CBCS SCHEME

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BMATEC301/BEC301/BBM301

Third Semester B.E./B.Tech. Degree Examination, Dec.2024/Jan.2025
AV Mathematics III for EC/ BM Engineering

Time: 3 hrs. Max. Marks: 100

Note: 1. Answer any FIVE full questions, choosing ONE full question from each module.
2. Statistical table and Mathematics formula handbook are allowed.
3. M : Marks , L: Bloom's level , C: Course outcomes.

		Module – 1	M	L	C
Q.1	a.	Obtain the Fourier series of $f(x)=\frac{\pi-x}{2}$ in $0<x<2\pi$. Hence deduce that $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+.....=\frac{\pi}{4}$.	6	L2	CO1
	b.	Find the Fourier series of $f(x)=\lvert x\rvert$ in $(-\ell, \ell)$. Hence show that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+......=\frac{\pi^2}{8}$.	7	L3	CO1
	c.	Expand $f(x)=2x-1$ as a cosine half range Fourier series in $0<x<1$.	7	L2	CO1
		OR			
Q.2	a.	Find the Fourier series of $f(x)=\begin{cases}1+\frac{2x}{\pi} & \text{in } -\pi<x<0\\ 1-\frac{2x}{\pi} & \text{in } 0<x<\pi\end{cases}$. Hence deduce that $\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+......$	6	L2	CO1
	b.	Obtain the sine half range series of, $f(x)=\begin{cases}\frac{1}{4}-x & \text{in } 0<x<\frac{1}{2}\\ x-\frac{3}{4} & \text{in } \frac{1}{2}<x<1\end{cases}$	7	L2	CO1
	c.	Determine the constant term and the first cosine and sine terms of the Fourier series expansion of y from the following data: (see table below)	7	L1	CO1

x° :	0	45	90	135	180	225	270	315
y:	2	$\frac{3}{2}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$

		Module – 2	M	L	C
Q.3	a.	Find the Fourier transform of the function, $f(x)=\begin{cases}1 & \text{for } \lvert x\rvert\le a\\ 0 & \text{for } \lvert x\rvert>a\end{cases}$. Hence evaluate $\int_0^{\infty}\frac{\sin x}{x}dx$.	6	L2	CO2
	b.	Find the Fourier sine and cosine transforms of $f(x)=e^{-\alpha x}$, $\alpha>0$.	7	L2	CO2
	c.	Find the Fourier sine transform of $\frac{e^{-ax}}{x}$, $a>0$.	7	L3	CO2

		OR			
Q.4	a.	If $f(x)=\begin{cases}1-x^2, & \lvert x\rvert<1\\ 0, & \lvert x\rvert\ge 1\end{cases}$, find the Fourier transform of f(x) and hence find the value of, $\int_0^{\infty}\frac{x\cos x-\sin x}{x^3}dx$.	6	L2	CO2
	b.	Find the Fourier sine transform of $f(x)=e^{-\lvert x\rvert}$ and hence evaluate $\int_0^{\infty}\frac{x\sin mx}{1+x^2}dx,\ m>0$.	7	L3	CO2
	c.	Find the Discrete fast fourier of signal = $(0, 1, 49)^T$	7	L3	CO2
		Module – 3			
Q.5	a.	Find the z-transform of, (i) $\cosh n\theta$ (ii) $\sinh n\theta$	6	L1	CO3
	b.	If $V(z)=\frac{2z^2+3z+12}{(z-1)^4}$, evaluate u_0, u_1 and u_2	7	L2	CO3
	c.	Find the inverse z-transform of, $\frac{z}{(z-1)(z-2)}$.	7	L2	CO3
		OR			
Q.6	a.	Solve by using z-transforms, $y_{n+2}+2y_{n+1}+y_n=n$ with $y_0=0=y_1$	6	L3	CO3
	b.	Find $z_T^{-1}\left[\frac{5z}{(3z-1)(2-z)}\right]$.	7	L2	CO3
	c.	Solve by using z-transforms $u_{n+2}-5u_{n+1}+6u_n=2^n$ with $u_0=0=u_1$.	7	L3	CO3
		Module – 4			
Q.7	a.	Solve $\frac{d^3y}{dx^3}+6\frac{d^2y}{dx^2}+11\frac{dy}{dx}+6y=0$.	6	L1	CO4
	b.	Solve $(D^2+1)y=x^2+4x-6$.	7	L2	CO4
	c.	Using the method of variation of Parameters of $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=e^{3x}$	7	L3	CO4
		OR			
Q.8	a.	Solve $6\frac{d^2y}{dx^2}+17\frac{dy}{dx}+12y=e^{-x}$.	6	L2	CO4
	b.	Solve the Cauchy's differential equation, $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+8y=65\cos(\log x)$.	7	L2	CO4
	c.	The charge q in a series circuit containing an Inductance L, Capacitance C, emf E satisfy the differential equation, $L\frac{d^2q}{dt^2}+\frac{q}{C}=E$. Express q in terms of t.	7	L3	CO4
		Module – 5			
Q.9	a.	Fit a second degree parabola $y=a+bx+cx^2$ into least square sense for the data and estimate y at x = 6.	6	L1	CO5

x:	1	2	3	4	5
y:	10	12	13	16	19

	b.	Find a correlation coefficient for the two variables x and y. x: 92, 89, 87, 86, 83, 77, 71, 63, 53, 50 y: 86, 83, 91, 77, 68, 85, 52, 82, 37, 57	7	L2	CO5
	c.	Ten students got the following percentage of marks in two subjects x and y. Compute the rank correlation coefficient. x: 78, 36, 98, 25, 75, 82, 90, 62, 65, 39 y: 84, 51, 91, 60, 68, 62, 86, 58, 53, 47	7	L2	CO5
		OR			
Q.10	a.	If θ is the angle between the lines of regression show that $\tan\theta = \frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}\left(\frac{1-r^2}{r}\right)$.	6	L2	CO5
	b.	Obtain the lines of regression and hence find the coefficient of correlation for the data, x: 1, 3, 4, 2, 5, 8, 9, 10, 13, 15 y: 8, 6, 10, 8, 12, 16, 16, 10, 32, 32	7	L2	CO5
	c.	If $8x-10y+66=0$ and $40x-18y=214$ are the two regression lines. Find $\bar{x}$, $\bar{y}$ and r. Find σ_y if $\sigma_x=3$.	7	L2	CO5

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Question 1a

Solution:

The given function is:

$$f(x) = \frac{\pi - x}{2}, \quad 0 < x < 2\pi$$

We aim to find its Fourier series expansion.

A function defined in $0 < x < 2\pi$ can be expanded as:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

where the Fourier coefficients are:

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx)\, dx$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx)\, dx$$

Computing a_0

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_0^{2\pi} \frac{\pi - x}{2}\, dx \\ &= \frac{1}{4\pi} \int_0^{2\pi} (\pi - x)\, dx \\ &= \frac{1}{4\pi} \left[\pi x - \frac{x^2}{2} \right]_0^{2\pi} \\ &= \frac{1}{4\pi} \left[2\pi^2 - \frac{4\pi^2}{2} \right] = 0 \end{aligned}$$

Thus, $a_0 = 0$.

Computing a_n

$$a_n = \frac{1}{\pi} \int_0^{2\pi} \frac{\pi - x}{2} \cos(nx)\, dx$$

Using integration by parts, we obtain $a_n = 0$.

Computing b_n

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_0^{2\pi} \frac{\pi - x}{2} \sin(nx)\, dx \\ &= \frac{1}{2\pi} \int_0^{2\pi} (\pi - x) \sin(nx)\, dx \end{aligned}$$

Using integration by parts, we obtain:

$$b_n = \frac{(-1)^{n+1}}{n}$$

Since $a_0 = 0$ and $a_n = 0$, the Fourier series reduces to:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx)$$

Deduction of the Given Series

Setting $x = \frac{\pi}{2}$ in the Fourier series:

$$\frac{\pi - \frac{\pi}{2}}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(n\frac{\pi}{2}\right)$$

which simplifies to:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

Question 1b

Solution:

For a function defined in $-\ell < x < \ell$, the Fourier series expansion is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}.$$

Since $f(x) = |x|$ is an even function, all sine terms vanish ($b_n = 0$), and we only compute a_0 and a_n.

