CMR INSTITUTE OF TECHNOLOGY	USN											CMRIT

Internal Assessment Test – II May 2025

Sub:	Discrete Mathematical Structures							Code:	BCS405A
Date:	23/05/2025	Duration:	90 mins	Max Marks:	50	Sem:	IV	Branch:	AIDS/CSDS/ISE/AIML/CSML

Question 1 is compulsory and answer any 6 from the remaining questions.

		Marks	OBE CO	OBE RBT
1	Let $A = \{1, 2, 3, 4, 6, 12, 18\}$ and R be a relation on A defined by aRb iff "a divides b". Write down the relation R, relation matrix M(R), draw its digraph, list out its indegree and outdegree and draw its Hasse diagram.	[8]	CO3	L3
2	State Pigeon-hole Principle. Prove that if 30 dictionaries in a library contain a total of 61,327 pages, then at least one of the dictionaries mush have at least 2045 pages.	[7]	CO3	L2
3	Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5, 6\}$. Find the number of one-to-one functions and onto functions from A to B. Let $f: R \to R$ be defined by, $f(x) = \begin{cases} 3x - 5, if\ x > 0 \\ 1 - 3x, if\ x \leq 0 \end{cases}$ find $f^{-1}([-6, 5])$ and $f^{-1}([-5, 5])$.	[3+4]	CO3	L2
4	Five teachers T_1, T_2, T_3, T_4, T_5 are to be made class teachers for five classes, C_1, C_2, C_3, C_4, C_5, one teacher for each class. T_1 and T_2 do not wish to become the class teachers for C_1 or C_2, T_3 and T_4 for C_4 or C_5, and T_5 for C_3 or C_4 or C_5. In how many ways can the teachers be assigned the work (without displeasing any teachers)?	[7]	CO4	L3

5	In how many ways can the 26 letters of the English alphabet be permuted so that none of the patterns CAR, DOG, PUN or BYTE occurs?	[7]	CO4	L2
6	Solve the recurrence relation $a_n = 2(a_{n-1} - a_{n-2})$, for $n \geq 2$, given that $a_0 = 1$ and $a_1 = 2$.	[7]	CO4	L2
7	Define Group. Show that fourth roots of unity is an abelian group under the operation multiplication.	[7]	CO5	L2
8	State and prove Lagrange's theorem.	[7]	CO5	L2

IAT-02

1. Let $A = \{1, 2, 3, 4, 6, 12, 18\}$

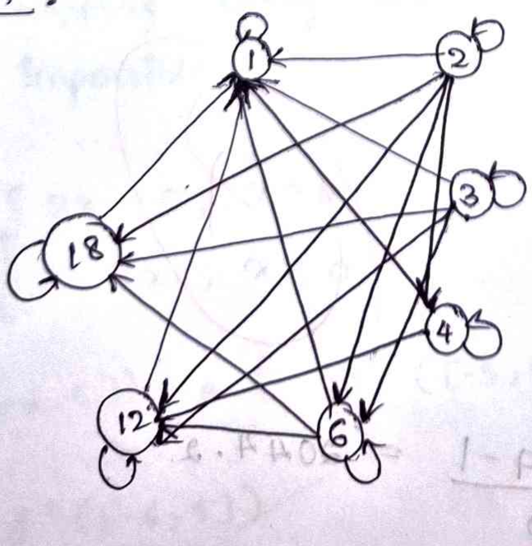
aRb = "a divides b".

$R = \{(1,1), (2,1), (3,1), (4,1), (6,1), (12,1), (18,1), (2,4), (2,2), (2,6), (2,12), (2,18), (3,3), (4,4), (6,6), (12,12), (18,18), (3,6), (3,12), (3,18), (4,12), (6,12), (6,18)\}$

Matrix representation

$$
M = \begin{array}{c|ccccccc}
 & 1 & 2 & 3 & 4 & 6 & 12 & 18 \\
\hline
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
3 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
4 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
6 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
12 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
18 & 1 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

Diagraph :-



No.	Indegree	Out degree
1	1	6
2	2	4
3	2	3
4	3	1
6	4	2
12	6	0
18	5	0

Hasse Diagram

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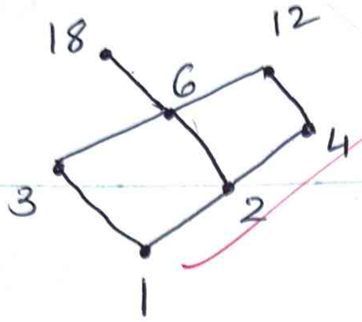

Qn 2

Pigeon hole principle states that if there are m pigeons and n pigeon holes and $m \geq n$ then atleast 1 pigeon ~~hold~~ hole will have more than 1 pigeon.

