CMR INSTITUTE OF TECHNOLOGY	USN											CMRIT

Internal Assessment Test – I April 2024 (Scheme & Solutions)

Sub:	**Mathematics-II for EC Engineering**							Code:	BMATE201	
Date:	08/04/2024	Duration:	90 mins	Max Marks:	50	Sem:	II	Branch:	**ECE Phy Cycle**	

Question 1 is compulsory and Answer any 6 from the remaining questions.

		Marks	OBE CO	OBE RBT
1	Verify Stoke's theorem for $\vec{F} = \left(x^2 + y^2\right)i - 2xyj$, taken around the rectangle bounded by the lines $x = \pm a,\ y = 0,\ y = b$. **(Stoke's theorem statement - 2m, Verification of LHS & RHS - 6 (3+3))**	[8]	CO1	L3
2	Find the area between the parabolas $y^2 = 4x$ and $x^2 = 4y$, with the help of Green's theorem in a plane. **(Green's theorem statement - 2, Graph & Point of intersections of given parabolas - 1m, Line integral evaluation along C1, C2 - 4m)**	[7]	CO1	L3
3	Find the work done in a moving particle in the force field $\vec{F} = 3x^2 i + (2xz - y)j + zk$, the straight line from (0, 0, 0) to (2, 1, 3). **(Line integral formula - 1m, Changing integral into parameter 't' - 2m, Evaluation of line integral and final answer - 4m)**	[7]	CO1	L3
4	Find the directional derivative of the function xyz, along the direction of the normal to the surface $xy^2 + yz^2 + zx^2 = 3$, at the point $(1,\ 1,\ 1)$. **(Finding Grad of xyx - 2m, Finding Grad of $xy^2 + yz^2 + zx^2 = 3$ - 2m, D.D formula & final answer- 3m)**	[7]	CO1	L3

5	Show that the vector $\vec{F} = \left(2xy^2 + yz\right)i + \left(2x^2y + xz + 2yz^2\right)j + \left(2y^2z + xy\right)k$ is a conservative force field. Find it's scalar potential. **(S.T $\vec{F}$ is a conservative field - 2m, Finding scalar potential - 5m)**	[7]	CO1	L3
6	Prove that intersection of two subspaces is also a subspace. **(W1& W2 belongs to O - 1m, Vector addition of W1& W2 - 3m, Scalar multiplication of W1& W2 - 3m)**	[7]	CO2	L3
7	Determine the dimension and basis of the subspace spanned by S = {(1, 2, 3), (3, 1, 0), (-2, 1, 3)}. **(Finding REF of S - 3m, Basis - 2m, Dimension - 2m)**	[7]	CO2	L3
8	Determine whether the matrix $[-1\ 7\ 8\ -1]$ is a linear combination of $[1\ 0\ 2\ 1],\ [2\ -3\ 0\ 2],\ [0\ 1\ 2\ 0]$ in the vector space M_{22} of (2×2) matrix. **(Writing the linear combination - 2m, Solving eqs and finding solution- 6m)**	[7]	CO2	L3

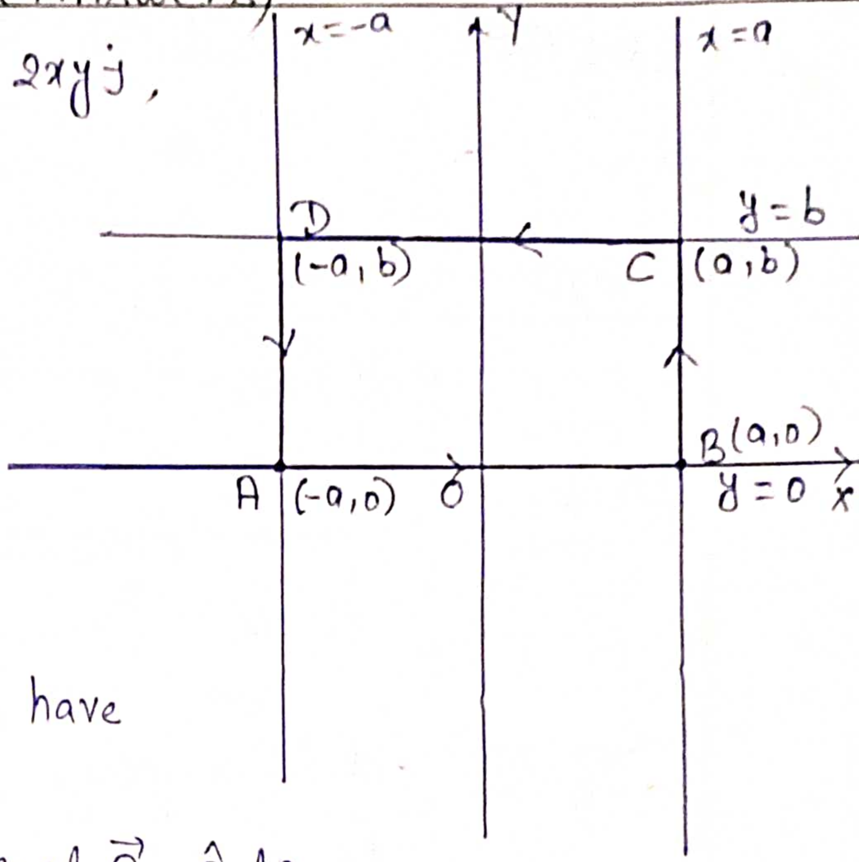
IAT 1 (Answers)