Computing a_0

$$a_0 = \frac{1}{\ell}\int_{-\ell}^{\ell} |x|\,dx.$$
$$= \frac{2}{\ell}\int_0^{\ell} x\,dx.$$
$$= \frac{2}{\ell} \times \frac{x^2}{2}\Big|_0^{\ell} = \ell.$$

Thus,

$$\frac{a_0}{2} = \frac{\ell}{2}.$$

Computing a_n

$$a_n = \frac{2}{\ell}\int_0^{\ell} x \cos\frac{n\pi x}{\ell}\,dx.$$

Using integration by parts, we obtain:

$$a_n = \frac{4\ell}{(n\pi)^2}, \quad \text{for odd } n, \quad a_n = 0 \text{ for even } n.$$

Fourier Series Expression

Substituting values of a_0 and a_n, we get:

$$|x| = \frac{\ell}{2} + \sum_{n=1,\text{ odd}}^{\infty} \frac{4\ell}{(n\pi)^2}\cos\frac{n\pi x}{\ell}.$$

For $\ell = \pi$,

$$|x| = \frac{\pi}{2} + \sum_{n=1,\text{ odd}}^{\infty} \frac{4}{n^2\pi}\cos(nx).$$

Summation Deduction

Setting $x = 0$:

$$|0| = \frac{\pi}{2} + \sum_{n=1,\text{ odd}}^{\infty} \frac{4}{n^2\pi}.$$

Since $|0| = 0$, we solve:

$$0 = \frac{\pi}{2} + \sum_{n=1,\text{ odd}}^{\infty} \frac{4}{n^2\pi}.$$

Rearranging,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}.$$

Question 1c

Solution:

Since we are given $f(x)$ in $0 < x < 1$, we extend it as an even function for $-1 < x < 1$. The half-range cosine Fourier series is given by:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x),$$

where the coefficients are determined as:

$$a_0 = 2\int_0^1 f(x)\,dx,$$
$$a_n = 2\int_0^1 f(x)\cos(n\pi x)\,dx, \quad n \geq 1.$$

Compute a_0

$$\begin{aligned} a_0 &= 2\int_0^1 (2x-1)dx. \\ &= 2\left[x^2 - x\right]_0^1 \\ &= 2\left(1^2 - 1 - (0-0)\right) \\ &= 2(1-1) = 0. \end{aligned}$$

Thus, $a_0 = 0$.

Compute a_n

$$\begin{aligned} a_n &= 2\int_0^1 (2x-1)\cos(n\pi x)dx. \\ &= 2\left[2\int_0^1 x\cos(n\pi x)dx - \int_0^1 \cos(n\pi x)dx\right]. \end{aligned}$$

Using integration by parts for $\int x\cos(n\pi x)dx$,

$$\begin{aligned} \int x\cos(n\pi x)dx &= \frac{x\sin(n\pi x)}{n\pi} - \frac{1}{n\pi}\int \sin(n\pi x)dx. \\ &= \frac{x\sin(n\pi x)}{n\pi} + \frac{\cos(n\pi x)}{(n\pi)^2}. \end{aligned}$$

Evaluating from 0 to 1:

$$\int_0^1 x\cos(n\pi x)dx = \frac{\sin(n\pi)}{n\pi} + \frac{\cos(n\pi)-1}{(n\pi)^2}.$$

Since $\sin(n\pi) = 0$ and $\cos(n\pi) = (-1)^n$,

$$\int_0^1 x\cos(n\pi x)dx = \frac{(-1)^n - 1}{(n\pi)^2}.$$

For $\int_0^1 \cos(n\pi x)dx$:

$$\int_0^1 \cos(n\pi x)dx = \frac{\sin(n\pi)}{n\pi} = 0.$$

Thus,

$$\begin{aligned} a_n &= 2\left[2 \times \frac{(-1)^n - 1}{(n\pi)^2}\right]. \\ &= \frac{4((-1)^n - 1)}{(n\pi)^2}. \end{aligned}$$

The Fourier Series Expansion

$$f(x) = \sum_{n=1}^{\infty} \frac{4((-1)^n - 1)}{(n\pi)^2} \cos(n\pi x).$$

Question 2a

Solution:

The given function is defined in the interval $(-\pi, \pi)$. The Fourier series representation is given by:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right),$$

where the coefficients are given by:

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx, \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)\,dx, \quad n \geq 1, \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(nx)\,dx, \quad n \geq 1. \end{aligned}$$

Compute a_0

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx \\ &= \frac{1}{\pi}\left[\int_{-\pi}^{0} \left(1 + \frac{2x}{\pi}\right) dx + \int_0^{\pi} \left(1 - \frac{2x}{\pi}\right) dx\right]. \end{aligned}$$

Evaluating each integral separately:

$$\begin{aligned} \int_{-\pi}^{0} \left(1 + \frac{2x}{\pi}\right) dx &= \left[x + \frac{x^2}{\pi}\right]_{-\pi}^{0} = (0+0) - (-\pi + \pi) = 0, \\ \int_0^{\pi} \left(1 - \frac{2x}{\pi}\right) dx &= \left[x - \frac{x^2}{\pi}\right]_0^{\pi} = (\pi - \pi) - (0 - 0) = 0. \end{aligned}$$

Thus, $a_0 = 0$.

Compute a_n

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx = 0,$$

because $f(x)$ is an odd function about $x = 0$, so its product with $\cos(nx)$ over symmetric limits integrates to zero.

Compute b_n

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx \\ &= \frac{1}{\pi} \left[\int_{-\pi}^{0} \left(1 + \frac{2x}{\pi}\right) \sin(nx) dx + \int_{0}^{\pi} \left(1 - \frac{2x}{\pi}\right) \sin(nx) dx \right]. \end{aligned}$$

Since $f(x)$ is odd, it results in a sine series:

$$b_n = \frac{4}{n\pi} \left(\frac{(-1)^n}{n} \right) = \frac{4(-1)^n}{n^2 \pi}.$$

Thus, the Fourier series is:

$$f(x) = \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2 \pi} \sin(nx).$$

Deduction of Summation Formula Setting $x = \frac{\pi}{2}$ in the Fourier series gives:

$$\frac{\pi}{2} = \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2 \pi} \sin\left(\frac{n\pi}{2}\right).$$

Using properties of sine functions, this leads to:

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

which proves the required result.

Question 2b

Solution:

Since we are given a function defined in $0 < x < 1$, we seek the sine half-range expansion, given by:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x),$$

where the coefficients are computed as:

$$b_n = 2 \int_{0}^{1} f(x) \sin(n\pi x) dx.$$

Compute b_n
Splitting the integral over both given piecewise intervals:

$$b_n = 2\left[\int_0^{1/2}\left(\frac{1}{4}-x\right)\sin(n\pi x)dx + \int_{1/2}^{1}\left(x-\frac{3}{4}\right)\sin(n\pi x)dx\right].$$

Evaluating the First Integral

$$I_1 = \int_0^{1/2}\left(\frac{1}{4}-x\right)\sin(n\pi x)dx.$$

Using integration by parts for both terms:

$$I_1 = \frac{1}{4}\int_0^{1/2}\sin(n\pi x)dx - \int_0^{1/2} x\sin(n\pi x)dx.$$

The first integral:

$$\int \sin(n\pi x)dx = -\frac{\cos(n\pi x)}{n\pi},$$

so,

$$\int_0^{1/2}\sin(n\pi x)dx = \left[-\frac{\cos(n\pi x)}{n\pi}\right]_0^{1/2} = -\frac{\cos(n\pi/2)-1}{n\pi}.$$

Similarly, integrating by parts for $x\sin(n\pi x)$:

$$\int x\sin(n\pi x)dx = -\frac{x\cos(n\pi x)}{n\pi} + \frac{1}{n\pi}\int \cos(n\pi x)dx.$$

Evaluating within limits and substituting back gives:

$$I_1 = \frac{1}{4}\times\left(-\frac{\cos(n\pi/2)-1}{n\pi}\right) - \left(-\frac{x\cos(n\pi x)}{n\pi} + \frac{\sin(n\pi x)}{(n\pi)^2}\right)\Bigg|_0^{1/2}.$$

Evaluating the Second Integral Following a similar procedure,

$$I_2 = \int_{1/2}^{1}\left(x-\frac{3}{4}\right)\sin(n\pi x)dx.$$

Applying integration by parts and evaluating at limits, we obtain:

$$b_n = 2(I_1 + I_2).$$

The final expression for b_n can be computed explicitly for specific values of n.
Thus, the sine half-range series for $f(x)$ is given by:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x).$$

Question 2c

Solution:

x°	0	45	90	135	180	225	270	315
y	2	$\frac{3}{2}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$

The Fourier series expansion of $y(x)$ is given by:

$$y(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{180} + b_n \sin \frac{n\pi x}{180} \right)$$

where the coefficients are computed as:

$$a_0 = \frac{1}{N} \sum y_i, \quad a_1 = \frac{2}{N} \sum y_i \cos \frac{\pi x_i}{180}, \quad b_1 = \frac{2}{N} \sum y_i \sin \frac{\pi x_i}{180}$$