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Given,

total number of pages = m = 61327

number of dictionary = n = 30

by pigeon hole principle,

$$p = \left\lfloor \frac{m-1}{n} \right\rfloor = \left\lfloor \frac{61327-1}{30} \right\rfloor = \lfloor 2044.2 \rfloor = 2044$$

3) $A = \{1, 2, 3, 4\}$ $B = \{1, 2, 3, 4, 5, 6\}$

No. of one-to-one functions is $\frac{n!}{(n-m)!}$

$|A| = 4$ $|B| = 6$

$\Rightarrow \frac{6!}{(6-4)!} = \frac{6!}{2!} = \frac{720}{2} = 360$

$\therefore$ No. of one-to-one functions are 360.

No. of onto functions $= \sum_{k=0}^{n} (-1)^k \, {}^nC_k (n-k)^m$

Here, $m = 4$ & $n = 6$

It is ~~impossible~~ to map each element of m to each element of n.

$\therefore$ No. of ~~o~~onto functions are 0

Given $f(x) = \begin{cases} 3x-5, & \text{if } x > 0 \\ 1-3x, & \text{if } x \le 0 \end{cases}$

$f^{-1}([-6, 5])$

$\Rightarrow -6 \le 3x-5 \le 5$

$\Rightarrow -1 \le 3x \le 10$

$\Rightarrow \frac{-1}{3} \le x \le \frac{10}{3} \rightarrow (1)$

$-6 \le 1-3x \le 5$

$-7 \le -3x \le 4$

$\frac{7}{3} \ge x \ge -\frac{4}{3}$

$-\frac{4}{3} \le x \le \frac{7}{3} \rightarrow (2)$

from (1) & (2)

$\Rightarrow \boxed{-\frac{1}{3} \le x \le \frac{10}{3}}$ for $f^{-1}([-6, 5])$

$f^{-1}([-5,5])$

$f^{-1}(x) \Rightarrow -5 \le 3x-5 \le 5$

$0 \le 3x \le 10$

$0 \le x \le \frac{10}{3} \rightarrow (1)$

$\Rightarrow -5 \le 1-3x \le 5$

$\Rightarrow -6 \le -3x \le 4$

$\Rightarrow -2 \le -x \le \frac{+4}{3}$ $\quad -\frac{4}{3} \le x \le 2 \rightarrow (2)$

(or)

$\Rightarrow 2 \ge x \ge -\frac{4}{3} \rightarrow (2)$

from (1) & (2)

$0 \le x \le \frac{10}{3}$ & $-\frac{4}{3} \le x \le 2$

$\Rightarrow \boxed{-\frac{4}{3} \le x \le \frac{10}{3} \text{ for } f^{-1}([-5,5])}$

4) Given T_1, T_2, T_3, T_4, T_5 & C_1, C_2, C_3, C_4, C_5

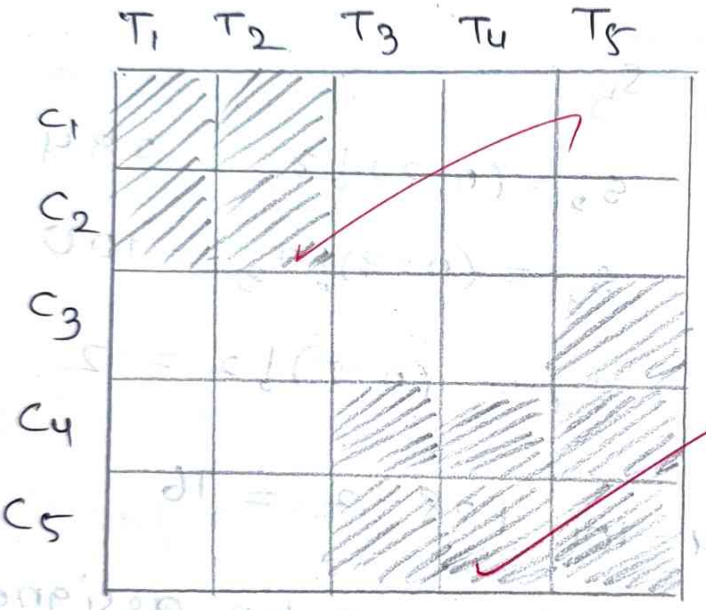
$n = 5$ & $m = 5$.

