CMR INSTITUTE OF TECHNOLOGY	USN											CMRIT

Internal Assessment Test – II May 2025

Sub:	Discrete Mathematical Structures						Code:	BCS405A
Date:	23/05/2025	Duration:	90 mins	Max Marks:	50	Sem: IV	Branch:	AIDS/CSDS/ISE/AIML/CSML

Question 1 is compulsory and answer any 6 from the remaining questions.

		Marks	OBE CO	OBE RBT
1	Let $A = \{1, 2, 3, 4, 6, 12, 18\}$ and R be a relation on A defined by aRb iff "a divides b". Write down the relation R, relation matrix M(R), draw its digraph, list out its indegree and outdegree and draw its Hasse diagram.	[8]	CO3	L3
2	State Pigeon-hole Principle. Prove that if 30 dictionaries in a library contain a total of 61,327 pages, then at least one of the dictionaries mush have at least 2045 pages.	[7]	CO3	L2
3	Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5, 6\}$. Find the number of one-to-one functions and onto functions from A to B. Let $f: R \to R$ be defined by, $f(x) = \begin{cases} 3x - 5, if\ x > 0 \\ 1 - 3x, if\ x \leq 0 \end{cases}$ find $f^{-1}([-6, 5])$ and $f^{-1}([-5, 5])$.	[3+4]	CO3	L2
4	Five teachers T_1, T_2, T_3, T_4, T_5 are to be made class teachers for five classes, C_1, C_2, C_3, C_4, C_5, one teacher for each class. T_1 and T_2 do not wish to become the class teachers for C_1 or C_2, T_3 and T_4 for C_4 or C_5, and T_5 for C_3 or C_4 or C_5. In how many ways can the teachers be assigned the work (without displeasing any teachers)?	[7]	CO4	L3

5	In how many ways can the 26 letters of the English alphabet be permuted so that none of the patterns CAR, DOG, PUN or BYTE occurs?	[7]	CO4	L2
6	Solve the recurrence relation $a_n = 2(a_{n-1} - a_{n-2})$, for $n \geq 2$, given that $a_0 = 1$ and $a_1 = 2$.	[7]	CO4	L2
7	Define Group. Show that fourth roots of unity is an abelian group under the operation multiplication.	[7]	CO5	L2
8	State and prove Lagrange's theorem.	[7]	CO5	L2

$a_n = 2a_{n-1} - 2a_{n-2}$

Q1

Given

$A = \{1, 2, 3, 4, 6, 12, 18\}$

It is given that R is a relation defined on A such that aRb iff "a divides b"

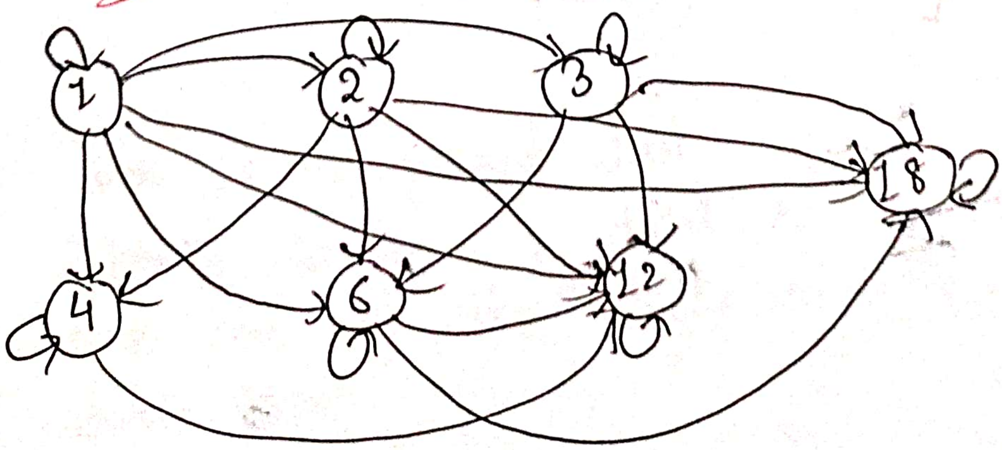
i.e $R = \{(a,b) \in R \mid a \text{ divides } b\}$

So,

$R = \{(1,1), (1,2), (1,3), (1,4), (1,6), (1,12), (1,18),$ & $(2,2), (2,4), (2,6), (2,18), (2,12), (3,6), (3,3), (3,12), (3,18), (4,4), (4,12), (6,6), (6,12), (6,18), (12,12), (18,18)\}$