Q.1 Given $\vec{F} = (x^2+y^2)\,i - 2xy\,j$,

$x = \pm a$

$y = 0$

$y = b$



The bounded region on xy plane is the rectangle ABCD.

By Stoke's theorem, we have

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot \hat{n}\, ds,$$

where $\hat{n}\,ds = dy\,dz\, i + dz\,dx\, j + dx\,dy\, k$.

$$L.H.S = \int_C \vec{F} \cdot d\vec{r} = \int_{AB} \vec{F} \cdot d\vec{r} + \int_{BC} \vec{F} \cdot d\vec{r} + \int_{CD} \vec{F} \cdot d\vec{r} + \int_{DA} \vec{F} \cdot d\vec{r} \quad — ①$$

Where $\vec{F} \cdot d\vec{r} = \left[(x^2+y^2)\, i - 2xy\, j + 0\, k\right] \cdot \left[dx\, i + dy\, j + dz\, k\right]$

$$\Rightarrow \vec{F} \cdot d\vec{r} = (x^2+y^2)\,dx - 2xy\,dy \quad — ②$$

Along AB :- $y = 0 \Rightarrow dy = 0$

$x : -a \rightarrow a.$

Along BC :- $x = a,\ dx = 0$

$y : 0 - b$

Along CD :- $y = b,\ dy = 0$

$x : a \rightarrow -a$

Along DA :- $x = -a \Rightarrow dx = 0$

$y : b \rightarrow 0$

From ① & ②

$$LHS = \int_{-a}^{a}\left[(x^2+0^2)\,dx - 2\cdot x\cdot 0\cdot 0\right] + \int_{0}^{b}\left[(a^2+y^2)\cdot 0 - 2\cdot a\cdot y\,dy\right]$$

$$+ \int_{a}^{-a}\left[(x^2+b^2)\,dx - 2\cdot x\cdot b\cdot 0\right] + \int_{b}^{0}\left[((-a)^2+y^2)\cdot 0 - 2(-a)y\,dy\right]$$

$$= \int_{-a}^{a} x^2\,dx + \int_{0}^{b} -2a\,y\,dy + \int_{a}^{-a}(x^2+b^2)\,dx + \int_{b}^{0} 2ay\,dy$$

$$L.H.S = \int_{-a}^{a} \cancel{x^2 dx} - 2a\int_{0}^{b} y\,dy - \int_{-a}^{a} \cancel{x^2 dx} + b^2\int_{a}^{-a} dx - 2a\int_{0}^{b} y\,dy$$

$$= -4a\int_{0}^{b} y\,dy + b^2\int_{a}^{-a} dx$$

$$= -4a\left[\frac{y^2}{2}\right]_0^b + b^2\left[x\right]_a^{-a}$$

$$= -\frac{4a}{2}(b^2 - 0^2) + b^2(-a-a)$$

$$= -2ab^2 - 2ab^2 = -4ab^2$$

$$\boxed{LHS = -4ab^2} \quad \text{— (3)}$$

Next we find curl of $\vec{F}$ for R.H.S.

$$\text{Curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$= i[0-0] - j[0-0] + k[-2y-2y]$$

$$\text{Curl } \vec{F} = -4yk$$

$$\text{curl}\,\vec{F}\cdot\hat{n}\,ds = (0\cdot i + 0\cdot j - 4y\,k)\cdot(dy\,dz\,i + dz\,dx\,j + dx\,dy\,k)$$

$$\Rightarrow \text{curl}\,\vec{F}\cdot\hat{n}\,ds = -4y\,dx\,dy$$

$$\text{R.H.S} = \iint_{ABCD} \text{curl}\,\vec{F}\cdot\hat{n}\,ds = \int_{y=0}^{b}\int_{x=-a}^{a} -4y\,dx\,dy$$

$$= -4\int_{y=0}^{b} y\,dy \times \int_{-a}^{a} dx$$

$$= -4\left[\frac{y^2}{2}\right]_0^b \times \left[x\right]_{-a}^{a}$$

$$= \frac{-4}{2}(b^2)\times(a+a) = -2b^2(2a)$$

$$\boxed{RHS = -4ab^2} \quad - \quad (4)$$

From (3) & (4), Stoke's theorem is verified.

