CBCS SCHEME

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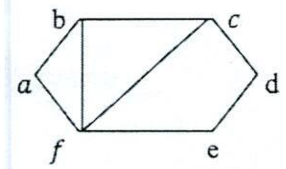
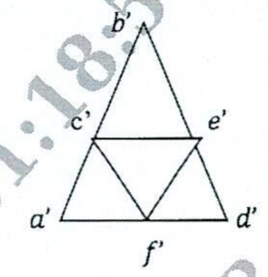
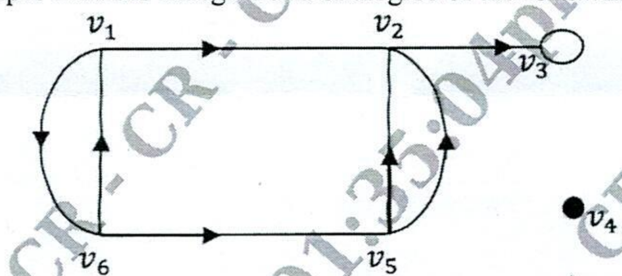
MMC102

First Semester MCA Degree Examination, Dec.2024/Jan.2025
Discrete Mathematics and Graph Theory

Time: 3 hrs. Max. Marks: 100

Note: 1. Answer any FIVE full questions, choosing ONE full question from each module.
2. M : Marks , L: Bloom's level , C: Course outcomes.

		Module – 1	M	L	C
Q.1	a.	If $U = \{1,2,3,4,5,6,7,8,9\}$, $A = \{1,2,3,7\}$, $B = \{4,5,6,7\}$ and $C = \{1,3,6\}$. Compute the following: (i) $\overline{A \cup C}$ (ii) $\bar{A} \cap \bar{B}$ (iii) $A \cap B \cap C$ (iv) $B - A$ (v) $A - B$	6	L1	CO1
	b.	Let $A = \{1,2,3,4,5,6\}$, $B = \{6,7,8,9,10\}$ and $f: A \to B$ be a function defined by $f = \{(1,7)(2,7)(3,8)(4,6)(5,9)(6,9)\}$. Determine $f^{-1}(6)$ and $f^{-1}(9)$. Also if $B_1 = \{7,8\}$, $B_2 = \{8,9,10\}$ then find $f^{-1}(B_1)$ and $f^{-1}(B_2)$.	7	L2	CO1
	c.	Find the eigen values and eigen vectors of the matrix $\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$.	7	L2	CO1
		OR			
Q.2	a.	For any two sets A and B, prove the Demorgan's laws.	6	L1	CO1
	b.	State pigeon-hole principle. Show that if 50 books in a library contain a total of 27551 pages, one of the books must have atleast 552 pages.	7	L2	CO1
	c.	In a class of 52 students, 30 are studying C++, 28 are studying pascal and 13 are studying both languages. How many in this class are studying at least one of these languages? How many are studying neither of these languages?	7	L2	CO1
		Module – 2			
Q.3	a.	Define tautology. Show that $[(p \vee q) \wedge (p \to r) \wedge (q \to r)] \to r$ is a tautology	7	L2	CO2
	b.	Write the converse, inverse and the contra positive of the conditional statement: "If oxygen is a gas then Gold is compound".	6	L2	CO2
	c.	Prove the following is valid argument : $p \to r$; $\sim p \to q$; $q \to s$; $\therefore \sim r \to s$	7	L2	CO2
		OR			
Q.4	a.	Prove the following using the laws of logic: $p \to (q \to r) \Leftrightarrow (p \wedge q) \to r$	7	L2	CO2
	b.	Negate and simplify: (i) $\forall x, [p(x) \wedge \sim q(x)]$. $\exists x, [\{p(x) \vee q(x)\} \to r(x)]$.	6	L2	CO2
	c.	Give the direct proof of the following statement "*If n is an odd integer, then n^2 is odd.*"	7	L2	CO2
		Module – 3			
Q.5	a.	Define graph and explain the types of graph.	8	L1	CO3

