CMR INSTITUTE OF TECHNOLOGY	USN ☐☐☐☐☐☐☐☐☐☐	CMRIT

Internal Assessment Test – I November 2024

Sub:	AV Mathematics –III for EC Engineering							Code:	BMATEC301
Date:	06/11/2024	Duration:	90 mins	Max Marks:	50	Sem:	III	Section:	**ECE A, B, C & D**

Question 1 is compulsory and Answer any 6 from the remaining questions.

<table>
<tr><th></th><th></th><th rowspan="2">Marks</th><th colspan="2">OBE</th></tr>
<tr><th></th><th></th><th>CO</th><th>RBT</th></tr>
<tr><td>1.</td><td>Obtain the Fourier series of y up to 2nd harmonics $f(x)$ is given by
<table>
<tr><td>x</td><td>0</td><td>$\pi/3$</td><td>$2\pi/3$</td><td>π</td><td>$4\pi/3$</td><td>$5\pi/3$</td><td>2π</td></tr>
<tr><td>y=f(x)</td><td>1.98</td><td>1.30</td><td>1.05</td><td>1.30</td><td>-0.88</td><td>-0.25</td><td>1.98</td></tr>
</table></td><td>[8]</td><td>CO1</td><td>L3</td></tr>
<tr><td>2.</td><td>Obtain the Fourier series of $f(x) = \frac{\pi - x}{2}$ in $0 < x < 2\pi$. Hence deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\dots\dots = \frac{\pi}{4}$.</td><td>[7]</td><td>CO1</td><td>L2</td></tr>
<tr><td>3.</td><td>Obtain the sine half range Fourier series of $f(x) = x^2$ in $0 < x < \pi$.</td><td>[7]</td><td>CO1</td><td>L2</td></tr>
<tr><td>4.</td><td>Find the Fourier transform of $f(x) = \begin{cases} 1 - |x|, & for |x| \leq 1 \\ 0, & for \ |x| \geq 1 \end{cases}$ and hence deduce that $\int_0^\infty \frac{\sin^2 t}{t^2}\, dt = \frac{\pi}{2}$.</td><td>[7]</td><td>CO2</td><td>L2</td></tr>
</table>

5.	Find the Fourier transform of $f(x) = \begin{cases} 1, for \ \lvert x\rvert \leq a \\ 0, for \ \lvert x\rvert \geq a \end{cases}$.	[7]	CO2	L2
6.	Find the Z-transform of $\cos n\theta$ and $\sin n\theta$.	[7]	CO3	L2
7.	Solve the difference equation $u_{n+2} + 6u_{n+1} + 9u_n = 2^n$ with $u_0 = u_1 = 0$ using Z-transform.	[7]	CO3	L3
8.	Solve $(D^3 - 3D + 2)y = 0$.	[7]	CO4	L1

1. Clearly the function f(x) in the interval (0, 2π)

x	$y=f(x)$	$\cos x$	$\cos 2x$	$y\cos x$	$y\cos 2x$
0	1.98	1	1	1.98	1.98
$\frac{\pi}{3}=60°$	1.30	0.5	-0.5	0.65	-0.65
$\frac{2\pi}{3}=120°$	1.05	-0.5	-0.5	-0.525	-0.525
$\pi=180°$	1.30	-1	1	-1.3	1.3
$\frac{4\pi}{3}=240°$	-0.88	-0.5	-0.5	0.44	0.44
$\frac{5\pi}{3}=300°$	-0.25	0.5	-0.5	-0.125	0.125
	$\Sigma y = 4.5$			$\Sigma y\cos x = 1.12$	$\Sigma y\cos 2x = 2.67$

x	$y=f(x)$	$\sin x$	$\sin 2x$	$y\sin x$	$y\sin 2x$
0	1.98	0	0	0	0
60°	1.30	0.866	0.866	1.1258	1.1258
120°	1.05	0.866	-0.866	0.9093	-0.9093
180°	1.30	0	0	0	0
240°	-0.88	-0.866	0.866	0.762	-0.762
300°	-0.25	-0.866	-0.866	0.2165	0.2165
				$\Sigma y\sin x = 3.0136$	$\Sigma y\sin 2x = -0.329$

