\left[y_0 + 2\sum_{i=1}^{n-1} y_i + y_n\right]$$

where $h = \frac{b-a}{n} = \frac{5-0}{5} = 1$. The ordinates are:

$$x_0 = 0, \quad x_1 = 1, \quad x_2 = 2, \quad x_3 = 3, \quad x_4 = 4, \quad x_5 = 5$$

and

$$y_i = \frac{1}{4x_i + 5}$$

So:

$$y_0 = \frac{1}{5}, \quad y_1 = \frac{1}{9}, \quad y_2 = \frac{1}{13}, \quad y_3 = \frac{1}{17}, \quad y_4 = \frac{1}{21}, \quad y_5 = \frac{1}{25}$$

Now, applying the formula:

$$I \approx \frac{1}{2}\left[\frac{1}{5} + 2\left(\frac{1}{9} + \frac{1}{13} + \frac{1}{17} + \frac{1}{21}\right) + \frac{1}{25}\right]$$

$$I \approx \frac{1}{2}\left[\frac{1}{5} + 2\,(0.1111 + 0.0769 + 0.0588 + 0.0476) + 0.04\right]$$

$$I \approx \frac{1}{2}\left[0.2 + 2 \cdot 0.2944 + 0.04\right] = \frac{1}{2}\left[0.2 + 0.5888 + 0.04\right]$$

$$I \approx \frac{1}{2} \times 0.8288 = 0.4144$$

Thus, the value of the integral is:

$$I \approx 0.414$$

Q9

(a)

The Taylor's series expansion for $y(x)$ about $x_0 = 0$ is:

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \dots$$

We need to find the derivatives of y. Given:

$$\frac{dy}{dx} = 2y + 3e^x,$$

At $x = 0$, $y(0) = 0$:

$$y'(0) = 2 \cdot 0 + 3e^0 = 3.$$

Next, differentiate $y'(x) = 2y + 3e^x$ with respect to x:

$$y''(x) = 2y' + 3e^x.$$

At $x = 0$, $y'(0) = 3$:

$$y''(0) = 2 \cdot 3 + 3e^0 = 6 + 3 = 9.$$

Now, differentiate $y''(x) = 2y' + 3e^x$:

$$y'''(x) = 2y'' + 3e^x.$$

At $x = 0$, $y''(0) = 9$:

$$y'''(0) = 2 \cdot 9 + 3e^0 = 18 + 3 = 21.$$

Thus, the Taylor series expansion becomes:

$$y(0.2) = 0 + 0.2 \cdot 3 + \frac{(0.2)^2}{2} \cdot 9 + \frac{(0.2)^3}{6} \cdot 21 + \dots$$

$$y(0.2) = 0.6 + 0.18 + 0.028 = 0.808.$$

Thus, $y(0.2) \approx 0.808$.

(b)

The formula for the modified Euler's method is:

$$y_{n+1} = y_n + \frac{h}{2}\left(f(x_n, y_n) + f(x_{n+1}, y^*)\right),$$

where $y^* = y_n + hf(x_n, y_n)$.

First iteration (n=0):

$$x_0 = 0, \quad y_0 = 1, \quad f(0,1) = 0 - 1^2 = -1.$$

$$y^* = y_0 + h \cdot f(0,1) = 1 + 0.1 \cdot (-1) = 0.9.$$

$$f(0.1, 0.9) = 0.1 - 0.9^2 = 0.1 - 0.81 = -0.71.$$

$$y_1 = 1 + \frac{0.1}{2}\left(-1 + (-0.71)\right) = 1 + 0.05 \cdot (-1.71) = 1 - 0.085 = 0.9150.$$

Second iteration (n=1):

$$x_1 = 0.1, \quad y_1 = 0.9150, \quad f(0.1, 0.915) = 0.1 - 0.915^2 = -0.736.$$

$$y^* = 0.915 + 0.1 \cdot (-0.736) = 0.915 - 0.0736 = 0.8414.$$

$$f(0.2, 0.8414) = 0.2 - 0.8414^2 = 0.2 - 0.708 = -0.508.$$

$$y_2 = 0.915 + \frac{0.1}{2}\left(-0.736 + (-0.508)\right) = 0.915 + 0.05 \cdot (-1.244) = 0.915 - 0.0622 = 0.8528.$$

Thus, $y(0.1) \approx 0.8528$.

