Internal Assessment Test – January - 2025

Sub:	Mathematics-1 forECE Stream solution							**Code:**	**BMATE101**
Date:	**16-1-2025**	**Duration :**	**90 mins**	**Max Marks:**	**50**	**Sem :**	**I**	**section**	**M to P**

Question 1 is compulsory and Answer any 6 from the remaining questions.

		Marks	OBE CO	OBE RBT
1	Find the extreme values of the function $f(x, y) = x^3 + y^3 - 3x - 12y + 20$	[08]	CO2	L3
2	Evaluate(i) $k = \lim_{x \to \frac{\pi}{2}} \frac{log(sinx)}{(\frac{\pi}{2}-x)^2}$ (ii)$k = \lim_{x \to 0} \frac{x^2+2cosx-2)}{x\,sin^3x}$	[07]	CO2	L3
3	If $x + y + z = u, y + z = v\ and\ z = uvw\ find\ the\ value\ of\ \frac{\partial(x,y,z)}{\partial(u,v,w)}$	[07]	CO2	L3

4	Solve $(x^2 + y^2 + x)dx + xy\,dy = 0$	[07]	CO3	L3
5	Solve by reducing to clairaut's form $(px - y)(py + x) = 2p$ using $X = x^2\ Y = y^2$	[07]	CO3	L3
6	Find the volume of tetrahedron bounded by the planes $x = 0, y = 0, z = 0, \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$	[07]	CO4	L3
7	Evaluate $\int_0^{\infty}\int_0^{\infty} \frac{e^{-y}}{y} dydx$ by changing the order of integration	[07]	CO4	L3
8	Show that $\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{sin\,\theta}} * \int_0^{\frac{\pi}{2}} d\theta\sqrt{sin\theta} = \pi$	[07]	CO4	L3

1)

Find the extreme values of the function

$$f(x, y) = x^3 + y^3 - 3x - 12y + 20$$

$f_x = 3x^2 - 3,\ f_y = 3y^2 - 12$

We shall find points (x, y) such that $f_x = 0$ and $f_y = 0$.

$3x^2 - 3 = 0$ and $3y^2 - 12 = 0$ or $x^2 - 1 = 0$ and $y^2 - 4 = 0$

$x = \pm 1,\ y = \pm 2$

$(1,2)(1,-2)(-1,2)(-1,-2)$ are the stationary points.

Let $A = f_{xx},\ B = f_{xy},\ C = f_{yy}$

	$(1, 2)$	$(1, -2)$	$(-1, 2)$	$(-1, -2)$
$A = 6x$	$6 > 0$	6	-6	$-6 < 0$
$B = 0$	0	0	0	0
$C = 6y$	12	-12	12	-12
$AC - B^2$	$72 > 0$	$-72 < 0$	$-72 < 0$	$72 > 0$
Conclusion	Min. pt.	Saddle pt.	Saddle pt.	Max. pt.

Maximum value of $f(x, y)$ is,

$$f(-1, -2) = -1 - 8 + 3 + 24 + 20 = 38$$

Minimum value of $f(x, y)$ is $f(1, 2) = 1 + 8 - 3 - 24 + 20 = 2$

Thus, **Maximum value is 38 and Minimum value is 2.**

2) (i)

Let $k = \lim_{x \to \pi/2} \dfrac{\log(\sin x)}{(\pi/2 - x)^2} \cdots \left(\dfrac{0}{0}\right)$

Applying L' Hospital's rule, $k = \lim_{x \to \pi/2} \dfrac{\cos x / \sin x}{-2(\pi/2 - x)}$

ie., $= \lim_{x \to \pi/2} \frac{\cot x}{-2(\pi/2 - x)} \cdots \left(\frac{0}{0}\right)$

Now, $k = \lim_{x \to \pi/2} \frac{-\operatorname{cosec}^2 x}{2} = \boxed{\frac{-1}{2}}$

(ii)

Let $k = \lim_{x \to 0} \frac{x^2 + 2\cos x - 2}{x \sin^3 x} \cdots \left(\frac{0}{0}\right)$

$$= \lim_{x \to 0} \frac{x^2 + 2\cos x - 2}{x \cdot \frac{\sin^3 x}{x^3} \cdot x^3} = \lim_{x \to 0} \frac{x^2 + 2\cos x - 2}{x^4} \cdot \lim_{x \to 0} \left(\frac{x}{\sin x}\right)^3$$

Hence, $k = \lim_{x \to 0} \frac{x^2 + 2\cos x - 2}{x^4} \cdot 1 \cdots \left(\frac{0}{0}\right)$