			M	L	C
	b.	Prove that the number of vertices of odd degree in a graph is always even.	6	L2	CO3
	c.	Define isomorphic graph and verify the following graphs are isomorphic or not.  	6	L2	CO3
		OR			
Q.6	a.	Explain the following graphs: (i) Bi- partite graph (ii) Sub graphs (iii) Walk (iv) Path	10	L1	CO3
	b.	Prove that a simple graph with n vertices and K components can have at most (n-k)(n-k+1)/2 edges.	10	L2	CO3
		Module – 4			
Q.7	a.	State and prove necessary condition of a graph to be a Euler graph.	10	L2	CO4
	b.	List and explain the different operations on graph.	10	L2	CO4
		OR			
Q.8	a.	Define digraph. Find the indegree and outdegree of the following graph: 	8	L2	CO4
	b.	Illustrate the travelling salesman problem using a graph.	6	L2	CO4
	c.	List and explain different digraphs and binary relations.	6	L2	CO4
		Module – 5			
Q.9	a.	Prove that every tree with two or more vertices is 2- Chromatic	10	L2	CO5
	b.	Explain the following for chromatic polynomial: (i) Finding a maximal independent set. (ii) Finding all maximal independent set.	10	L2	CO5
		OR			
Q.10	a.	Prove that the vertices of every planar graph can be properly colored with five colors.	10	L2	CO5
	b.	Explain the Greedy coloring algorithm.	10	L2	CO5

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MMC102

Discrete Mathematics & Graph Theory

1 a. (i) $A \cup C = \{1,2,3,7\} \cup \{1,3,6\} = \{1,2,3,6,7\}$

(ii) $\overline{A} \cap \overline{B} = \{8,9\}$

(iii) $A \cap B \cap C = \emptyset$

(iv) $B - A = \{4,5,6\}$

(v) $A - B = \{1,2,3\}$

b. $f^{-1}(6) = \{4\}$

$f^{-1}(9) = \{5,6\}$

$f^{-1}(B_1) = \{1,2,3\}$

$f^{-1}(B_2) = \{3,5,6\}$

c. $A = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$

Characteristic eqn.

$|A - \lambda I| = 0$

$\begin{vmatrix} 7-\lambda & 3 \\ 3 & -1-\lambda \end{vmatrix} = \lambda^2 - 6\lambda - 16 = 0$

$\lambda = 8, -2$

~~For $\lambda=8$~~ $\Rightarrow$ Consider $(A - \lambda I)X = 0$

For $\lambda = 8$

$-x + 3y = 0$

$\Rightarrow x = 3y$

$\frac{x}{3} = \frac{y}{1}$

$(3, 1)$

For $\lambda = -2$

$9x + 3y = 0$

$x = -\frac{1}{3}y$

$\frac{x}{-1} = \frac{y}{3}$

$(-1, 3)$

2 (a) (i) $\overline{A \cup B} = \bar{A} \cap \bar{B}$

consider $\bar{A} \cap \bar{B} = \{x \mid x \in \bar{A} \text{ and } x \in \bar{B}\}$

$= \{x \mid x \notin A \text{ and } x \notin B\}$

$= \{x \mid x \notin A \cup B\}$

$= \{x \mid x \in \overline{A \cup B}\}$

$= \overline{A \cup B}$

(ii) $\overline{A \cap B} = \bar{A} \cup \bar{B}$

$\bar{A} \cup \bar{B} = \{x \mid x \in \bar{A} \text{ or } x \in \bar{B}\}$

$= \{x \mid x \notin A \text{ or } x \notin B\}$

$= \{x \mid x \notin A \cap B\}$

$= \overline{A \cap B}$

(b) If m pigeons occupy n pigeonholes with $m > n$ then atleast one pigeon-hole must have $p+1$ or more pigeons in it where $p = \lfloor \frac{m-1}{n} \rfloor$.

Let $m = 27,551$, $n = 50$

$p = \lfloor \frac{m-1}{n} \rfloor = \lfloor \frac{27550}{50} \rfloor = 551$

2(c) ~~Let $|C|=30$~~

Let S be the set of all students. Let C & P be the set of all students studying C++ & pascal.