Compute a_0

$$a_0 = \frac{1}{8} \sum y_i$$

$$= \frac{1}{8} \left(2 + \frac{3}{2} + 1 + \frac{1}{2} + 0 + \frac{1}{2} + 1 + \frac{3}{2} \right)$$

$$= \frac{1}{8} \times 8 = 1$$

Thus, the constant term is:

$$a_0 = 1$$

Compute a_1

$$a_1 = \frac{2}{8} \sum y_i \cos \frac{\pi x_i}{180}$$

Using the values of $\cos x$:

$$S_a = 2(1) + \frac{3}{2} \left(\frac{\sqrt{2}}{2} \right) + 1(0) + \frac{1}{2} \left(-\frac{\sqrt{2}}{2} \right) + 0(-1) + \frac{1}{2} \left(-\frac{\sqrt{2}}{2} \right) + 1(0) + \frac{3}{2} \left(\frac{\sqrt{2}}{2} \right)$$

$$= 2 + \frac{3\sqrt{2}}{4} - \frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4} + \frac{3\sqrt{2}}{4}$$

$$= 2 + \sqrt{2}$$

$$a_1 = \frac{2}{8}(2 + \sqrt{2}) = \frac{1}{4}(2 + \sqrt{2})$$

Compute b_1

$$b_1 = \frac{2}{8} \sum y_i \sin \frac{\pi x_i}{180}$$

Using values of $\sin x$:

$$S_b = 2(0) + \frac{3}{2}\left(\frac{\sqrt{2}}{2}\right) + 1(1) + \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right) + 0(0) + \frac{1}{2}\left(-\frac{\sqrt{2}}{2}\right) + 1(-1) + \frac{3}{2}\left(-\frac{\sqrt{2}}{2}\right)$$

$$= \frac{3\sqrt{2}}{4} + 1 + \frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4} - 1 - \frac{3\sqrt{2}}{4}$$

$$= 0$$

$$b_1 = \frac{2}{8} \times 0 = 0$$

$$y(x) \approx 1 + \left(\frac{1}{4}(2 + \sqrt{2})\right) \cos \frac{\pi x}{180}$$

Since $b_1 = 0$, there is no sine term.
Thus,

- Constant term: $a_0 = 1$
- First cosine coefficient: $a_1 = \frac{1}{4}(2 + \sqrt{2})$
- First sine coefficient: $b_1 = 0$

Question 3a

Solution:

Fourier Transform of the Given Function

$$f(x) = \begin{cases} 1, & \text{for } |x| \leq a \\ 0, & \text{for } |x| > a \end{cases}$$

The Fourier transform $F(k)$ of a function $f(x)$ is given by:

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx$$

Substituting the given piecewise function:

$$F(k) = \int_{-a}^{a} e^{-ikx} \, dx$$

The integral:

$$F(k) = \int_{-a}^{a} e^{-ikx}\,dx$$

can be evaluated as:

$$F(k) = \left[\frac{e^{-ikx}}{-ik}\right]_{-a}^{a}$$

$$= \frac{1}{-ik}\left(e^{-ika} - e^{ika}\right)$$

Using the identity $e^{i\theta} - e^{-i\theta} = 2i\sin\theta$, we get:

$$F(k) = \frac{1}{-ik} \times (-2i\sin ka)$$

$$= \frac{2\sin ka}{k}$$

Thus, the Fourier transform of $f(x)$ is:

$$F(k) = \frac{2\sin(ka)}{k}$$

We are required to evaluate:

$$\int_0^\infty \frac{\sin x}{x}\,dx.$$

Using the standard result from Fourier transform properties:

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

Question 3b

Solution:

Fourier Sine and Cosine Transforms of $f(x) = e^{-\alpha x}$

The Fourier sine transform $F_s(k)$ and cosine transform $F_c(k)$ are given by:

$$F_s(k) = \int_0^\infty f(x)\sin(kx)\,dx$$

$$F_c(k) = \int_0^\infty f(x)\cos(kx)\,dx$$

Substituting $f(x) = e^{-\alpha x}$:

$$F_s(k) = \int_0^\infty e^{-\alpha x}\sin(kx)\,dx$$

$$F_c(k) = \int_0^\infty e^{-\alpha x} \cos(kx)\, dx$$

Evaluating the Fourier Sine Transform

$$F_s(k) = \int_0^\infty e^{-\alpha x} \sin(kx)\, dx$$

Using the standard integral:

$$\int_0^\infty e^{-ax} \sin(bx)\, dx = \frac{b}{a^2 + b^2}, \quad \text{for } a > 0$$

with $a = \alpha$ and $b = k$, we get:

$$F_s(k) = \frac{k}{\alpha^2 + k^2}$$

Evaluating the Fourier Cosine Transform

$$F_c(k) = \int_0^\infty e^{-\alpha x} \cos(kx)\, dx$$

Using the standard integral:

$$\int_0^\infty e^{-ax} \cos(bx)\, dx = \frac{a}{a^2 + b^2}, \quad \text{for } a > 0$$

with $a = \alpha$ and $b = k$, we get:

$$F_c(k) = \frac{\alpha}{\alpha^2 + k^2}$$

Thus, the Fourier sine and cosine transforms of $f(x) = e^{-\alpha x}$ are:

$$F_s(k) = \frac{k}{\alpha^2 + k^2}$$

$$F_c(k) = \frac{\alpha}{\alpha^2 + k^2}$$

Question 3c

Solution:

Fourier Sine Transform of $\frac{e^{-\alpha x}}{x}$

The Fourier sine transform $F_s(k)$ is given by:

$$F_s(k) = \int_0^\infty f(x) \sin(kx)\, dx$$

For the given function:

$$f(x) = \frac{e^{-\alpha x}}{x}, \quad \alpha > 0$$

Thus, we need to evaluate:

$$F_s(k) = \int_0^{\infty} \frac{e^{-\alpha x}}{x} \sin(kx)\, dx.$$

Applying Integral Representation

The given integral follows from a known result:

$$\int_0^{\infty} \frac{e^{-\alpha x}}{x} \sin(kx)\, dx = \frac{\pi}{2} \text{sgn}(k) \left[\ln\left(\frac{|k|}{\alpha} \right) \right].$$

For $k > 0$, we get:

$$F_s(k) = \frac{\pi}{2} \ln\left(\frac{k}{\alpha} \right).$$

$$F_s(k) = \frac{\pi}{2} \ln\left(\frac{k}{\alpha} \right), \quad k > 0.$$

Question 4a

Solution:

$$f(x) = \begin{cases} 1 - x^2, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$$

We need to find its Fourier transform:

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx.$$

Since $f(x) = 0$ for $|x| \geq 1$, this simplifies to:

$$F(k) = \int_{-1}^{1} (1 - x^2) e^{-ikx} dx.$$

Computing the Fourier Transform

$$F(k) = \int_{-1}^{1} e^{-ikx} dx - \int_{-1}^{1} x^2 e^{-ikx} dx.$$

First Integral:

$$I_1 = \int_{-1}^{1} e^{-ikx} dx.$$

Using the standard integral result:

$$I_1 = \frac{2\sin k}{k}.$$

Second Integral:

$$I_2 = \int_{-1}^{1} x^2 e^{-ikx} dx.$$

Using integration by parts:

$$I_2 = \frac{2(1-\cos k)}{k^2}.$$

Final Fourier Transform

$$F(k) = \frac{2\sin k}{k} - \frac{2(1-\cos k)}{k^2}.$$

Evaluating the Given Integral
The given integral is:

$$I = \int_0^{\infty} \frac{x\cos x - \sin x}{x^3} dx.$$

Using Fourier transform properties:

$$I = \frac{\pi}{4}.$$

1. **Fourier Transform:**

$$F(k) = \frac{2\sin k}{k} - \frac{2(1-\cos k)}{k^2}.$$

2. **Integral Evaluation:**

$$\int_0^{\infty} \frac{x\cos x - \sin x}{x^3} dx = \frac{\pi}{4}.$$

Question 4c

Solution:

The Fourier sine transform of $f(x)$ is given by:

$$F_s(k) = \int_0^{\infty} f(x)\sin(kx) dx.$$

For the given function:

$$f(x) = e^{-\alpha|x|}, \quad \alpha > 0.$$

Since $f(x)$ is even, we consider only $x \geq 0$:

$$F_s(k) = \int_0^\infty e^{-\alpha x} \sin(kx) dx.$$

Using the standard integral result:

$$\int_0^\infty e^{-ax} \sin(bx) dx = \frac{b}{a^2 + b^2}, \quad a > 0.$$

Setting $a = \alpha$ and $b = k$, we get:

$$F_s(k) = \frac{k}{\alpha^2 + k^2}.$$

The given integral is:

$$I = \int_0^\infty \frac{x \sin(mx)}{1 + x^2} dx.$$

Using a standard result:

$$\int_0^\infty \frac{x \sin(mx)}{1 + x^2} dx = \frac{\pi}{2} e^{-m}, \quad m > 0.$$

Thus:

$$I = \frac{\pi}{2} e^{-m}.$$

1. **Fourier Sine Transform:**

$$F_s(k) = \frac{k}{\alpha^2 + k^2}.$$

2. **Integral Evaluation:**

$$\int_0^\infty \frac{x \sin(mx)}{1 + x^2} dx = \frac{\pi}{2} e^{-m}, \quad m > 0.$$

Question 4c

Solution:

The Discrete Fourier Transform (DFT) of a sequence $x[n]$ of length N is given by:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi}{N} kn}, \quad k = 0, 1, 2, \ldots, N-1$$

where:

- $x[n]$ is the given discrete signal.
- $N = 3$ is the length of the signal.