Applying L′ Hospital's rule,

$$k = \lim_{x \to 0} \frac{2x - 2\sin x}{4x^3} \cdots \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{2 - 2\cos x}{12x^2} \cdots \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{2\sin x}{24x} = \frac{1}{12} \lim_{x \to 0} \frac{\sin x}{x} = \frac{1}{12} \cdot 1 = \boxed{\frac{1}{12}}$$

3)

If $x + y + z = u$, $y + z = v$ *and* $z = uvw$, *find the value of* $\frac{\partial(x, y, z)}{\partial(u, v, w)}$

☞ $$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

It is evident that we should have x, y, z in terms of u, v, w.

Consider $x + y + z = u$ . . . (1), $y + z = v$ . . . (2), $z = uvw$ . . . (3)

Using (2) in (1) we have, $x + v = u$ $\therefore x = u - v$

Also by using (3) in (2) we have, $y + uvw = v$ $\therefore y = v - uvw$

Hence, the given data is modified into the form,

$x = u - v,\ y = v - uvw,\ z = uvw$

Substituting for the partial derivatives, we have,

$$\therefore \quad J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1 & -1 & 0 \\ -vw & (1-uw) & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= 1\{(1-uw)uv-(uw)(-uv)\}+1\{(-vw)(uv)-(vw)(-uv)\}$$

$$= uv - u^2 vw + u^2 vw - uv^2 w + uv^2 w = uv;$$ Thus, $\boxed{J = uv}$

4)

Let $M = x^2 + y^2 + x$ and $N = xy$

$$\frac{\partial M}{\partial y} = 2y \text{ and } \frac{\partial N}{\partial x} = y$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2y - y = y \ldots$$ near to N.

Now, $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{y}{xy} = \frac{1}{x} = f(x)$

Hence, $I \cdot F = e^{\int f(x)\,dx} = e^{\int \frac{1}{x}\,dx} = e^{\log x} = x$

Multiplying the given equation by x, we now have,

$$M = x^3 + xy^2 + x^2 \text{ and } N = x^2 y$$

The solution is $\int M\,dx + \int N(y)\,dy = c$

ie., $\int (x^3 + xy^2 + x^2)\,dx + \int 0\,dy = c$ or $\boxed{\frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^3}{3} = c}$

5)

Solve the equation $(px - y)(py + x) = 2p$ *by reducing into Clairaut's form, taking the substitution* $X = x^2, Y = y^2$

☞ $X = x^2 \Rightarrow \frac{dX}{dx} = 2x$

$$Y = y^2 \Rightarrow \frac{dY}{dy} = 2y$$

Now, $p = \frac{dy}{dx} = \frac{dy}{dY}\frac{dY}{dX}\frac{dX}{dx}$ and let $P = \frac{dY}{dX}$

ie., $p = \frac{1}{2y}\cdot P\cdot 2x$ or $p = \frac{x}{y}P$. That is $p = \frac{\sqrt{X}}{\sqrt{Y}}P$

Consider, $(px-y)(py+x) = 2p$

ie., $\left[\frac{\sqrt{X}}{\sqrt{Y}}P\sqrt{X}-\sqrt{Y}\right]\left[\frac{\sqrt{X}}{\sqrt{Y}}P\sqrt{Y}+\sqrt{X}\right] = 2\frac{\sqrt{X}}{\sqrt{Y}}P$

or $\frac{(PX-Y)}{\sqrt{Y}}(P+1)\sqrt{X} = 2\frac{\sqrt{X}}{\sqrt{Y}}P$

ie., $(PX-Y)(P+1) = 2P$ or $PX-Y = \frac{2P}{P+1}$

ie., $Y = PX - \frac{2P}{P+1}$

This is in the Clairaut's form and hence the associated general solution is

$$Y = cX - \frac{2c}{c+1} \quad \text{or} \quad y^2 = cx^2 - \frac{2c}{c+1}$$

Thus the required general solution of the given equation is $\boxed{y^2 = cx^2 - \frac{2c}{c+1}}$

6)