Q.2 By Green's theorem,

$$\int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

If $M = -\frac{y}{2}$ & $N = \frac{x}{2}$

then

$$\int_C \left(-\frac{y}{2}dx + \frac{x}{2}dy\right) = \iint_R \left(\frac{1}{2} + \frac{1}{2}\right) dx\, dy$$

$$\Rightarrow \frac{1}{2}\int_C (x\,dy - y\,dx) = \iint_R dx\, dy = \text{Area}.$$

$$\text{Area} = \frac{1}{2}\int_C x\,dy - y\,dx \quad -①$$

The bounded region is intersection of $y^2 = 4x$ & $x^2 = 4y$.

Point of intersection :-

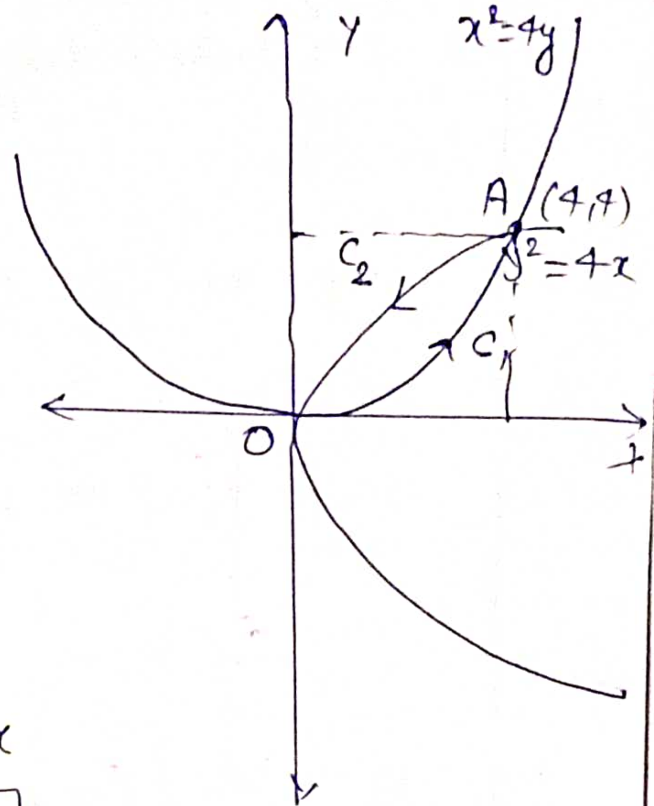
$y^2 = 4x$ & $x^2 = 4y \Rightarrow y = \frac{x^2}{4}$

$\Rightarrow \left(\frac{x^2}{4}\right)^2 = 4x \Rightarrow \frac{x^4}{16} = 4x$

$\Rightarrow x^4 = 64x \Rightarrow x(x^3 - 64) = 0 \Rightarrow x = 0, x = 4$

& $y = 0, y = 4$

Therefore the points of intersection are $(0,0)$ & $(4,4)$.



Along OA i.e C_1 :-

$$x^2 = 4y \Rightarrow y = \frac{x^2}{4}$$

$$\Rightarrow 2x\,dx = 4\,dy$$

$$\Rightarrow \boxed{dy = \frac{x\,dx}{2}}$$

$x : 0 - 4$

Along AO i.e C_2 :-

$$y^2 = 4x \Rightarrow 2y\,dy = 4\,dx$$

$$\Rightarrow x = \frac{y^2}{4} \qquad \Rightarrow \boxed{dx = \frac{y\,dy}{2}}$$

$y : 4 - 0$.