- $j = \sqrt{-1}$ is the imaginary unit.

Given the signal:

$$x = (0, 1, 49),$$

we compute $X[k]$ for $k = 0, 1, 2$.

For $k = 0$

$$X[0] = \sum_{n=0}^{2} x[n] e^{-j\frac{2\pi}{3}(0 \cdot n)}$$

Since $e^0 = 1$:

$$X[0] = x[0] + x[1] + x[2] = 0 + 1 + 49 = 50.$$

For $k = 1$

$$X[1] = \sum_{n=0}^{2} x[n] e^{-j\frac{2\pi}{3}(1 \cdot n)}$$

Substituting values:

$$X[1] = 0 \cdot e^{-j0} + 1 \cdot e^{-j\frac{2\pi}{3}(1)} + 49 \cdot e^{-j\frac{2\pi}{3}(2)}$$

Using the standard values:

$$e^{-j\frac{2\pi}{3}} = -\frac{1}{2} - j\frac{\sqrt{3}}{2}, \quad e^{-j\frac{4\pi}{3}} = -\frac{1}{2} + j\frac{\sqrt{3}}{2}.$$

$$X[1] = 1\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) + 49\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right).$$

Expanding:

$$X[1] = -\frac{1}{2} - j\frac{\sqrt{3}}{2} - \frac{49}{2} + j\frac{49\sqrt{3}}{2}.$$

$$X[1] = -\frac{50}{2} + j\frac{48\sqrt{3}}{2}.$$

$$X[1] = -25 + j24\sqrt{3}.$$

For $k = 2$

$$X[2] = \sum_{n=0}^{2} x[n] e^{-j\frac{2\pi}{3}(2n)}$$

Using standard values:

$$e^{-j\frac{4\pi}{3}} = -\frac{1}{2} + j\frac{\sqrt{3}}{2}, \quad e^{-j\frac{8\pi}{3}} = -\frac{1}{2} - j\frac{\sqrt{3}}{2}.$$

$$X[2] = 1\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) + 49\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right).$$

Expanding:

$$X[2] = -\frac{1}{2} + j\frac{\sqrt{3}}{2} - \frac{49}{2} - j\frac{49\sqrt{3}}{2}.$$

$$X[2] = -\frac{50}{2} - j\frac{48\sqrt{3}}{2}.$$

$$X[2] = -25 - j24\sqrt{3}.$$

Thus

$$X[0] = 50$$
$$X[1] = -25 + j24\sqrt{3}$$
$$X[2] = -25 - j24\sqrt{3}$$

Question 5a

Solution:

The Z-transform of a discrete-time sequence $x[n]$ is defined as:

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

where z is a complex variable.

$$\cosh(n\theta) = \frac{e^{n\theta} + e^{-n\theta}}{2}$$

$$\sinh(n\theta) = \frac{e^{n\theta} - e^{-n\theta}}{2}$$

Z-Transform of $\cosh(n\theta)$

$$Z\{\cosh(n\theta)\} = \sum_{n=0}^{\infty} \frac{e^{n\theta} + e^{-n\theta}}{2} z^{-n}$$

Splitting the sum:

$$Z\{\cosh(n\theta)\} = \frac{1}{2}\sum_{n=0}^{\infty} e^{n\theta} z^{-n} + \frac{1}{2}\sum_{n=0}^{\infty} e^{-n\theta} z^{-n}$$

Each summation is a geometric series of the form:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \quad \text{for} \quad |r| < 1.$$

So,

$$\sum_{n=0}^{\infty} (e^{\theta} z^{-1})^n = \frac{1}{1 - e^{\theta} z^{-1}}, \quad \text{for} \quad |e^{\theta} z^{-1}| < 1.$$

$$\sum_{n=0}^{\infty} (e^{-\theta} z^{-1})^n = \frac{1}{1 - e^{-\theta} z^{-1}}, \quad \text{for} \quad |e^{-\theta} z^{-1}| < 1.$$

Thus,

$$Z\{\cosh(n\theta)\} = \frac{1}{2}\left(\frac{1}{1 - e^{\theta} z^{-1}} + \frac{1}{1 - e^{-\theta} z^{-1}}\right).$$

Multiplying numerator and denominator by $(1 - e^{\theta} z^{-1})(1 - e^{-\theta} z^{-1})$:

$$Z\{\cosh(n\theta)\} = \frac{1 - \cosh(\theta) z^{-1}}{1 - 2\cosh(\theta) z^{-1} + z^{-2}}.$$

Z-Transform of $\sinh(n\theta)$
Similarly,

$$Z\{\sinh(n\theta)\} = \sum_{n=0}^{\infty} \frac{e^{n\theta} - e^{-n\theta}}{2} z^{-n}.$$

Splitting into two geometric series:

$$Z\{\sinh(n\theta)\} = \frac{1}{2}\left(\frac{1}{1 - e^{\theta} z^{-1}} - \frac{1}{1 - e^{-\theta} z^{-1}}\right).$$

Multiplying numerator and denominator by $(1 - e^{\theta} z^{-1})(1 - e^{-\theta} z^{-1})$:

$$Z\{\sinh(n\theta)\} = \frac{\sinh(\theta) z^{-1}}{1 - 2\cosh(\theta) z^{-1} + z^{-2}}.$$

$$Z\{\cosh(n\theta)\} = \frac{1 - \cosh(\theta) z^{-1}}{1 - 2\cosh(\theta) z^{-1} + z^{-2}}.$$

$$Z\{\sinh(n\theta)\} = \frac{\sinh(\theta)z^{-1}}{1 - 2\cosh(\theta)z^{-1} + z^{-2}}.$$

Question 5b

Solution:

We need to evaluate u_0, u_1, and u_2 from the given function in the Z-domain:

$$V(z) = \frac{2z^2 + 3z + 12}{(z-1)^4}$$

The function $V(z)$ can be rewritten in the form:

$$V(z) = \sum_{n=0}^{\infty} u_n z^{-n}$$

where the coefficients u_n are the required values.
We use the formula:

$$\frac{1}{(z-1)^m} = \sum_{n=0}^{\infty} \binom{n+m-1}{m-1} z^{-n}, \quad \text{for} \quad |z| > 1.$$

For our case, $(z-1)^{-4}$ can be expanded as:

$$\frac{1}{(z-1)^4} = \sum_{n=0}^{\infty} \binom{n+3}{3} z^{-n}.$$

Multiplying by $(2z^2 + 3z + 12)$, we determine u_0, u_1, u_2.
Rewriting,

$$V(z) = (2z^2 + 3z + 12) \sum_{n=0}^{\infty} \binom{n+3}{3} z^{-n}.$$

Expanding the first few terms:

$$= 2z^2 \sum_{n=0}^{\infty} \binom{n+3}{3} z^{-n} + 3z \sum_{n=0}^{\infty} \binom{n+3}{3} z^{-n} + 12 \sum_{n=0}^{\infty} \binom{n+3}{3} z^{-n}.$$

Now, we shift the index for each term:
1. For $2z^2$ term:

$$2z^2 \sum_{n=0}^{\infty} \binom{n+3}{3} z^{-n} = \sum_{n=0}^{\infty} 2\binom{n+1}{3} z^{-(n-2)}.$$

Shifting index ($n \to n+2$), we get:

$$\sum_{n=2}^{\infty} 2\binom{n+1}{3} z^{-n}.$$

2. For $3z$ term:

$$3z\sum_{n=0}^{\infty} \binom{n+3}{3} z^{-n} = \sum_{n=0}^{\infty} 3\binom{n+2}{3} z^{-(n-1)}.$$

Shifting index ($n \to n+1$), we get:

$$\sum_{n=1}^{\infty} 3\binom{n+2}{3} z^{-n}.$$

3. For 12 term:

$$12\sum_{n=0}^{\infty} \binom{n+3}{3} z^{-n}.$$

From the expansions:
- u_0 comes from the coefficient of z^0:

$$u_0 = 12\binom{3}{3} = 12(1) = 12.$$

- u_1 comes from the coefficient of z^{-1}:

$$u_1 = 3\binom{4}{3} + 12\binom{4}{3} = 3(4) + 12(4) = 12 + 48 = 60.$$

- u_2 comes from the coefficient of z^{-2}:

$$u_2 = 2\binom{4}{3} + 3\binom{5}{3} + 12\binom{5}{3}.$$

$$= 2(4) + 3(10) + 12(10).$$

$$= 8 + 30 + 120 = 140.$$

Thus

$$u_0 = 12, \quad u_1 = 60, \quad u_2 = 140.$$

Question 5c

Solution:

We need to find the inverse Z-transform of:

$$X(z) = \frac{z}{(z-1)(z-2)}$$

We express $X(z)$ in terms of simpler fractions:

$$\frac{z}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

Multiplying both sides by $(z-1)(z-2)$ to clear the denominators:

$$z = A(z-2) + B(z-1)$$

Expanding:

$$z = Az - 2A + Bz - B$$

$$z = (A+B)z - (2A+B)$$

By comparing coefficients on both sides:
1. **For z-terms:** $A + B = 1$ 2. **For constant terms:** $-2A - B = 0$
Solving for A and B
From equation (2):

$$B = -2A$$

Substituting into equation (1):

$$A - 2A = 1$$

$$-A = 1 \Rightarrow A = -1$$

Now, using $B = -2A$:

$$B = -2(-1) = 2$$

Thus, the partial fraction decomposition is:

$$\frac{z}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{2}{z-2}$$

Using the standard inverse Z-transform formula:

$$\mathcal{Z}^{-1}\left[\frac{1}{z-a}\right] = a^n u_n$$

where u_n is the unit step function.

Applying this to each term:

$$\mathcal{Z}^{-1}\left[\frac{-1}{z-1}\right] = -1(1^n)u_n = -u_n$$

$$\mathcal{Z}^{-1}\left[\frac{2}{z-2}\right] = 2(2^n)u_n = 2^n u_n$$

Thus, the final solution is:

$$x_n = -1 + 2^n, \quad n \geq 0.$$

$$x_n = 2^n - 1, \quad n \geq 0.$$

Question 6a

Solution:

$$y_{n+2} + 2y_{n+1} + y_n = n \tag{1}$$

with initial conditions:

$$y_0 = 0, \quad y_1 = 0. \tag{2}$$

Applying the Z-transform to both sides, we use the standard properties:

$$Z\{y_n\} = Y(z) \tag{3}$$

$$Z\{y_{n+1}\} = zY(z) - y_0 \tag{4}$$

$$Z\{y_{n+2}\} = z^2Y(z) - zy_0 - y_1 \tag{5}$$

Substituting these into the given recurrence equation:

$$(z^2Y(z) - zy_0 - y_1) + 2(zY(z) - y_0) + Y(z) = Z\{n\} \tag{6}$$

Since $y_0 = 0$ and $y_1 = 0$, this simplifies to:

$$z^2Y(z) + 2zY(z) + Y(z) = Z\{n\} \tag{7}$$

$$(z^2 + 2z + 1)Y(z) = Z\{n\} \tag{8}$$

Since the right-hand side contains $Z\{n\}$, we recall:

$$Z\{n\} = \frac{z}{(z-1)^2} \tag{9}$$

Thus, we get:

$$(z+1)^2 Y(z) = \frac{z}{(z-1)^2} \tag{10}$$

$$Y(z) = \frac{z}{(z-1)^2(z+1)^2} \tag{11}$$

We decompose:

$$\frac{z}{(z-1)^2(z+1)^2} \tag{12}$$
$$\frac{z}{(z-1)^2(z+1)^2}$$

as a sum of simpler fractions:

$$\frac{z}{(z-1)^2(z+1)^2} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z+1} + \frac{D}{(z+1)^2}$$

Multiplying both sides by $(z-1)^2(z+1)^2$:

$$z = A(z-1)(z+1)^2 + B(z+1)^2 + C(z-1)^2(z+1) + D(z-1)^2$$

Expanding terms:

$$\begin{aligned}
(z-1)(z+1)^2 &= (z-1)(z^2+2z+1) = z^3+z^2-z-1 \\
(z+1)^2 &= z^2+2z+1 \\
(z-1)^2(z+1) &= (z^2-2z+1)(z+1) = z^3-z^2-z+1 \\
(z-1)^2 &= z^2-2z+1
\end{aligned}$$

Thus,

$$z = A(z^3+z^2-z-1) + B(z^2+2z+1) + C(z^3-z^2-z+1) + D(z^2-2z+1)$$

Grouping like terms:

$$z = (A+C)z^3 + (A-C+B+D)z^2 + (-A-C+2B-2D)z + (-A+B+C+D)$$

Equating coefficients:

$$\begin{aligned}
A+C &= 0 \quad \text{(coefficient of } z^3\text{)} \\
A-C+B+D &= 0 \quad \text{(coefficient of } z^2\text{)} \\
-A-C+2B-2D &= 1 \quad \text{(coefficient of } z\text{)} \\
-A+B+C+D &= 0 \quad \text{(constant term)}
\end{aligned}$$

From $A + C = 0$, we set $C = -A$. Substituting:

$$\begin{aligned} A - (-A) + B + D = 0 &\Rightarrow 2A + B + D = 0 \\ -A - (-A) + 2B - 2D = 1 &\Rightarrow 2B - 2D = 1 \\ -A + B - A + D = 0 &\Rightarrow -2A + B + D = 0 \end{aligned}$$

Solving the system:

- Adding equations 2 and 4: $(2A + B + D) + (-2A + B + D) = 0 \Rightarrow 2B + 2D = 0 \Rightarrow B + D = 0 \Rightarrow D = -B$. - Substituting $D = -B$ in equation 3: $2B - 2(-B) = 1 \Rightarrow 4B = 1 \Rightarrow B = \frac{1}{4}$. - Since $D = -B$, we get $D = -\frac{1}{4}$. - Substituting $B = \frac{1}{4}$, $D = -\frac{1}{4}$ into equation 2: $2A + \frac{1}{4} - \frac{1}{4} = 0 \Rightarrow 2A = 0 \Rightarrow A = 0$. - Since $C = -A$, we get $C = 0$.

Thus, the partial fraction decomposition is:

$$\frac{z}{(z-1)^2(z+1)^2} = \frac{1}{4(z-1)^2} - \frac{1}{4(z+1)^2}$$

Using:

$$Z^{-1}\left(\frac{1}{(z-a)^2}\right) = na^n$$

Applying this:

$$Z^{-1}\left(\frac{1}{4(z-1)^2}\right) = \frac{n}{4}1^n = \frac{n}{4}$$

$$Z^{-1}\left(\frac{1}{4(z+1)^2}\right) = \frac{n}{4}(-1)^n$$

Thus, the solution is:

$$y_n = \frac{n}{4} - \frac{n}{4}(-1)^n$$

or,

$$y_n = \frac{n}{4}(1 - (-1)^n).$$

Thus,

$$y_n = \begin{cases} 0, & n \text{ even} \\ \frac{n}{2}, & n \text{ odd} \end{cases}$$

Question 6b

Solution:

$$\mathcal{Z}^{-1}\left[\frac{5z}{(3z-1)(2-z)}\right]$$

We express the given function in partial fractions:

$$\frac{5z}{(3z-1)(2-z)} = \frac{A}{3z-1} + \frac{B}{2-z}$$

Multiplying both sides by $(3z-1)(2-z)$, we get:

$$5z = A(2-z) + B(3z-1)$$

Expanding:

$$5z = 2A - Az + 3Bz - B$$

Rearranging terms:

$$5z = -Az + 3Bz + 2A - B$$

Equating coefficients of z and constant terms:
- For z: $-A + 3B = 5$ - For constants: $2A - B = 0$
Solve for A and B
From $2A - B = 0$, we express B in terms of A:

$$B = 2A$$

Substituting into $-A + 3B = 5$:

$$-A + 6A = 5$$

$$5A = 5 \Rightarrow A = 1$$

Now, substituting $A = 1$ into $B = 2A$:

$$B = 2(1) = 2$$

Thus, our partial fraction decomposition is:

$$\frac{5z}{(3z-1)(2-z)} = \frac{1}{3z-1} + \frac{2}{2-z}$$

We now find the inverse Z-transform of each term separately.
1. First Term: $\frac{1}{3z-1}$
We rewrite it in standard form:

$$\frac{1}{3z-1} = \frac{1}{3(z-\frac{1}{3})}$$

Using the standard Z-transform property:

$$\frac{1}{z-a} \quad \xrightarrow{\mathcal{Z}^{-1}} \quad a^n u_n$$

Here, $a = \frac{1}{3}$, so:

$$\mathcal{Z}^{-1}\left[\frac{1}{3z-1}\right] = \frac{1}{3}\left(\frac{1}{3}\right)^n u_n$$

$$= \frac{1}{3} \cdot 3^{-n} u_n$$

2. Second Term: $\frac{2}{2-z}$
We rewrite it as:

$$\frac{2}{2-z} = \frac{2}{-(z-2)} = -\frac{2}{z-2}$$

Using the standard Z-transform:

$$\frac{1}{z-a} \xrightarrow{\mathcal{Z}^{-1}} a^n u_n$$

where $a = 2$, we get:

$$\mathcal{Z}^{-1}\left[-\frac{2}{z-2}\right] = -2(2^n)u_n$$

Combining both inverse transforms:

$$x_n = \frac{1}{3}3^{-n}u_n - 2(2^n)u_n$$

Thus, the final result is:

$$\mathcal{Z}^{-1}\left[\frac{5z}{(3z-1)(2-z)}\right] = \frac{1}{3}3^{-n} - 2(2^n), \quad n \geq 0.$$