Given $|C|=30$, $|P|=28$, $|C\cap P|=13$

(i) $|C\cup P| = |C| + |P| - |C\cap P| = 45$

(ii) students studying neither:

$$|\overline{C\cup P}| = 52-45=7$$

3(a) Tautology is a compound statement which is always true, regardless of truth values of its components.

p	q	r	$p\vee q$	$p\to r$	$q\to r$	$\underbrace{(p\vee q)\wedge(p\to r)\wedge(q\to r)}_{①}$	$①\to r$
0	0	0	0	1	1	0	1
0	0	1	0	1	1	0	1
0	1	0	1	1	0	0	1
0	1	1	1	1	1	1	1
1	0	0	1	0	1	0	1
1	0	1	1	1	1	1	1
1	1	0	1	0	0	0	1
1	1	1	1	1	1	1	1

Since all the entries of last column are 1, it is a tautology.

3(b) Let p: Oxygen is a gas.

q: Gold is compound.

Given $p \to q$

Converse: $q \to p$

ie If Gold is a compound then oxygen is a gas.

Inverse: $\neg p \to \neg q$

ie., If Oxygen is not a gas then Gold is not compound

Contrative: $\neg q \to \neg p$

ie., If Gold is not a compound then Oxygen is not a gas.

3(c)

$$\begin{array}{l} p \to r \\ \neg p \to q \\ \underline{q \to s} \\ \therefore \neg r \to s \end{array} \quad \Rightarrow \quad \begin{array}{l} p \to r \\ \underline{\neg p \to s} \\ \therefore \neg r \to s \end{array}$$

Rule of syllogism

$$\Leftrightarrow \quad \begin{array}{l} \neg r \to \neg p \\ \underline{\neg p \to s} \\ \therefore \neg r \to s \end{array}$$

$p \to r \Leftrightarrow \neg r \to \neg p$.

Valid by rule of syllogism.

4(a) LHS $= p \to (q \to r)$

$\neg p \vee (q \to r)$

$\neg p \vee (\neg q \vee r)$

$(\neg p \vee \neg q) \vee r$

$\neg (p \wedge q) \vee r$

$(p \wedge q) \to r \quad = \text{RHS}$

4(b) (i) Negn: $\neg \{ \forall x, [p(x) \wedge q(x)] \}$

$\Leftrightarrow \exists x, \neg [p(x) \wedge q(x)]$

$\Leftrightarrow \exists x, \neg p(x) \vee \neg q(x)$

(ii) Negn : $\neg \{ \exists x, [\{p(x) \vee q(x)\} \to r(x)] \}$

$\forall x, \neg [\{p(x) \vee q(x)\} \to r(x)]$

$\forall x, \{p(x) \vee q(x)\} \wedge \neg r(x)$

4(c) Let p: n is an odd int.

q: n^2 is odd

Given $p \to q$

Assume p is true.

$\Rightarrow$ n is odd.

$\Rightarrow n = 2k+1 \qquad k \in Z$

$$n^2 = (2k+1)^2$$
$$= 4k^2 + 1 + 4k$$
$$= 2(2k^2+2k)+1 = 2l+1 \quad \text{which is odd}$$

$\Rightarrow q$ is true.

Hence $p \rightarrow q$ is true.

5(a) Two graphs $G_1 = (V_1, E_1)$ & $G_2 = (V_2, E_2)$ are said to be isomorphic if there exists a bijection $f : V_1 \rightarrow V_2$ such that $(u, v) \in E_1 \Rightarrow (f(u), f(v)) \in E_2$

Here, G_1 and G_2 have 6 vertices.

G_1 has 8 edges. But G_2 has 9 edges.

$\therefore$ G_1 & G_2 are not isomorphic.

5(b) Consider a graph with n vertices. Suppose k of these vertices are of odd degree so that remaining $n-k$ vertices are of even degree. Denote the vertices with odd degree by $v_1, v_2 \dots v_k$ and vertices with even deg by $v_{k+1}, v_{k+2} \dots v_n$.

$$\sum_{i=1}^{n} \deg(v_i) = \sum_{i=1}^{k} \deg(v_i) + \sum_{i=k+1}^{n} \deg(v_i) \quad \text{—(1)}$$

By Handshaking property,

Sum in the LHS of (1) is even.

~~Sum~~ second term in the RHS is even.

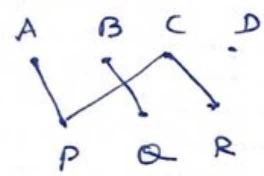
So, first term in the RHS must also be even.