From (1),

$$\text{Area} = \frac{1}{2}\int_C x\,dy - y\,dx = \frac{1}{2}\left[\int_{OA}(x\,dy - y\,dx) + \int_{AO}(x\,dy - y\,dx)\right]$$

$$= \frac{1}{2}\left[\int_{x=0}^{4}\left(x\cdot\frac{x\,dx}{2} - \frac{x^2}{4}\,dx\right) + \int_{y=4}^{0}\left(\frac{y^2}{4}dy - y\cdot\frac{y\,dy}{2}\right)\right]$$

$$= \frac{1}{2}\left[\int_{x=0}^{4}\frac{1}{4}x^2\,dx + \int_{y=4}^{0} -\frac{y^2}{4}\,dy\right]$$

$$\Rightarrow \text{Area} = \frac{1}{2}\cdot\frac{1}{4}\left\{\left[\frac{x^3}{3}\right]_0^4 - \left[\frac{y^3}{3}\right]_4^0\right\}$$

$$= \frac{1}{8}\left(\frac{4^3}{3}+\frac{4^3}{3}\right) = \frac{1}{\cancel{8}_4}\times\cancel{2}\times\frac{4^3}{3}$$

$$\boxed{\text{Area} = \frac{4^2}{3} = \frac{16}{3}\ \text{sq. unit}}$$

Q.3 Eq: of st. line from (0,0,0) to (2,1,3) is

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0}$$

$$\Rightarrow \frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t \text{ (say)}.$$

$\Rightarrow x = 2t, y = t, z = 3t \Rightarrow dx = 2dt, dy = dt, dz = 3dt$

$t: 0 - 1.$

$$\text{Work done} = \int_C \vec{F} \cdot d\vec{r}$$

where $\vec{F} = 3x^2 i + (2xz - y) j + z k$

& $d\vec{r} = dx\, i + dy\, j + dz\, k$

$$\vec{F} \cdot d\vec{r} = 3x^2 dx + (2xz - y) dy + z dz.$$

$$= 3(2t)^2 \cdot 2dt + (2 \cdot 2t \cdot 3t - t) dt + 3t \cdot 3dt$$

$$= 24t^2 dt\ (12t^2 - t) dt + 9t\, dt$$

$$= (24t^2 + 12t^2 - t + 9t) dt$$

$$\Rightarrow \vec{F} \cdot d\vec{r} = (36t^2 + 8t) dt$$

$$\text{Work done} = \int_0^1 (36t^2 + 8t) dt = \left[\frac{36t^3}{3} + \frac{8t^2}{2}\right]_0^1$$

$$\Rightarrow \boxed{W = 16 J}$$

Q.4) Let $\phi = xyz$

$$\nabla\phi = \frac{\partial}{\partial x}(xyz)\,i + \frac{\partial}{\partial y}(xyz)\,j + \frac{\partial}{\partial z}(xyz)\,k$$

$$\Rightarrow \nabla\phi = yz\,i + xz\,j + xy\,k$$

at $(1,1,1)$

$$\nabla\phi = i + j + k \quad \text{--- (1)}$$

$$D.D = \nabla\phi \cdot \hat{n}$$

To calculate $\hat{n}$,

Let $\psi = xy^2 + yz^2 + zx^2$

$$\nabla\psi = (y^2 + 2xz)\,i + (2xy + z^2)\,j + (2yz + x^2)\,k$$

at $(1,1,1)$

$$\nabla\psi = 3i + 3j + 3k = 3(i + j + k)$$

$$|\nabla\psi| = \sqrt{3^2 + 3^2 + 3^2} = 3\sqrt{3}$$

$$\hat{n} = \frac{\nabla\psi}{|\nabla\psi|} = \frac{\cancel{3}(i+j+k)}{\cancel{3}\sqrt{3}} = \frac{i+j+k}{\sqrt{3}} \quad \text{--- (2)}$$

$$DD = \nabla\phi \cdot \hat{n} = (i+j+k) \cdot \frac{(i+j+k)}{\sqrt{3}} \quad (\text{from (1) \& (2)})$$

$$\Rightarrow \boxed{D.D = \frac{1+1+1}{\sqrt{3}} = \sqrt{3}}$$

Q.5 $\vec{F} = (2xy^2 + yz)\,i + (2x^2y + xz + 2yz^2)\,j + (2y^2z + xy)\,k$

$$\text{Curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^2+yz & 2x^2y+xz+2yz^2 & 2y^2z+xy \end{vmatrix}$$

$$= i\left[4yz + x - x - 4yz\right]$$

$$- j\left[y - y\right] + k\left[4xy + z - 4xy - z\right]$$

$\Rightarrow$ curl $\vec{F} = 0$

$\Rightarrow$ $\vec{F}$ is a conservative force field.