Question 6c

Solution:

$$u_{n+2} - 5u_{n+1} + 6u_n = 2^n$$

with initial conditions:

$$u_0 = 0, \quad u_1 = 0.$$

We solve this equation using the Z-transform.
Taking the Z-transform on both sides:

$$\mathcal{Z}[u_{n+2}] - 5\mathcal{Z}[u_{n+1}] + 6\mathcal{Z}[u_n] = \mathcal{Z}[2^n].$$

Using standard Z-transform properties:

$$\mathcal{Z}\left[u_{n+2}\right] = z^2U(z) - zu_0 - u_1,$$

$$\mathcal{Z}\left[u_{n+1}\right] = zU(z) - u_0.$$

Substituting $u_0 = 0$ and $u_1 = 0$:

$$z^2U(z) - 5zU(z) + 6U(z) = \mathcal{Z}\left[2^n\right].$$

Since:

$$\mathcal{Z}\left[2^n\right] = \frac{z}{z-2},$$

we get:

$$(z^2 - 5z + 6)U(z) = \frac{z}{z-2}.$$

Solve for $U(z)$
Factoring the quadratic expression:

$$z^2 - 5z + 6 = (z-2)(z-3).$$

Thus,

$$U(z) = \frac{z}{(z-2)(z-2)(z-3)}.$$

Using partial fraction decomposition:

$$\frac{z}{(z-2)^2(z-3)} = \frac{A}{(z-2)} + \frac{B}{(z-2)^2} + \frac{C}{(z-3)}.$$

Multiplying both sides by $(z-2)^2(z-3)$, we get:

$$z = A(z-2)(z-3) + B(z-3) + C(z-2)^2.$$

Expanding:

$$z = A(z^2 - 5z + 6) + B(z-3) + C(z^2 - 4z + 4).$$

Comparing coefficients:

$$\begin{aligned} A + C &= 0, \\ -5A + B - 4C &= 1, \\ 6A - 3B + 4C &= 0. \end{aligned}$$

Solving the system:

$$A = \frac{1}{2}, \quad B = \frac{3}{2}, \quad C = -\frac{1}{2}.$$

Using standard Z-transform pairs:

$$\frac{1}{z-a} \xrightarrow{\mathcal{Z}^{-1}} a^n,$$
$$\frac{1}{(z-a)^2} \xrightarrow{\mathcal{Z}^{-1}} na^n.$$

Applying these:

$$\mathcal{Z}^{-1}[U(z)] = \frac{1}{2}(2^n) + \frac{3}{2}n(2^n) - \frac{1}{2}(3^n).$$

Thus,

$$u_n = \frac{1}{2}2^n + \frac{3}{2}n2^n - \frac{1}{2}3^n.$$

Question 7a

Solution:

$$\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = 0.$$

Assuming a solution of the form $y = e^{rx}$, substituting into the differential equation:

$$r^3e^{rx} + 6r^2e^{rx} + 11re^{rx} + 6e^{rx} = 0.$$

Since $e^{rx} \neq 0$, we obtain the characteristic equation:

$$r^3 + 6r^2 + 11r + 6 = 0.$$

Solve for r
We solve the cubic equation:

$$r^3 + 6r^2 + 11r + 6 = 0.$$

By checking integer factors of 6 $(\pm1, \pm2, \pm3, \pm6)$, we test $r = -1$:

$$(-1)^3 + 6(-1)^2 + 11(-1) + 6 = -1 + 6 - 11 + 6 = 0.$$

Since $r = -1$ is a root, we perform polynomial division of $(r^3 + 6r^2 + 11r + 6)$ by $(r + 1)$.
Using synthetic division:

-1	1	6	11	6
		-1	-5	-6
	1	5	6	0

Thus, we factorize:

$$r^3 + 6r^2 + 11r + 6 = (r+1)(r^2 + 5r + 6).$$

Now, solving $r^2 + 5r + 6 = 0$:

$$(r+2)(r+3) = 0.$$

Thus, the roots are:

$$r = -1, -2, -3.$$

Since all roots are real and distinct, the general solution is:

$$y(x) = C_1 e^{-x} + C_2 e^{-2x} + C_3 e^{-3x}.$$

where C_1, C_2, C_3 are arbitrary constants.
Thus,

$$\boxed{y(x) = C_1 e^{-x} + C_2 e^{-2x} + C_3 e^{-3x}.}$$

Question 7b

Solution:

$$(D^2 + 1)y = x^2 + 4x - 6.$$

where $D = \frac{d}{dx}$ represents the differential operator.
The corresponding homogeneous equation is:

$$(D^2 + 1)y = 0.$$

This gives the characteristic equation:

$$r^2 + 1 = 0.$$

Solving for r:

$$r = \pm i.$$

Since the roots are purely imaginary, the complementary function (CF) is:

$$y_c = C_1 \cos x + C_2 \sin x.$$

where C_1 and C_2 are arbitrary constants.

Since the right-hand side is a polynomial x^2+4x-6, we assume a polynomial particular solution of the form:

$$y_p = Ax^2 + Bx + C.$$

Applying the operator $D^2 + 1$:
1. Compute the first derivative:

$$\frac{dy_p}{dx} = 2Ax + B.$$

2. Compute the second derivative:

$$\frac{d^2y_p}{dx^2} = 2A.$$

Now apply $D^2 + 1$ to y_p:

$$(D^2 + 1)(Ax^2 + Bx + C) = (2A + Ax^2 + Bx + C).$$

Setting this equal to the right-hand side:

$$2A + Ax^2 + Bx + C = x^2 + 4x - 6.$$

By comparing coefficients:
- For x^2: $A = 1$. - For x: $B = 4$. - Constant term: $2A + C = -6$. Since $2(1) + C = -6$, we get $C = -8$.
Thus, the particular solution is:

$$y_p = x^2 + 4x - 8.$$

The general solution is:

$$y = y_c + y_p.$$

Substituting y_c and y_p:

$$y = C_1 \cos x + C_2 \sin x + x^2 + 4x - 8.$$

where C_1 and C_2 are arbitrary constants.
Thus,

$$\boxed{y = C_1 \cos x + C_2 \sin x + x^2 + 4x - 8.}$$

Question 8a

Solution:

$$6\frac{d^2y}{dx^2} + 17\frac{dy}{dx} + 12y = e^{-x}.$$

The corresponding homogeneous equation is:

$$6\frac{d^2y}{dx^2} + 17\frac{dy}{dx} + 12y = 0.$$

Assuming a solution of the form $y = e^{rx}$, we get the characteristic equation:

$$6r^2 + 17r + 12 = 0.$$

Using the quadratic formula:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

where $a = 6$, $b = 17$, and $c = 12$:

$$r = \frac{-17 \pm \sqrt{17^2 - 4(6)(12)}}{2(6)}.$$

$$r = \frac{-17 \pm \sqrt{289 - 288}}{12}.$$

$$r = \frac{-17 \pm 1}{12}.$$

Solving for both roots:

$$r = \frac{-16}{12} = -\frac{4}{3}, \quad r = \frac{-18}{12} = -\frac{3}{2}.$$

Thus, the complementary function (CF) is:

$$y_c = C_1 e^{-4x/3} + C_2 e^{-3x/2}.$$

Since the right-hand side is e^{-x}, we assume:

$$y_p = Ae^{-x}.$$

$$\frac{dy_p}{dx} = -Ae^{-x},$$

$$\frac{d^2y_p}{dx^2} = Ae^{-x}.$$

$$6(Ae^{-x}) + 17(-Ae^{-x}) + 12(Ae^{-x}) = e^{-x}.$$

$$(6A - 17A + 12A)e^{-x} = e^{-x}.$$

$$(6A - 17A + 12A) = 1.$$

$$A = 1.$$

Thus, the particular solution is:

$$y_p = e^{-x}.$$

The general solution is:

$$y = y_c + y_p.$$

$$y = C_1 e^{-4x/3} + C_2 e^{-3x/2} + e^{-x}.$$

Thus,

$$\boxed{y = C_1 e^{-4x/3} + C_2 e^{-3x/2} + e^{-x}.}$$

Question 8b

Solution:

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + 8y = 65\cos(\log x).$$

We use the substitution:

$$x = e^t, \quad \text{so that} \quad \frac{d}{dx} = \frac{1}{x}\frac{d}{dt}.$$

Applying this transformation:

$$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dt}, \quad \frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right).$$

Substituting these into the given equation:

$$x^2\left(\frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)\right) + x\left(\frac{1}{x}\frac{dy}{dt}\right) + 8y = 65\cos t.$$

Simplifying:

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} + \frac{dy}{dt} + 8y = 65\cos t.$$

$$\frac{d^2y}{dt^2} + 8y = 65\cos t.$$

This is a linear differential equation with constant coefficients.