<u>Forbidden places</u> :—

T_1 & $T_2 \longrightarrow (\times)\ C_1, C_2$

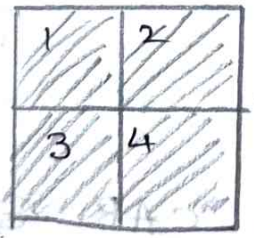
T_3 & $T_4 \longrightarrow (\times)\ C_4, C_5$

$T_5 \longrightarrow (\times)\ C_3, C_4, C_5$



Board-1

<u>$r(C_1, x)$</u> :—



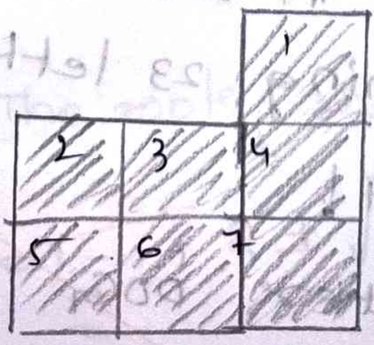
$r_1 = 4 = (n)$

$r_2 = 2$

$r(C_1, x) = 1 + 4x + x^2$

<u>Board-2</u>

<u>$r(C_2, x)$</u>:—



$r_1 = 7 = n$

$r_2 = 10$

$r_3 = 2$

$r(C_2, x) = 1 + 7x + 10x^2 + 2x^3$

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$R(C_1x) = 1 + 4x + 2x^2$

$R(C_2x) = 1 + 7x + 9x^2 + 2x^3$

$$R(Cx) = R(C_1x) \times R(C_2x)$$

$$= (1 + 4x + 2x^2)(1 + 7x + 9x^2 + 2x^3)$$

$$= \cancel{1 + 11x + 11x^2 + 2x^3}$$

$$= 1 + 7x + 9x^2 + 2x^3 + 4x + 28x^2 + 36x^3 + 8x^4 +$$

$$2x^2 + 14x^3 + 18x^4 + 4x^6$$

$$= 1 + 11x + 39x^2 + 52x^3 + 26x^4 + 4x^6$$

Let $r_1 = 11$, $r_2 = 39$, $r_3 = 52$, $r_4 = 26$, $r_5 = 4$.

$S_n = (n-k)!\, r_n$

$S_0 = 5! = 120$

$S_1 = (5-1)! \times 11 = 264$

$S_2 = (5-2)! \times 39 = 234$

$S_3 = (5-3)! \times 52 = 104$

$S_4 = (5-4)! \times 26 = 26$

$S_5 = (5-5)! \times 4 = 0$

$N_0 = S_0 - S_1 + S_2 - S_3 \cdots + S_n$

$= S_0 - S_1 + S_2 - S_3 + S_4 - S_5$

$= 120 - 264 + 234 - 104 + 26 - 0$

$= 8$

$\boxed{\therefore \ N_0 = 8}$

5) Given total letters = 26

let us Assume a, has letters CAR & set of all permutations of a in which CAR (has 3 letters) form 1 block and remaining 23 letters is

$a = (23+1)! = 24!$

simillarly b has letters 'DOG' c has letters 'PUN'

$b = (23+1)! = 24!$ $c = (23+1) = 24!$

d has letter BYTE $= (22+1)! = 23!$

The set of all permutations that has a & b together is $a \cap b = (26-6+2)! = 22!$

similarly $b \cap c = 22! = a \cap c$

$a \cap d = b \cap d = c \cap d = (26-7+2)! = 21!$

$a \cap b \cap c = (26-9+3)! = 20!$

$a \cap b \cap d = a \cap c \cap d = b \cap c \cap d = (26-10+3)! = 19!$

$a \cap b \cap c \cap d = (26-13+4)! = 17!$

The ways to be permutated so that none of the pattern occurs is

$\Rightarrow 26! - (3 \times 24! + 23!) + (3 \times 22! + 3 \times 21!)$
$- (20! + 3 \times 19!) + 17!$

6) Given recurrence relation is $a_n = 2(a_{n-1} - a_{n-2})$,

$a_0 = 1$ & $a_1 = 2$

$a_n = 2a_{n-1} - 2a_{n-2}$

$a_n - 2a_{n-1} + 2a_{n-2} = 0$

standard form for above equation is

$k^2 - 2k + 2 = 0$ The roots are

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow \frac{2 \pm \sqrt{4 - 4(1)(2)}}{2}$$

$$\frac{2 \pm \sqrt{-4}}{2} \Rightarrow \frac{+2 \pm \sqrt{(2i)^2}}{2} \Rightarrow \frac{2 \pm 2i}{2} \Rightarrow \frac{\cancel{2}(1 \pm i)}{\cancel{2}}$$

The roots are $1+i$, $1-i$ which is in form of $A+ib, a-ib$.