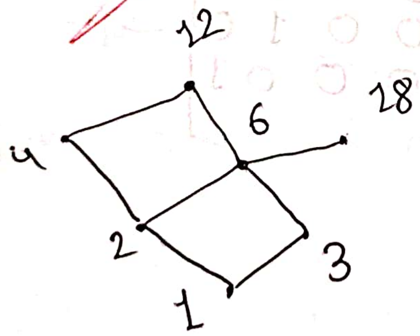
$$M(R) = \begin{array}{c|c} & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 6 & 12 & 18 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 6 \\ 12 \\ 18 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Digraph



Vertex	Indegree	Outdegree
1	~~0~~ 1	~~6~~ 7
2	2	5
3	~~3~~ 2	4
4	3	2
6	4	3
12	6	1
18	5	1

Hasse diagram



Pigeon-hole Principle Q2

If there are 'm' pigeons and 'n' pigeon holes such that $m > n$ then there will be atleast one pigeon hole which will have more than $\left(\left\lfloor \frac{m-1}{n} \right\rfloor + 1\right)$ pigeons in it.

Given,

$n = 30$, $m = 61,327$

To prove $\Rightarrow$ $p+1 = \left\lfloor \frac{m-1}{n} \right\rfloor + 1 = 2045$

By Pigeon Hole Principle we have

$$\left\lfloor \frac{m-1}{n} \right\rfloor + 1 = \left\lfloor \frac{61327-1}{30} \right\rfloor + 1$$

$$= \cancel{\lfloor 2044.2 \rfloor + 1}$$

$$= \boxed{2045}$$

$\therefore$ Thus it is proved that if 30 dictionaries in a library contain a total of 61,327 pages, then at least one of the dictionaries must have at least 2045 pages

$A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5, 6\}$ Q3

Let m denote the number of elements of set A
Let n denote the number of elements of set B

So,
$$|A| = m = 4$$
$$|B| = n = 6$$

No. of one-to-one functions $= {}^nP_m$ where $n \geq m$

$$= {}^6P_4$$
$$= \frac{6!}{(6-4)!}$$
$$= \frac{6!}{2!}$$
$$= \frac{6 \times 5 \times 4 \times 3 \times \cancel{2!}}{\cancel{2!}}$$

$\boxed{\text{No. of one-to one functions} = 360}$

Onto function

For onto function $m \geq n$ but here in the question $m < n$ i.e $4 < 6$. Therefore onto functions are not possible because in onto function $\forall b \in B \; \exists \, a \in A$ such that $f(a) = b$

Now,

$$f(x) = \begin{cases} 3x-5, & \text{if } x > 0 \\ 1-3x, & \text{if } x \le 0 \end{cases}$$

To find $f^{-1}([-6,5])$ and $(f^{-1}[-5,5])$

So,

$f^{-1}([-6,5])$

We have

$$f^{-1}([-6,5]) = \{x \in R \mid f(x) \in [-6,5])$$
$$= \{x \in R \mid -6 \le \cancel{3x}\, f(x) \le 5\}$$

So,

$$-6 \le f(x) \le 5$$
$$-6 \le 3x-5 \le 5$$
$$-6+5 \le 3x-5+5 \le 5+5$$
$$-1 \le 3x \le 10$$
$$-1/3 \le x \le 10/3 \quad -(1)$$

And

$$f^{-1}([-6,5]) = \{x \in R \mid f(x) \in [-6,5])$$
$$= \{x \in R \mid -6 \le 1-3x \le 5\}$$

So,

$$-6 \le 1-3x \le 5$$
$$-7 \le -3x \le 4$$
$$7/3 \ge x \ge -4/3 \quad -(2)$$
$$-4/3 \le x \le 7/3$$

From (1) and (2)

$$\boxed{f^{-1}([-6,5]) = [-4/3, 10/3]}$$

$f^{-1}([-5,5])$

We have,

$$f^{-1}([-5,5]) = \{x \in R \mid f(x) \in [-5,5]\}$$
$$= \{x \in R \mid -5 \le 3x-5 \le 5\}$$

So,

$$-5 \le 3x-5 \le 5$$
$$-5+5 \le 3x-\cancel{5}+\cancel{5} \le 5+5$$
$$0 \le 3x \le 10$$
$$0 \le x \le 10/3 \quad -(i)$$

Also,

$$f^{-1}([-5,5]) = \{x \in R \mid f(x) \in [-5,5]\}$$
$$= \{x \in R \mid -5 \le 1-3x \le 5\}$$