☞ $V = \iiint dx\,dy\,dz$

$x/a + y/b + z/c = 1 \quad \therefore \; z = c(1 - x/a - y/b)$

If $z = 0$, then $x/a + y/b = 1 \;\therefore\; y = b(1 - x/a)$

If $z = 0$, $y = 0$ then $x = a$

$$\therefore \quad V = \int_{x=0}^{a}\int_{y=0}^{b(1-x/a)}\int_{z=0}^{c(1-x/a-y/b)} dz\,dy\,dx$$

$$V = \int_0^a \int_0^{b(1-x/a)} (z)_0^{c(1-x/a-y/b)} \, dy \, dx$$

$$V = \int_0^a \int_0^{b(1-x/a)} c(1-x/a-y/b) \, dy \, dx$$

$$V = c\int_0^a \left(y - \frac{x}{a}(y) - \frac{1}{b}\left(y^2/2\right)\right)_{y=0}^{b(1-x/a)} dx$$

$$V = c\int_0^a b(1-x/a) - \frac{x}{a} \cdot b(1-x/a) - \frac{1}{2b} b^2(1-x/a)^2 \, dx$$

$$V = bc\int_0^a 1 - x/a - \frac{x}{a} + x^2/a^2 - \frac{1}{2}\left(1 + x^2/a^2 - \frac{2x}{a}\right) dx$$

$$V = bc\left[x - \frac{1}{a}\frac{x^2}{2} - \frac{1}{a}x^2/2 + \frac{1}{a^2}\left(x^3/3\right) - \frac{1}{2}(x) - \frac{1}{2a^2}\left(x^3/3\right) + \frac{1}{2}\cdot\frac{2}{a}\left(x^2/2\right)\right]_0^a$$

$$V = bc\left[a - a + \frac{1}{3a^2}a^3 - \frac{a}{2} - \frac{1}{6a^2}a^3 + \frac{1}{2a}a^2\right]$$

$$V = abc\left[\frac{1}{3} - \frac{1}{6}\right] = abc\left[\frac{2-1}{6}\right] = \frac{abc}{6} \text{ cubic units.}$$

7)

shaded region - entire first quadrant ,

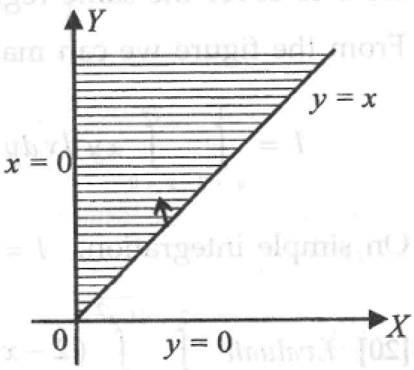
integral does not converge in this case since both limits are 0 to infinity

After changing order we get

$$\int_0^\infty \int_0^\infty \frac{e^{-y}}{y} dx dy$$

<u>or they can change question to</u> $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dy dx$

☞ $$I = \int_{x=0}^{\infty} \int_{y=x}^{\infty} \frac{e^{-y}}{y}\, dy\, dx$$



On changing the order we must have $y = 0$ to ∞ and $x = 0$ to y.

$$I = \int_{y=0}^{\infty} \int_{x=0}^{y} \frac{e^{-y}}{y}\, dx\, dy = \int_{y=0}^{\infty} \frac{e^{-y}}{y} [x]_0^y\, dy$$

$$I = \int_{y=0}^{\infty} \frac{e^{-y}}{y} \cdot y\, dy = \int_{y=0}^{\infty} e^{-y}\, dy = -[e^{-y}]_0^{\infty} = \boxed{1}$$

8)

Let, $$I_1 = \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \int_0^{\pi/2} \sin^{-1/2}\theta\, d\theta = \int_0^{\pi/2} \sin^{-1/2}\theta \cos^0\theta\, d\theta$$

and $I_2 = \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta\, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta \cos^0\theta\, d\theta$

We have, $\int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

Hence $I_1 = \frac{1}{2}\beta\left(\frac{-1/2+1}{2}, \frac{0+1}{2}\right) = \frac{1}{2}\beta\left(\frac{1}{4}, \frac{1}{2}\right)$

$I_2 = \frac{1}{2}\beta\left(\frac{1/2+1}{2}, \frac{0+1}{2}\right) = \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{2}\right)$

$\therefore \quad I_1 \times I_2 = \frac{1}{4}\beta(1/4, 1/2)\cdot\beta(3/4, 1/2)$

$I_1 \times I_2 = \frac{1}{4}\frac{\Gamma(1/4)\Gamma(1/2)}{\Gamma(3/4)}\cdot\frac{\Gamma(3/4)\cdot\Gamma(1/2)}{\Gamma(5/4)}$

This, $I_1 \times I_2 = \frac{1}{4}\Gamma(1/4)\sqrt{\pi}\cdot\frac{\sqrt{\pi}}{1/4\cdot\Gamma(1/4)} = \boxed{\pi}$