$a = 1$, $b = |-1| = 1$

$r = \sqrt{a^2+b^2} = \sqrt{1+1} = \sqrt{2}$ $\quad \pi = \tan^{-1}\left(\frac{b}{a}\right)$

$= \tan^{-1}\left(\frac{1}{1}\right)$

$r^n = A\cos n\frac{\pi}{4} + B\sin n\frac{\pi}{4}$ $\left(\theta = \frac{\pi}{4}\right]$ $\pi \Rightarrow \tan^{-1}(45°)$

$(\sqrt{2})^n = A\cos n\frac{\pi}{4} + B\sin n\frac{\pi}{4}$

$(\sqrt{2})^n = \frac{A_{n_0}\cos\pi}{\sqrt{4}} + \frac{B_{n_0}\sin n\pi}{\sqrt{2}} \Rightarrow \frac{A+B}{\sqrt{2}}$

Given $a_0 = 1$

$a_0 = \sqrt{2}^0 = \frac{A}{\sqrt{2}} + \frac{B}{\sqrt{2}} = \frac{A+B}{\sqrt{2}}$

$\sqrt{2} = A + B$ $\qquad A+B = \sqrt{2} \longrightarrow (1)$

Given $a_1 = 2$

$a_1 = (\sqrt{2})^1 \Rightarrow \sqrt{2} = \frac{A+B}{\sqrt{2}}$ $\qquad A+B = \sqrt{2}\cdot\sqrt{2}$

$A + B = 2 \longrightarrow (2)$

$\cancel{A}+B = \sqrt{2}$

$\underline{A + B = 2}$

$2B = 2 + \frac{\sqrt{2}}{2}$ $\qquad A+B = 2$

$B = 1 + \frac{1}{\sqrt{2}}$ $\qquad A+1+\frac{1}{\sqrt{2}} = 2$

$A + \frac{1}{\sqrt{2}} = 1$

$B = \frac{\sqrt{2}+1}{\sqrt{2}}$ $\qquad A = 1 - \frac{1}{\sqrt{2}}$

$A = \sqrt{2} - 1$

7) Group :-

let G be a group & * be the binary operation defined on it. Then algebraic structure $(G,*)$ is said to be group if it follows the below properties:-

① closure property :-

$a*b \in G \quad \forall a,b \in G$

② Associative property :-

$a*(b*c) = (a*b)*c \quad \forall a,b,c \in G$

③ Existence of Identity :-

$\exists e \in G, \quad a*e = a = e*a \quad \forall a \in G$

Here, e is the identity.

④ existence of Inverse of Each element :-

$\exists b \in G, \quad a*b = e = b*a$

Here, b is the Inverse of a.

Given that fourth roots of unity is

$A = \{1, -1, i, -i\}$

Composition table:—

*	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

closure property:—

from above table, we can see that $a*b \in A$. every element in each row & column belongs to Group A.

So, closure property is satisfied.

Associative property:—

let $1, -1, i \in A$.

$a*(b*c) = (a*b)*c$

$LHS = a*(b*c) = 1*(-1*i) = 1*(-i) = -i$

$RHS = (a*b)*c = (1*-1)*i = (-1)*i = -i$

$LHS = RHS$.

So, Associative property is satisfied.

existence of Identity:—

In table, we see that 1 is the identity element.

Existence of Inverse of each element :-

Inverse of 1 is 1

Inverse of -1 is -1

Inverse of i is -i

Inverse of -i is i (from the table)

Commutative property :-

$a * b = b * a$ let $1, i \in A$.

$1 * i = i * 1 = i$

So, commutative property is satisfied.

$\therefore$ The fourth roots of unity is an abelian group under the operation multiplication.

3) Statement :-

If subgroup H belongs to the finite group G then the order of subgroup H must divide order of group G.

$$k = \frac{O(G)}{O(H)}$$

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Proof :-

Given G is a finite set.

$\therefore$ H is also a finite set. ($\because H \in G$)

The no. of cosets in subgroup H are also finite.

$\exists\, a \in G$, There exists some element a belongs to group G

then Ha is the right coset of G

similarly, $Ha_1, Ha_2, Ha_3, \ldots Ha_k$ are some distinct right cosets of Group G.

According to the law of decomposition of cosets,

$$G = Ha_1 \cap Ha_2 \cap Ha_3 \ldots Ha_k \rightarrow (1)$$

$$G = Ha_1 + Ha_2 + Ha_3 + \ldots + Ha_k$$

from (1),

$$o(G) = Ha_1 \cap Ha_2 \cap Ha_3 + \ldots + Ha_k$$

we know that $Ha_1 = Ha_2 = Ha_3 = \ldots = Ha_k$

$$o(G) = Ha_1 + Ha_2 + Ha_3 + \ldots + Ha_k$$

$$= Ha_1 + Ha_1 + Ha_1 + \ldots + k(Ha_1)$$

$$o(G) = k \cdot o(H).$$

$$\boxed{k = \frac{o(G)}{o(H)}}$$

Therefore, the order of subgroup H divides the order of group G.