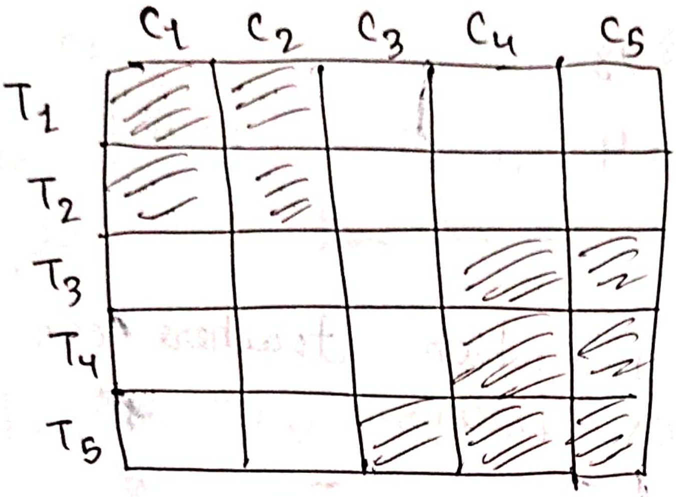
So,

$$-5 \le 1-3x \le 5$$
$$-6 \le -3x \le 4$$
$$2 \ge x \ge -4/3$$
$$-4/3 \le x \le 2 \quad -(ii)$$

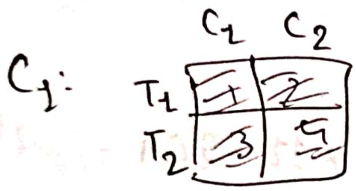
From eq (i) and eq (ii)

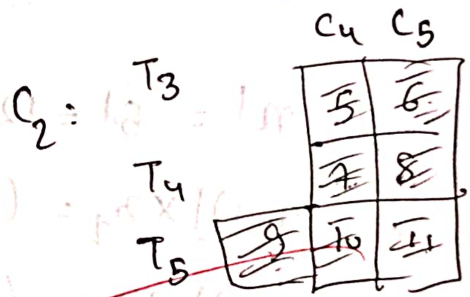
$$\boxed{f^{-1}([-5,5]) = [-4/3,\ 10/3]}$$

The given question can be solved using Rook Polynomial.
Consider the below board 'c' (Q4)



Here we have two disjoint boards:- C_1 and C_2





$n = 4$

$r(C_1, x) = 1 + 4x + 2x^2$

Two non-attacking $= (1,4)$ & $(2,3)$

$r_3 = r_4 = 0$

$r(C_2, x) = 1 + 7x + 10x^2 + 2x^3$

Two non-attacking pairs:
$(5,8), (5,11), (5,9), (6,7), (6,9),$
$(6,10), (7,9), (7,11), (8,9), (8,10)$

Three non-attacking pairs:
$(5,8,9), (6,7,9)$

By Product formula we have

$r(C, x) = r(C_1, x) \times r(C_2, x)$

$r(C, x) = \cancel{r}\ (1 + 4x + 2x^2) \times (1 + 7x + 10x^2 + 2x^3)$

$= (1 + 7x + 10x^2 + 2x^3 + 4x + 28x^2 + 40x^3 + 8x^4 + 2x^2 +$
$14x^3 + 20x^4 + 4x^5)$

$$r(C,x) = 1 + 11x + 40x^2 + 56x^3 + 28x^4 + 4x^5$$
$$= 1 + r_1 x + r_2 x^2 + r_3 x^3 + r_4 x^4 + r_5 x^5$$

So, $r_1 = 11$ $\quad r_4 = 28$

$r_2 = 40$ $\quad r_5 = 4$ $\quad \boxed{n = 5}$

$r_3 = 56$

To find no. of ways in which teachers can be assigned the work is given as follows:-

$$\bar{N} = S_0 - S_1 + S_2 - S_3 + S_4 - S_5$$

where

$$S_0 = n! = 5! = 120$$
$$S_1 = (n-1)! \times r_1 = (5-1)! \times 11 = 4! \times 11 = 264$$
$$S_2 = (n-2)! \times r_2 = (5-2)! \times 40 = 240$$
$$S_3 = (n-3)! \times r_3 = (5-3)! \times 56 = 112$$
$$S_4 = (n-4)! \times r_4 = (5-4)! \times 28 = 28$$
$$S_5 = (n-5)! \times r_5 = (5-5)! \times 4 = 4$$