We know that fourier series of y upto 2^{nd} Harmonic $f(x)$ in interval $(0,2\pi)$ is

~~$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_{0_1}\cos nx + a_2 \sin nx$~~ $\therefore N=6$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$f(x) = \frac{a_0}{2} + a_1\cos x + b_1 \sin x + a_2\cos 2x + b_2 \sin 2x$$

$$a_0 = \frac{2}{N}\Sigma y = \frac{\cancel{2}}{\cancel{6}_3}\overset{1.5}{(4.5)} = 1.5$$

$$a_1 = \frac{2}{N}\Sigma y\cos x = \frac{\cancel{2}}{\cancel{6}_3}\overset{0.373}{(1.12)} = 0.373$$

$$b_1 = \frac{2}{N}\Sigma y \sin x = \frac{2}{6}(3.0136) = 1.004$$

$$a_2 = \frac{2}{N}\Sigma y\, \cancel{\sin}\cos 2x = \frac{\cancel{2}}{\cancel{6}_3}\overset{0.89}{(2.67)} = 0.89$$

$$b_2 = \frac{2}{N}\Sigma y\sin 2x = \frac{2}{6}(-0.329) = -0.109$$

$\therefore$ The fourier series of y upto 2^{nd} Harmonic $f(x)$ is

$$f(x) = \frac{a_0}{2} + a_1\cos x + b_1 \sin x + a_2\cos 2x + b_2\sin 2x$$

$$= 0.75 + 0.373\cos x + 1.004\sin x + 0.89\cos 2x + (-0.109)\sin 2$$

constant term $\frac{a_0}{2} = 0.75$

first Harmonic $a_1\cos x + b_1\sin x = 0.373\cos x + 1.004\sin x$

second Harmonic $a_2\cos 2x + b_2\sin 2x = 0.89\cos 2x - 0.109\sin 2x$

2. Given $f(x) = \frac{\pi - x}{2} \to$ ① in $0 < x < 2\pi$

the fourier series of $f(x)$ in $(0, 2\pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow ②$$

find whether the given function $f(x)$ is even or odd

so $f(x) = \frac{\pi - x}{2}$

$$f(2\pi - x) = \frac{\pi - (2\pi - x)}{2} = \frac{\pi - 2\pi + x}{2} = \frac{-\pi + x}{2}$$

$$= -\frac{(\pi - x)}{2}$$

$f(2\pi - x) = -f(x)$

so given function is odd

$a_0 = 0; \; a_n = 0$

$$b_n = \frac{2}{\pi} \int_0^{\pi} \cancel{b} f(x) \sin nx \, dx$$

$\left[\because \int uv \, dx = u \int v - \int \int \left(\frac{du}{dx} \right) v \, dx \right]$

$$= \frac{2}{\pi} \int_0^{\pi} \left(\frac{\pi - x}{2} \right) \sin nx \, dx$$

$\left[\because \sin n\pi = 0 \right.$
$= \sin 0$

$$= \frac{2}{\pi} \left[\frac{\pi - x}{2} \frac{(-\cos nx)}{n} + \overset{0}{\cancel{\frac{(-\sin nx)}{2n^2}}} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\overset{0}{\cancel{\frac{(\pi - \pi)(-\cos n\pi)}{2n}}} + \frac{(\pi - 0)(\cos 0)}{2n} \right]$$

$$= \frac{\cancel{2}}{\cancel{\pi}} \left(\frac{\cancel{\pi}}{\cancel{2}n} \right) = \frac{1}{n}$$

sub $a_0 = a_n = 0; \; b_n = \frac{1}{n}$ in eq ②

the fourier series is

$$\boxed{f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin nx}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin nx$$

$$\frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{1}{n} \sin nx$$

let, $x = \pi/2$

$$\frac{\pi - \pi/2}{2} = \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2}$$