This can happen only if the no of the vertices with odd degree is even.

6(a) (i) Bipartite Graph:

A bipartite graph is a graph whose vertex set can be divided into two disjoint sets U and V such that every edge ~~contains~~ connects a vertex in U to a vertex in V and no edge exists between vertices of the same set.

Eg:-

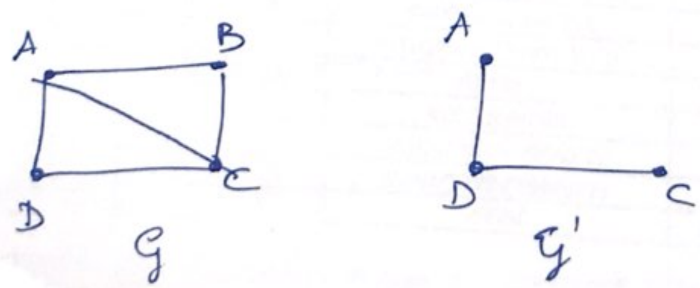


$V_1 = \{A, B, C, D\}$

$V_2 = \{P, Q, R\}$ are bipartite

(ii) Subgraph: A subgraph is a portion of a graph consisting of a subset of its vertices & edges. If $G = (V, E)$ then a subgraph $G' = (V', E')$ iff $V' \subseteq V$, $E' \subseteq E$.

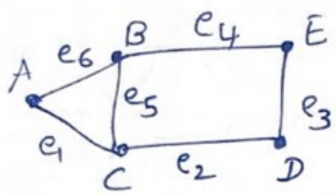
Eg:-



G' is a subgraph of G.

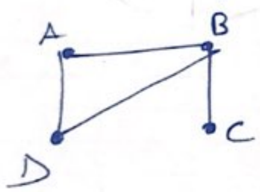
(iii) Walk: A walk in graph is a sequence of vertices and eges: ~~$v_0, v_1, v_2 \ldots v_k$~~

$v_0, e_1, v_1, e_2, v_2, \ldots v_k$. Vertices & edges can repeat in a walk.



$A e_1 C e_2 D e_3 E e_4 B e_5 C$ is a walk.

(iv) Path: A path is a walk in which no vertices are repeated. It is the simplest way of connecting two nodes without revisiting.



ABC is a path

6(b) A component is a connected subgraph. To maximize the edges in such a graph, each component should be a complete graph. If the graph as k components & n vertices then we partition n vertices into k parts such that sum of edges on each part maximized.

One component has $n-k+1$ vertices.

Remaining $k-1$ components have 1 vertex each.

$\therefore$ Max no of edges $= \dfrac{(n-k+1)(n-k)}{2}$

7(a) A graph is called an Euler graph if it contains an Euler circuit.

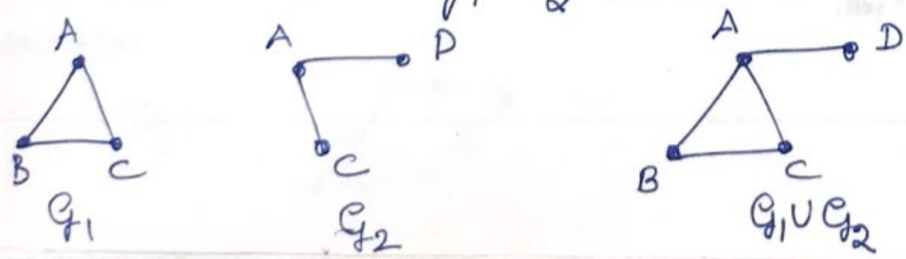
" A connected graph is an Euler graph if and only if the degree of every vertex is even."

<u>Proof:</u> Consider an Euler Circuit starting and ending at a vertex v. Every time the circuit enters a vertex, it must also leave it. So, the edges at each vertex must come in pairs — one for entry & one for exit. Hence each vertex is of even degree. If it visits that vertex again through 3^{rd} edge, it can not leave with any of these three edges (as no edge is repeated in a circuit) & so it has to leave through 4^{th} edge.

Hence, degree of every vertex is even.