So,

$$\bar{N} = 120 - 264 + 240 - 112 + 28 - 4$$
$$= 8$$

$$\boxed{\bar{N} = 8}$$

Let A_1 denote the pattern 'CAR' occurring. Q5

" A_2 " " " 'DOG' "

" A_3 " " " 'PUN' "

" A_4 " " " 'BYTE' "

We have

$|S| = 26!$ (As there are 26 letters)

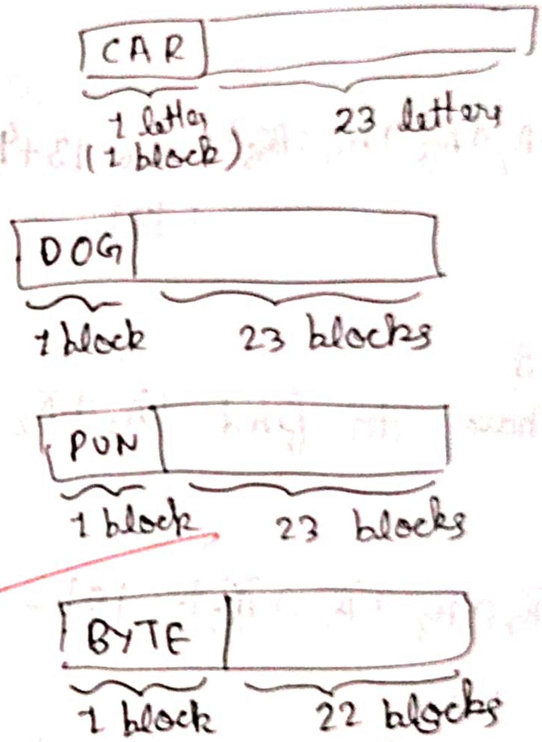
So,

$|A_1| = (26-3+1)! = 24!$

$|A_2| = (26-3+1)! = 24!$

$|A_3| = (26-3+1)! = 24!$

$|A_4| = (26-4+1)! = 23!$



Now,

$|A_1 \cap A_2| = (26-6+2) = 22!$

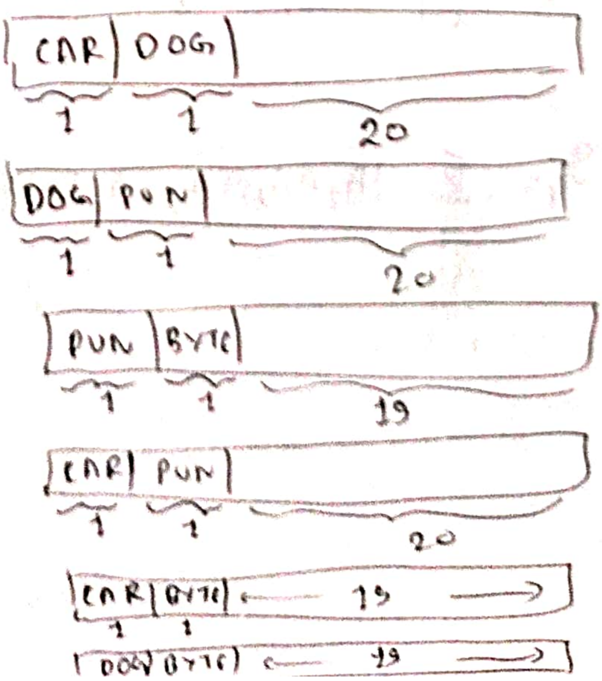
$|A_2 \cap A_3| = (26-6+2) = 22!$

$|A_3 \cap A_4| = (26-7+2) = 21!$

$|A_1 \cap A_3| = (26-6+2) = 22!$

$|A_1 \cap A_4| = (26-7+2) = 21!$

$|A_2 \cap A_4| = (26-7+2) = 21!$



Now,

$|A_1 \cap A_2 \cap A_3| = (26-9+3) = 20!$ $\qquad$ $\underbrace{\text{CAR}}_{1}\,|\,\underbrace{\text{DOG}}_{1}\,|\,\underbrace{\text{PUN}}_{1}\,|\,\underbrace{\qquad\qquad}_{17}$

$|A_2 \cap A_3 \cap A_4| = (26-10+3) = 19!$ $\qquad$ $\underbrace{\text{DOG}}_{1}\,|\,\underbrace{\text{PUN}}_{1}\,|\,\underbrace{\text{BYTE}}_{1}\,|\,\underbrace{\qquad\qquad}_{16}$