[∵ if n value is even then $\sin \frac{n\pi}{2} = 0$]

sub $n = 1, 3, 5, \ldots$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots$$

[$\sin \frac{\pi}{2} = 1$; $\sin \frac{3\pi}{2} = -1$; $\sin \frac{5\pi}{2} = 1 \ldots$]

$$\boxed{\therefore\ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots = \frac{\pi}{4}}$$ Hence proved

③ Given $f(x) = x^2 \rightarrow ①$ in $0 < x < \pi$

The sine half range fourier series in $(0,\pi)$ is given as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$\left(\int uv\,dx = u \int v - \frac{du}{dx} \iint v \right)$

find $b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$

$$= \frac{2}{\pi} \int_0^{\pi} x^2 \sin nx \, dx$$

[$\sin n\pi = 0 = \sin 0$]
[$\cos n\pi = (-1)^n$]

$$= \frac{2}{\pi} \left[\frac{x^2(-\cos nx)}{n} - \cancel{\frac{2x(-\sin nx)}{n^2}}^{\rightarrow 0} + \frac{2}{n^3}(\cos nx) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{-\pi^2(-1)^n}{n} + \frac{2}{n^3}(-1)^n + 0 - \frac{2}{n^3} \right]$$

$$= \frac{2}{\pi}\left[\frac{2}{n^3}\left[(-1)^n - 1\right] - \frac{\pi^2(-1)^n}{n}\right]$$

$$b_n = \frac{4}{\pi n^3}\left[(-1)^n - 1\right] + \frac{2\pi}{n}(-1)^{n+1}$$

∴ The required sine half range fourier series is

$$\boxed{f(x) = \sum_{n=1}^{\infty}\left[\frac{4}{\pi n^3}\left[(-1)^n - 1\right] + \frac{2\pi}{n}(-1)^{n+1}\right]\sin nx}$$

Given

4) $f(x) = \begin{cases} 1-|x| & \text{for } |x| \le 1 \Rightarrow -1 \le x \le 1 \\ 0 & \text{for } |x| > 1 \end{cases}$

$$f(x) = \begin{cases} 1+x & -1 \le x \le 0 \\ 1-x & 0 \le x \le 1 \\ 0 & |x| > 1 \end{cases} \rightarrow ①$$

The fourier transform of f(x) is is

$$F(u) = \int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx$$

$$= \int_{-\infty}^{-1} \cancelto{0}{f(x) e^{iux}}\, dx + \int_{-1}^{0} f(x)\, e^{iux}\, dx + \int_{0}^{1} f(x) e^{iux}\, dx + \int_{1}^{\infty} \cancelto{0}{f(x) e^{iux}}\, dx$$

$$= \int_{-1}^{0} (1+x) e^{iux}\, dx + \int_{0}^{1} (1-x) e^{iux}\, dx$$

$$= \left[\frac{(1+x)e^{iux}}{iu} - \frac{e^{iux}}{i^2u^2}\right]_{-1}^{0} + \left[\frac{(1-x)e^{iux}}{iu} + \frac{e^{iux}}{i^2u^2}\right]_{0}^{1}$$

$$= \left(\cancel{\frac{1}{iu}} + \frac{1}{u^2} \mp \frac{e^{-iu}}{u^2} \mp \frac{e^{iu}}{u^2} - \cancel{\frac{1}{iu}} + \frac{1}{u^2}\right)$$

$$= \frac{2}{u^2} - \frac{1}{u^2}\left(e^{iu} + e^{-iu}\right) \qquad \left[e^{i\theta} + e^{-i\theta} = 2\cos\theta\right]$$

$$= \frac{2}{u^2} - \frac{1}{u^2}(2\cos u) = \frac{2(1-\cos u)}{u^2} \qquad \left[1-\cos x = 2\sin^2\right.$$

$$= \frac{2\left(2\sin^2 \overset{u}{\cancel{x}}/2\right)}{u^2}$$

$$= \frac{4\sin^2 \overset{u}{\cancel{x}}/2}{u^2}$$

$$= \left(\frac{\sin \overset{u}{\cancel{x}}/2}{u/2}\right)^2$$

$$\boxed{F[u] = \left(\frac{\sin(u/2)}{(u/2)}\right)^2}$$

to bind $\int_0^\infty \frac{\sin^2 t}{t}\, dt$ Apply Inverse fourier transform

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F[u]\, e^{-iux}\, du$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\frac{\sin u/2}{u/2}\right)^2 e^{-iux}\, du$$