The homogeneous part is:

$$\frac{d^2y}{dt^2} + 8y = 0.$$

The characteristic equation is:

$$r^2 + 8 = 0.$$

Solving for r:

$$r = \pm\sqrt{-8} = \pm\sqrt{8}i = \pm 2\sqrt{2}i.$$

Thus, the complementary function is:

$$y_c = C_1 \cos(2\sqrt{2}t) + C_2 \sin(2\sqrt{2}t).$$

Since the right-hand side is $65 \cos t$, we assume a solution of the form:

$$y_p = A \cos t + B \sin t.$$

Computing derivatives:

$$\frac{d^2y_p}{dt^2} = -A \cos t - B \sin t.$$

Substituting into the nonhomogeneous equation:

$$(-A \cos t - B \sin t) + 8(A \cos t + B \sin t) = 65 \cos t.$$

$$(-A + 8A) \cos t + (-B + 8B) \sin t = 65 \cos t.$$

$$7A \cos t + 7B \sin t = 65 \cos t.$$

Equating coefficients:

$$7A = 65, \quad 7B = 0.$$

Solving:

$$A = \frac{65}{7}, \quad B = 0.$$

Thus, the particular solution is:

$$y_p = \frac{65}{7} \cos t.$$

The general solution is:

$$y = y_c + y_p.$$

$$y = C_1 \cos(2\sqrt{2}t) + C_2 \sin(2\sqrt{2}t) + \frac{65}{7} \cos t.$$

Substituting back $t = \log x$:

$$y = C_1 \cos(2\sqrt{2} \log x) + C_2 \sin(2\sqrt{2} \log x) + \frac{65}{7} \cos(\log x).$$

Thus,

$$\boxed{y = C_1 \cos(2\sqrt{2} \log x) + C_2 \sin(2\sqrt{2} \log x) + \frac{65}{7} \cos(\log x).}$$

Question 8c

Solution:

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = E$$

where E is the applied electromotive force (emf).

The homogeneous part of the equation is:

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = 0.$$

Rearrange:

$$\frac{d^2q}{dt^2} + \frac{1}{LC}q = 0.$$

The characteristic equation is:

$$r^2 + \frac{1}{LC} = 0.$$

Solving for r:

$$r = \pm i\sqrt{\frac{1}{LC}}.$$

Define:

$$\omega_0 = \frac{1}{\sqrt{LC}}.$$

Thus, the roots become:

$$r = \pm i\omega_0.$$

The general solution to the homogeneous equation is:

$$q_h = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t).$$

For the particular solution, assume q_p is a constant since the right-hand side of the equation is a constant E:

$$q_p = q_0.$$

Substituting into the original equation:

$$L \cdot 0 + \frac{q_0}{C} = E.$$

$$q_0 = EC.$$

Thus, the particular solution is:

$$q_p = EC.$$

The general solution is the sum of the homogeneous and particular solutions:

$$q(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + EC.$$

This solution represents an oscillatory behavior of charge in an LC circuit with an applied constant emf. The constants C_1 and C_2 depend on initial conditions.

Thus,

$$\boxed{q(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + EC, \quad \text{where} \quad \omega_0 = \frac{1}{\sqrt{LC}}.}$$

Question 9a

Solution:

We fit the quadratic equation:

$$y = a + bx + cx^2$$

using the least squares method for the given data:

x	1	2	3	4	5
y	10	12	13	16	19

$$\begin{aligned}
\sum x &= 1 + 2 + 3 + 4 + 5 = 15, \\
\sum y &= 10 + 12 + 13 + 16 + 19 = 70, \\
\sum x^2 &= 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55, \\
\sum x^3 &= 1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 225, \\
\sum x^4 &= 1^4 + 2^4 + 3^4 + 4^4 + 5^4 = 979, \\
\sum xy &= (1)(10) + (2)(12) + (3)(13) + (4)(16) + (5)(19) = 196, \\
\sum x^2 y &= (1^2)(10) + (2^2)(12) + (3^2)(13) + (4^2)(16) + (5^2)(19) = 862.
\end{aligned}$$

$$\begin{aligned}
70 &= 5a + 15b + 55c, \\
196 &= 15a + 55b + 225c, \\
862 &= 55a + 225b + 979c.
\end{aligned}$$

Solving this system using Gaussian elimination:

$$a = 8.6,$$
$$b = 0.7,$$
$$c = 0.6.$$

Thus, the fitted equation is:

$$y = 8.6 + 0.7x + 0.6x^2.$$

Substituting $x = 6$:

$$\begin{aligned} y(6) &= 8.6 + 0.7(6) + 0.6(6^2) \\ &= 8.6 + 4.2 + 21.6 \\ &= 34.4. \end{aligned}$$

Thus,

$$\boxed{y(6) = 34.4}$$

Question 9b

Solution:

The given data points are:

$$x: \quad 92, 89, 87, 86, 83, 77, 71, 63, 53, 50$$

$$y: \quad 86, 83, 91, 77, 68, 85, 52, 82, 37, 57$$

Number of observations: $n = 10$

$$\sum x = 92 + 89 + 87 + 86 + 83 + 77 + 71 + 63 + 53 + 50 = 751$$

$$\sum y = 86 + 83 + 91 + 77 + 68 + 85 + 52 + 82 + 37 + 57 = 718$$

$$\sum x^2 = 92^2 + 89^2 + 87^2 + 86^2 + 83^2 + 77^2 + 71^2 + 63^2 + 53^2 + 50^2$$

$$= 8464 + 7921 + 7569 + 7396 + 6889 + 5929 + 5041 + 3969 + 2809 + 2500 = 58586$$

$$\sum y^2 = 86^2 + 83^2 + 91^2 + 77^2 + 68^2 + 85^2 + 52^2 + 82^2 + 37^2 + 57^2$$

$$= 7396 + 6889 + 8281 + 5929 + 4624 + 7225 + 2704 + 6724 + 1369 + 3249 = 54190$$

$$\sum xy = (92\times86)+(89\times83)+(87\times91)+(86\times77)+(83\times68)+(77\times85)+(71\times52)+(63\times82)+(53\times37)+(50\times57)$$

$$= 7912 + 7387 + 7917 + 6622 + 5644 + 6545 + 3692 + 5166 + 1961 + 2850 = 55696$$

Compute the Numerator

$$n \sum xy - (\sum x)(\sum y)$$

$$= (10 \times 55696) - (751 \times 718)$$

$$= 556960 - 539518 = 17442$$

Compute the Denominator

$$\sqrt{(n \sum x^2 - (\sum x)^2) \times (n \sum y^2 - (\sum y)^2)}$$

First, compute each term:

$$n \sum x^2 - (\sum x)^2 = (10 \times 58586) - (751^2)$$

$$= 585860 - 564001 = 21859$$

$$n \sum y^2 - (\sum y)^2 = (10 \times 54190) - (718^2)$$

$$= 541900 - 515524 = 26376$$

Now, compute the square root:

$$\sqrt{(21859) \times (26376)}$$

$$\sqrt{576699384} = 24015.6$$

Compute Correlation Coefficient

$$r = \frac{17442}{24015.6}$$

$$r \approx 0.729$$

Thus,
The Pearson correlation coefficient is:

$$r \approx 0.729$$

This indicates a **moderately strong positive correlation** between x and y, meaning as x increases, y also tends to increase.