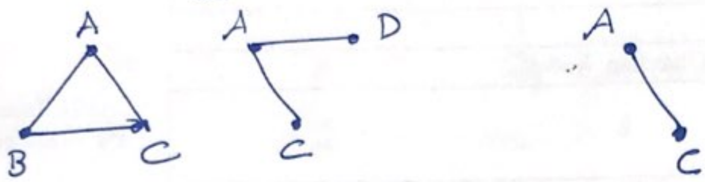
7(b) <u>Operations on Graphs:</u>

1. Union : If $G_1(V_1, E_1)$ & $G_2(V_2, E_2)$ are two graphs then their union is $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$



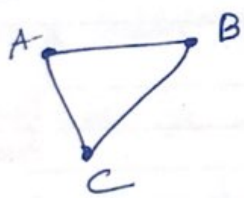
2. Intersection :

$$G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$$



G_1 G_2 $G_1 \cap G_2$

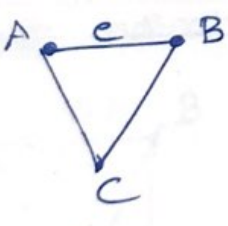
3. Vertex Deletion: Removing a vertex and all edges incident on it.



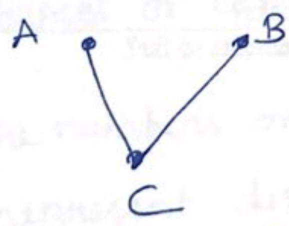
G

$G - \{B\}$

4. Edge Deletion : Removing a specific edge from a graph.



G



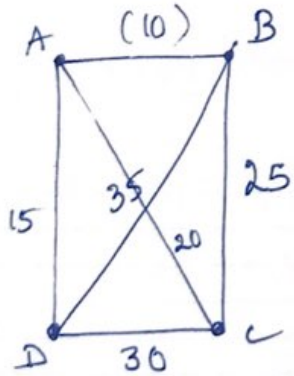
$G - \{e\}$

8(a) A digraph is a type of graph in which each edge has a direction.

Eg :-

vertex	In-degree	Out-degree
v_1	1	2
v_2	3	1
v_3	2	1
v_4	0	0
v_5	1	2
v_6	1	2

8b. Travelling Salesman Problem is a classic optimization problem where a salesman must visit each city exactly once, return to the starting city and do so with the minimum possible total distance (cost). Represent cities as vertices, paths between cities as edges with weights indicating distances or costs.



The numbers on the edges represent distances.

9(a) Out of syllabus

9(b) A maximal independent set is an independent set that can not be extended by including any adjacent vertex.

(i) To find one maximal independent set:

Start with an independent set S.

Pick any vertex v from the graph.

Add v to S and remove v and all its neighbors from the graph.

Repeat with the remaining graph until no vertices are left.

The set S is a maximal independent set.

(ii) To find all maximal independent set:

1. ~~Use back~~ Choose a vertex to include in the current set and exclude its neighbors.

2. Continue until no more vertices can be added.

3. Each path that leads to a dead-end gives one maximal independent set.

4- Collect all such paths to get all maximal sets.

10 (a) Let's use Mathematical Induction on the no of vertices in a planar graph.

Basis step: For small graphs (with ≤ 5 vertices), the statement is clearly true. Each vertex can be colored with a different color.

Induction step:

Assume every planar graph with n vertices can be colored using five colors.

Let G be a planar graph with n+1 vertices.

Use Euler's formula for planar graphs.

$$v - e + f = 2$$

Using this, it can be shown that every planar graph has atleast one vertex of degree ≤ 5.

So, remove a vertex v with degree ≤ 5 from G. Let the resulting graph be G' with n vertices.

By assumption, G' can be colored with five colors.

Now add back vertex V. Since v as atmost 5 neighbors & only 5 colors are used, if fewer than 5 colors are used among neighbors then assign the unused color to V.

If all 5 colors are used, we apply a color rearrangem
-ent trick to free up a color for v.

10(b) 1. Order the vertices – Lable the vertices in some order.

2. Initialize colors : Assign the first available color to each vertex.

3. Coloring rule : While coloring a vertex, choose the smallest numbered color that has not been assigned to its adjacent vertices.

4. Repeat until all vertices are colored.