$x = 0$; $f(0) = 1$

$$1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\frac{\sin u/2}{u/2}\right)^2 du$$

$$\left[f(\overset{u}{\cancel{x}}) = \left(\frac{\sin u/2}{u/2}\right)^2 \Rightarrow f(-u) = \left(\frac{\sin u/2}{u/2}\right)^2 = f(u) \Rightarrow \text{even function} \right]$$

$$1 = \frac{1}{\pi}\int_{0}^{\infty} \left(\frac{\sin u/2}{u/2}\right)^2 du$$

$\frac{u}{2} = t$

$u = 2t$

$du = 2dt$

$$1 = \frac{1}{\pi} \int_0^{\infty} \left(\frac{\sin t}{t} \right)^2 (2dt)$$

$$\boxed{\int_0^{\infty} \frac{\sin^2 t}{t^2}\, dt = \frac{\pi}{2}}$$ Hence proved

⑤ Given $f(x) = \begin{cases} 1 & \text{for } |x| \le a \Rightarrow -a \le x \le a \\ 0 & \text{for } |x| > a \end{cases}$

The fourier transform of $f(x)$ is

$$F(u) = \int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx$$

$$= \int_{-a}^{0} f(x)\, e^{iux}\, dx + \int_{0}^{a} f(x)\, e^{iux}\, dx$$

$$= \int_{-a}^{0} e^{iux}\, dx + \int_{0}^{a} e^{iux}\, dx$$

$$= \left[\frac{e^{iux}}{iu} \right]_{-a}^{0} + \left[\frac{e^{iux}}{iu} \right]_{0}^{a}$$

$$= \cancel{\frac{1}{iu}} - \frac{e^{-iua}}{iu} + \frac{e^{iua}}{iu} - \cancel{\frac{1}{iu}}$$

$e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

$$= \frac{1}{iu} \left[e^{iua} - e^{-iua} \right]$$

$$= \frac{1}{\cancel{i}u} \left[2\cancel{i}\sin ua \right] = \underline{\underline{\frac{2\sin au}{u}}}$$

∴ the fourier transform of f(x) is

$$\boxed{F(u) = \frac{2\sin au}{u}}$$

⑥ The z-transform of cosnθ and sinnθ

we know that, $e^{i\theta} = \cos\theta + i\sin\theta$

$$e^{in\theta} = \cos n\theta + i\sin n\theta$$

apply z-transform

$[Z_T\{k^n\}] = \frac{z}{z-k}$

$$Z_T[\cos n\theta] + i\, Z_T[\sin n\theta] = Z_T[e^{in\theta}]$$

$$= \frac{z}{z-e^{i\theta}}$$

$$= \frac{z}{z-e^{i\theta}} \times \frac{z-e^{-i\theta}}{z-e^{-i\theta}}$$

$$= \frac{z(z-e^{-i\theta})}{z^2 - ze^{-i\theta} - ze^{i\theta} + 1}$$

$$= \frac{z^2 - ze^{-i\theta}}{z^2 - z(2\cos\theta) + 1}$$

$$= \frac{z^2 - z(\cos\theta - i\sin\theta)}{z^2 - z(2\cos\theta) + 1}$$

$$= \frac{z^2 - z\cos\theta}{z^2 - 2z\cos\theta + 1} + i\,\frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

compare real & imaginary parts

$$Z_T[\cos n\theta] = \frac{z^2 - z\cos\theta}{z^2 - 2z\cos\theta + 1}$$

$$Z_T[\sin n\theta] = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

⑦ Given $u_{n+2} + 6u_{n+1} + 9u_n = 2^n$; $u_0 = u_1 = 0$.