$\Rightarrow$ $\exists$ a scalar potential ϕ such that

$$\nabla \phi = F$$

Now,

$$\nabla \phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$$

$$\Rightarrow \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k = (2xy^2 + yz)\,i + (2x^2y + xz + 2yz^2)\,j + (2y^2z + xy)\,k$$

Therefore,

$$\frac{\partial \phi}{\partial x} = 2xy^2 + yz \Rightarrow \partial\phi = (2xy^2+yz)\,\partial x \quad -(1)$$

$$\frac{\partial \phi}{\partial y} = 2x^2y + xz + 2yz^2 \Rightarrow \partial\phi = (2x^2y+xz+2yz^2)\,\partial y \quad -(2)$$

$$\frac{\partial \phi}{\partial z} = 2y^2z + xy \Rightarrow \partial\phi = (2y^2z+xy)\,\partial z \quad -(3)$$

Integrating eq: (1), (2) & (3), we get

$$\phi = x^2y^2 + xyz + f_1(y,z)$$

$$\phi = x^2y^2 + xyz + y^2z^2 + f_2(x,z)$$

$$\phi = y^2z^2 + xyz + f_3(x,y)$$

From above we analyse,

$$f_1(y,z) = y^2z^2, \quad f_2(x,z) = 0, \quad f_3(x,y) = x^2y^2.$$

Hence the scalar potential,

$$\boxed{\phi = x^2y^2 + xyz + y^2z^2}$$

Q.6 Let W_1 & W_2 be two subspaces of a vector space $V(F)$.

To prove $W_1 \cap W_2$ is also a subspace.

(i) Since W_1 & W_2 are subspace then

$0 \in W_1$ & $0 \in W_2$

$\Rightarrow 0 \in W_1 \cap W_2$

(ii) Let $x, y \in W_1 \cap W_2$

$\Rightarrow x, y \in W_1$ & $x, y \in W_2$

$\Rightarrow x+y \in W_1$ & $x+y \in W_2$ (Since W_1 & W_2 are subspaces)

$\Rightarrow x+y \in W_1 \cap W_2$

(iii) Let $\alpha \in R$ & $x \in W_1 \cap W_2$

$\Rightarrow x \in W_1$ & $x \in W_2$

$\Rightarrow \alpha x \in W_1$ & $\alpha x \in W_2$ (as W_1 & W_2 are closed under scalar multiplication)

$\Rightarrow \alpha x \in W_1 \cap W_2$.

Hence $W_1 \cap W_2$ is also a subspace of $V(F)$.

Q.7) Determine the dimension and basis of the subspace spanned by $S = \{(1,2,3), (3,1,0), (-2,1,3)\}$.

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ -2 & 1 & 3 \end{bmatrix}$

$R_2 \rightarrow R_2 - 3R_1$

$R_3 \rightarrow R_3 + 2R_1$

$$\simeq \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -9 \\ 0 & 5 & 9 \end{bmatrix}$$

$R_3 \rightarrow R_3 + R_2$

$$\simeq \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -9 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence dimension of the subspace $= 2$

& basis of the subspace is $\{(1,2,3), (0,-5,-9)\}$

Q.8 Let α, β, γ be any scalars such that

$$\alpha\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix} + \beta\begin{bmatrix}2 & -3\\ 0 & 2\end{bmatrix} + \gamma\begin{bmatrix}0 & 1\\ 2 & 0\end{bmatrix} = \begin{bmatrix}-1 & 7\\ 8 & -1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}\alpha+2\beta & -3\beta+\gamma\\ 2\alpha+2\gamma & \alpha+2\beta\end{bmatrix} = \begin{bmatrix}-1 & 7\\ 8 & -1\end{bmatrix}$$

$\alpha + 2\beta = -1$ — (1)

$-3\beta + \gamma = 7$ — (2)

$2\alpha + 2\gamma = 8$ — (3)

$\alpha + 2\beta = -1$ — (4)

from (3),

$$2(\alpha+\gamma) = 8$$

$$\Rightarrow \alpha+\gamma = 4 \quad \text{— (5)}$$

Sub (5) in (1), we get

$$4-\gamma+2\beta = -1 \Rightarrow 2\beta - \cancel{\gamma} = -5$$

$$-3\beta + \cancel{\gamma} = 7 \quad (\text{from } (2))$$

$$-\beta = 2$$

$\Rightarrow \boxed{\beta = -2}$

from (1),

$\boxed{\alpha = 3}$

& $\boxed{\gamma = 1}$

Hence the matrix $\begin{bmatrix}1 & 7\\ 8 & -1\end{bmatrix}$ is a L.C of $\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}, \begin{bmatrix}2 & -3\\ 0 & 2\end{bmatrix}, \begin{bmatrix}0 & 1\\ 2 & 0\end{bmatrix}$.