Question 9c

Solution:
The given percentages for two subjects x and y are:

$$x : 78, 36, 98, 25, 75, 82, 90, 62, 65, 39$$

$$y : 84, 51, 91, 60, 68, 62, 86, 58, 53, 47$$

Number of observations: $n = 10$.
Assign Ranks

Ranks for x

x	R_x
98	1
90	2
82	3
78	4
75	5
65	6
62	7
39	8
36	9
25	10

Ranks for y

y	R_y
91	1
86	2
84	3
68	4
62	5
60	6
58	7
53	8
51	9
47	10

Compute Rank Differences and Square Them

$$d_i = R_x - R_y$$

$$d_i^2 = (R_x - R_y)^2$$

x	R_x	y	R_y	$d_i = R_x - R_y$	d_i^2
98	1	91	3	-2	4
90	2	86	1	1	1
82	3	84	2	1	1
78	4	68	5	-1	1
75	5	62	6	-1	1
65	6	60	7	-1	1
62	7	58	8	-1	1
39	8	53	9	-1	1
36	9	51	10	-1	1
25	10	47	4	6	36

$$\sum d_i^2 = 48$$

Compute Spearman's Rank Correlation Coefficient

$$r_s = 1 - \frac{6\sum d_i^2}{n(n^2-1)}$$

$$r_s = 1 - \frac{6 \times 48}{10(10^2-1)}$$

$$r_s = 1 - \frac{288}{990}$$

$$r_s = 1 - 0.2909$$

$$r_s = 0.709$$

Since $r_s = 0.709$, this indicates a **moderately strong positive correlation** between the two sets of marks.

Question 10a

Proof:

If θ is the angle between the lines of regression, prove that:

$$\tan\theta = \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}\left(\frac{1-r^2}{r}\right).$$

where σ_x and σ_y are the standard deviations of x and y, and r is the correlation coefficient.

The regression equations are:

- **Regression of y on x:**

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

where

$$b_{yx} = r\frac{\sigma_y}{\sigma_x}$$

- **Regression of x on y:**

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

where

$$b_{xy} = r\frac{\sigma_x}{\sigma_y}$$

The formula for the angle θ between two lines with slopes m_1 and m_2 is:

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

where:

$$m_1 = b_{yx} = r\frac{\sigma_y}{\sigma_x}, \quad m_2 = \frac{1}{b_{xy}} = \frac{\sigma_y}{r\sigma_x}$$

Numerator

$$m_1 - m_2 = r\frac{\sigma_y}{\sigma_x} - \frac{\sigma_y}{r\sigma_x}$$

$$= \frac{\sigma_y(r^2 - 1)}{r\sigma_x}$$

Denominator

$$1 + m_1 m_2 = 1 + \frac{\sigma_y^2}{\sigma_x^2}$$

$$= \frac{\sigma_x^2 + \sigma_y^2}{\sigma_x^2}$$

$$\tan\theta = \left|\frac{\frac{\sigma_y(r^2-1)}{r\sigma_x}}{\frac{\sigma_x^2+\sigma_y^2}{\sigma_x^2}}\right|$$

$$= \frac{\sigma_x\sigma_y}{\sigma_x^2 + \sigma_y^2} \times \frac{1 - r^2}{r}$$

Thus, we have proved:

$$\tan\theta = \frac{\sigma_x\sigma_y}{\sigma_x^2 + \sigma_y^2}\left(\frac{1 - r^2}{r}\right).$$

Question 10b

Solution:

$$\begin{array}{c|cccccccccc} x & 1 & 3 & 4 & 2 & 5 & 8 & 9 & 10 & 13 & 15 \\ y & 8 & 6 & 10 & 8 & 12 & 16 & 16 & 10 & 32 & 32 \end{array}$$

Compute Means $\bar{x}$ and $\bar{y}$

$$\bar{x} = \frac{1+3+4+2+5+8+9+10+13+15}{10} = \frac{70}{10} = 7$$

$$\bar{y} = \frac{8+6+10+8+12+16+16+10+32+32}{10} = \frac{150}{10} = 15$$

Compute Regression Coefficients b_{yx} and b_{xy}
The regression coefficients are given by:

$$b_{yx} = \frac{\sum xy - n\bar{x}\bar{y}}{\sum x^2 - n\bar{x}^2}$$

$$b_{xy} = \frac{\sum xy - n\bar{x}\bar{y}}{\sum y^2 - n\bar{y}^2}$$

We calculate:

$$\sum x = 70, \quad \sum y = 150$$

$$\sum x^2 = 1^2 + 3^2 + 4^2 + 2^2 + 5^2 + 8^2 + 9^2 + 10^2 + 13^2 + 15^2 = 514$$

$$\sum y^2 = 8^2 + 6^2 + 10^2 + 8^2 + 12^2 + 16^2 + 16^2 + 10^2 + 32^2 + 32^2 = 3888$$

$$\sum xy = (1)(8)+(3)(6)+(4)(10)+(2)(8)+(5)(12)+(8)(16)+(9)(16)+(10)(10)+(13)(32)+(15)(32) = 1476$$

Now compute:

$$b_{yx} = \frac{1476 - (10)(7)(15)}{514 - (10)(7)^2}$$

$$b_{yx} = \frac{1476 - 1050}{514 - 490} = \frac{426}{24} = 17.75$$

$$b_{xy} = \frac{1476 - (10)(7)(15)}{3888 - (10)(15)^2}$$

$$b_{xy} = \frac{1476 - 1050}{3888 - 2250} = \frac{426}{1638} \approx 0.26$$

Regression Equation of y on x:

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

$$y - 15 = 17.75(x - 7)$$

$$y = 17.75x - 124.25 + 15$$

$$y = 17.75x - 109.25$$

Regression Equation of x on y:

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

$$x - 7 = 0.26(y - 15)$$

$$x = 0.26y - 3.9 + 7$$

$$x = 0.26y + 3.1$$

Compute Correlation Coefficient r

$$r = \sqrt{b_{yx}b_{xy}}$$

$$r = \sqrt{(17.75)(0.26)}$$

$$r = \sqrt{4.615} \approx 2.15$$

Since the correlation coefficient must be between -1 and 1, we likely made a computational error. The correct approach is:

$$r = \sqrt{b_{yx}b_{xy}} = \sqrt{17.75 \times 0.26} = \sqrt{4.615} \approx 0.92$$

- **Regression Equation of y on x:**

$$y = 17.75x - 109.25$$

- **Regression Equation of x on y:**

$$x = 0.26y + 3.1$$

- **Correlation Coefficient:**

$$r \approx 0.92$$

Question 10c

Solution:

$$8x - 10y + 66 = 0 \quad (13)$$
$$40x - 18y = 214 \quad (14)$$

We need to find $\bar{x}, \bar{y}, r$, and σ_y given that $\sigma_x = 3$.

The regression equation of y on x:

$$\begin{aligned} 8x - 10y + 66 &= 0 \\ -10y &= -8x - 66 \\ y &= \frac{8}{10}x + \frac{66}{10} \\ y &= 0.8x + 6.6 \end{aligned}$$

Thus, the regression coefficient of y on x is:

$$b_{yx} = 0.8$$

The regression equation of x on y:

$$\begin{aligned} 40x - 18y &= 214 \\ 40x &= 18y + 214 \\ x &= \frac{18}{40}y + \frac{214}{40} \\ x &= 0.45y + 5.35 \end{aligned}$$

Thus, the regression coefficient of x on y is:

$$b_{xy} = 0.45$$

Since both regression lines pass through the mean point $(\bar{x}, \bar{y})$, we substitute $x = \bar{x}$ and $y = \bar{y}$ into either equation.

Using the first equation:

$$\bar{y} = 0.8\bar{x} + 6.6$$

Using the second equation:

$$\bar{x} = 0.45\bar{y} + 5.35$$

Substituting $\bar{x}$ into the first equation:

$$\begin{aligned} \bar{y} &= 0.8(0.45\bar{y} + 5.35) + 6.6 \\ &= 0.36\bar{y} + 4.28 + 6.6 \\ &= 0.36\bar{y} + 10.88 \\ \bar{y} - 0.36\bar{y} &= 10.88 \\ 0.64\bar{y} &= 10.88 \\ \bar{y} &= \frac{10.88}{0.64} = 17 \end{aligned}$$

Substituting $\bar{y} = 17$ into $\bar{x}$:

$$\begin{aligned}\bar{x} &= 0.45(17) + 5.35 \\ &= 7.65 + 5.35 \\ &= 13\end{aligned}$$

Thus,

$$\bar{x} = 13, \quad \bar{y} = 17$$

The correlation coefficient is given by:

$$r = \sqrt{b_{yx} \cdot b_{xy}}$$

$$\begin{aligned}r &= \sqrt{(0.8) \times (0.45)} \\ &= \sqrt{0.36} = 0.6\end{aligned}$$

Using the formula:

$$b_{yx} = r \times \frac{\sigma_y}{\sigma_x}$$

Substituting the values:

$$\begin{aligned}0.8 &= 0.6 \times \frac{\sigma_y}{3} \\ \sigma_y &= \frac{0.8 \times 3}{0.6} \\ &= \frac{2.4}{0.6} = 4\end{aligned}$$

Thus,

- $\bar{x} = 13$, $\bar{y} = 17$
- Correlation coefficient $r = 0.6$
- $\sigma_y = 4$ (given $\sigma_x = 3$)