$\left[Z_T[k^n] = \frac{z}{z-k}\right]$

apply z-transform

$$Z_T[u_{n+2}] + 6 \cdot Z_T[u_{n+1}] + 9\, Z_T[u_n] = Z_T[2^n]$$

By left shifting property,

$$z^2\left(\bar{u}(z) - u_0 - \frac{u_1}{z}\right) + 6z\left(\bar{u}(z) - u_0\right) + 9\left(\bar{u}(z)\right) = \frac{z}{z-2}$$

$$(z^2 + 6z + 9)\,\bar{u}(z) - u_0(z^2 + 6z) - zu_1 = \frac{z}{z-2}$$

Given u_0 and u_1 is zero

$$(z^2 + 6z + 9)\,\bar{u}(z) = \frac{z}{z-2}$$

$$\bar{u}(z) = \frac{z}{(z-2)(z+3)^2}$$

$$\frac{\bar{u}(z)}{z} = \frac{1}{(z-2)(z+3)^2}$$

$$\frac{1}{(z-2)(z+3)^2} = \frac{A}{z-2} + \frac{B}{z+3} + \frac{C}{(z+3)^2}$$

$$1 = A(z+3)^2 + B(z-2)(z+3) + C(z-2)$$

if $z = 2 \Rightarrow 25A = 1 \Rightarrow \boxed{A = 1/25}$

$z = -3 \Rightarrow -5C = 1 \Rightarrow \boxed{C = -1/5}$

$1 = A(z+3)^2 + B(z-2)(z+3) + C(z-2)$

$= A(z^2+9+6z) + B(z^2+z-6) + C(z-2)$

compare z^2 coefficients

$0 = A+B$

$B = -A = \frac{-1}{25}$

$$\therefore \frac{\bar{u}(z)}{z} = \frac{1}{25(z-2)} - \frac{1}{25(z+3)} - \frac{1}{5(z+3)^2}$$

$$\bar{u}(z) = \frac{1}{4}\left(\frac{z}{z-2} - \frac{1}{z+3}\right) - \frac{1}{5}\frac{z}{(z+3)^2}$$

$$= \frac{1}{25}\left[\frac{z}{z-2} - \frac{z}{z+3}\right] + \frac{1}{15}\frac{-3z}{(z+3)^2}$$

apply inverse z-transform

$$Z_T^{-1}[\bar{u}(z)] = \frac{1}{25}\left(Z_T^{-1}\left(\frac{z}{z-2}\right) - Z_T^{-1}\left(\frac{z}{z+3}\right)\right) + \frac{1}{15} Z_T^{-1}\left(\frac{-3z}{(z+3)^2}\right)$$

$$\boxed{u_n = \frac{1}{25}\left[2^n - (-3)^n\right] + \frac{1}{15}(-3)^n \cdot n}$$

$\left[\because Z_T^{-1}\left(\frac{kz}{(z-k)^2}\right) = k^n \cdot n\right]$

⑧ $(D^3 - 3D + 2)y = 0.$ — ①

The auxillary equation of eq ① is

$m^3 - 3m + 2 = 0$

$m = 1$

$(1)^3 - 3(1) + 2 = 0$

$3 - 3 = 0$

$0 = 0$

$\therefore$ $m = 1$ is one of the root of the equation

$$\begin{array}{c|cccc} 1 & 1 & 0 & -3 & 2. \\ & 0 & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & \boxed{0} \\ & m^2 & m & c & \end{array}$$

$\therefore$ $m^2 + m - 2 = 0$

$m^2 + 2m - 1m - 2 = 0$

$m(m+2) - 1(m+2).$

$(m+2).(m-1) = 0.$

-2

$\wedge$

$2 \cdot 1$

-2

1

$\therefore$ The roots are $m = -2$, $m = 1$, $m = 1$

The solution for the equation ① is

$$\boxed{y = (c_1 + c_2 x)e^{x} + c_3 e^{-2x}}$$