(c)

The predictor formula for Milne's method is:

$$y_{n+1} = y_{n-3} + \frac{4h}{3}\left(2f_{n-2} - f_{n-1} + 2f_n\right).$$

Given:

$$h = 0.1, \quad f(x, y) = x^2(1 + y).$$

Compute f values:

$$f(1, 1) = 1^2(1 + 1) = 2, \quad f(1.1, 1.233) = 1.21 \cdot 2.233 = 2.704,$$

$$f(1.2, 1.548) = 1.44 \cdot 2.548 = 3.667, \quad f(1.3, 1.979) = 1.69 \cdot 2.979 = 5.035.$$

Predictor step:

$$y(1.4)_{\text{pred}} = 1 + \frac{4 \cdot 0.1}{3}\left(2 \cdot 2.704 - 3.667 + 2 \cdot 5.035\right).$$

$$y(1.4)_{\text{pred}} = 1 + \frac{0.4}{3}(5.408 - 3.667 + 10.07) = 1 + \frac{0.4}{3} \cdot 11.811.$$

$$y(1.4)_{\text{pred}} = 1 + 1.5748 = 2.5748.$$

Corrector step:

$$y(1.4)_{\text{corr}} = y(1.2) + \frac{h}{3}\left(f(1.2, y(1.2)) + 4f(1.3, y(1.3)) + f(1.4, y(1.4)_{\text{pred}})\right).$$

$$y(1.4)_{\text{corr}} = 1.548 + \frac{0.1}{3}(3.667 + 4 \cdot 5.035 + 1.96 \cdot 3.5748)$$

Q10

(a)

The Modified Euler's method formula is:

$$y_{n+1} = y_n + \frac{h}{2}\left(f(x_n, y_n) + f(x_{n+1}, y^*)\right),$$

where $y^* = y_n + h \cdot f(x_n, y_n)$.

Initial Step (for $x = 0.2$):

Given $y_0 = 0$ (assumed for simplicity),

$$f(0, 0) = \log_{10}(0 + 0) = \log_{10}(0) \to \text{undefined}.$$

(We cannot proceed unless $x \neq 0$. Let's assume $y(0.2) \approx 0.01$ for meaningful iterations.)

For the first iteration:

$$y^* = y_0 + 0.2 \cdot \log_{10}(0.2 + 0) = 0 + 0.2 \cdot (-0.698) = -0.1396.$$

$$f(0.4, -0.1396) = \log_{10}(0.4 - 0.1396) = \log_{10}(0.2604) \approx -0.585.$$

$$y_1 = 0 + \frac{0.2}{2}(-0.698 + (-0.585)) = 0 + 0.1 \cdot (-1.283) = -0.1283.$$

For the second iteration:

$$y^* = -0.1283 + 0.2 \cdot \log_{10}(0.4 - 0.1283) \approx -0.1283 - 0.117 = -0.2453.$$

$$f(0.6, -0.2453) = \log_{10}(0.6 - 0.2453) \approx \log_{10}(0.3547) \approx -0.451.$$

$$y_2 = -0.1283 + 0.1 \cdot (-1.036) = -0.2319.$$

Thus, after two iterations at $x = 0.4$, $y(0.4) \approx -0.2319$.

—

(b)

The 4th order Runge-Kutta method formula is:

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

where:

$$k_1 = h \cdot f(x_n, y_n), \quad k_2 = h \cdot f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right),$$

$$k_3 = h \cdot f\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right), \quad k_4 = h \cdot f\left(x_n + h, y_n + k_3\right).$$

Given:

$$y' = \frac{1}{9x + y}, \quad y(0.4) = 1, \quad h = 0.1.$$

Compute the following:

$$k_1 = 0.1 \cdot \frac{1}{9 \cdot 0.4 + 1} = 0.1 \cdot \frac{1}{4.6} = 0.02174,$$

$$k_2 = 0.1 \cdot \frac{1}{9 \cdot (0.4 + 0.05) + (1 + 0.01087)} = 0.1 \cdot \frac{1}{5.05587} \approx 0.01978,$$

$$k_3 = 0.1 \cdot \frac{1}{9 \cdot (0.4 + 0.05) + (1 + 0.00989)} \approx 0.01979,$$

$$k_4 = 0.1 \cdot \frac{1}{9 \cdot (0.5) + (1 + 0.01979)} \approx 0.01852.$$

Now, compute $y(0.5)$:

$$y(0.5) = 1 + \frac{0.1}{6}(0.02174 + 2 \cdot 0.01978 + 2 \cdot 0.01979 + 0.01852) \approx 1.0198.$$

Thus, $y(0.5) \approx 1.0198$.

—

(c)

```python
import math

def taylor_series(x0, y0, h, x):
    # Taylor series expansion: y(x) = y(0) + h * y'(0) + (h^2 / 2!) * y''(0) + ...

    def dy_dx(x, y):
        return x - y

    def d2y_dx2(x, y):
        return 1 - dy_dx(x, y)

    # Calculate terms of Taylor series
    y1 = y0 + h * dy_dx(x0, y0)
    y2 = y1 + (h**2 / 2) * d2y_dx2(x0, y0)

    return round(y2, 4)

# Example usage
x0 = 0
y0 = 1
h = 0.1
x = 0.1

result = taylor_series(x0, y0, h, x)
print(f"y({x}) = {result}")
```

—