$|A_1 \cap A_2 \cap A_4| = (26-10+3) = 19!$ $\qquad$ $\underbrace{\text{CAR}}_{1}\,|\,\underbrace{\text{DOG}}_{1}\,|\,\underbrace{\text{BYTE}}_{1}\,|\,\underbrace{\qquad\qquad}_{16}$

$|A_1 \cap A_3 \cap A_4| = (26-10+3) = 19!$ $\qquad$ $\underbrace{\text{CAR}}_{1}\,|\,\underbrace{\text{PUN}}_{1}\,|\,\underbrace{\text{BYTE}}_{1}\,|\,\underbrace{\qquad\qquad}_{16}$

$|A_1 \cap A_2 \cap A_3 \cap A_4 \cap \cancel{A_5}| = (26-13+4)!$

$= 17!$ $\qquad$ $\underbrace{\text{CAR}}_{1}\,|\,\underbrace{\text{PUN}}_{1}\,|\,\underbrace{\text{BYTE}}_{1}\,|\,\underbrace{\text{DOG}}_{1}\,|\,\underbrace{\qquad\qquad}_{13}$

A.T.Q

We have to find $|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4 \cap \bar{A}_5|$

So,

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4 \cap \cancel{\bar{A}_5}| = |S| - S_1 + S_2 - S_3 + S_4$$

where

$$S_1 = \sum_{i=1}^{4} A_i \quad , \quad S_2 = \sum_{\substack{i=1\\j=1}}^{4} (A_i \cap A_j) \qquad S_3 = \sum_{\substack{i=1\\j=1\\k=1}}^{4} (A_i \cap A_j \cap A_k)$$

$$S_4 = \cancel{\sum_{\substack{i=1\\j=1\\k=1\\l=1}}} |A_1 \cap A_2 \cap A_3 \cap A_4| = 17!$$

So,

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4| = 26! - (3\times 24! + 23!) + (3\times 22! + 3\times 21!) - (20! + 3\times 19!) + 17!$$

$$= \boxed{4.014077864 \times 10^{26}}$$

Q6

Given,

$a_n = 2a_{n-1} - 2a_{n-2}$ for $n \geq 2$, $a_0 = 1$ & $a_1 = 2$

$a_n - 2a_{n-1} + 2a_{n-2} = 0$

This is a second order linear recurrence relation.

C.E corresponding to this

$c_n k^2 + c_{n-1} k + c_{n-2} = 0$

So, $1(k^2) - 2k + 2 = 0$

$k^2 - 2k + 2 = 0$ — (1)

Now,

Roots of eq (1) $= \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)}$

$= \dfrac{2 \pm \sqrt{4-8}}{2}$

$= \dfrac{2 \pm \sqrt{-4}}{2}$

$= \dfrac{2 \pm 2i}{2} = 1 \pm i$

So, $k_1 = 1 + i \Rightarrow$ where $p = 1$, $q = 1$ $\qquad k_1 = p + qi$

$k_2 = 1 - i \Rightarrow$ where $p = 1$, $q = 1$ $\qquad k_2 = p - qi$

Since the roots are not real, solution for this is given by,

$$a_n = r^n (A\cos n\theta + B\sin n\theta)$$

where

$$r = \sqrt{p^2 + q^2} \qquad \theta = \tan^{-1}\left(\frac{q}{p}\right)$$

$$r = \sqrt{(1)^2 + (1)^2} \qquad \theta = \tan^{-1}\left(\frac{1}{1}\right)$$

$$r = \sqrt{2} \qquad \theta = \tan^{-1}(1) = \frac{\pi}{4}$$

So, $$a_n = (\sqrt{2})^n \left(A\cos\left(\frac{n\pi}{4}\right) + B\sin\left(\frac{n\pi}{4}\right)\right) \quad \text{— ②}$$

We have been given initial conditions $a_0 = 1$ and $a_1 = 2$

At $n = 0$, eq ② becomes

$$1 = (\sqrt{2})^0 \left(A\cos\left(\frac{0\times\pi}{4}\right) + B\sin\left(\frac{0\times\pi}{4}\right)\right)$$

$$1 = \sqrt{2}\left(A\cos(0) + B\sin(0)\right)$$

$$1 = \sqrt{2}\left(A(1) + B(0)\right)$$

$$1 = \sqrt{2}(A + 0)$$

$$1 = \sqrt{2}A$$

$$\boxed{A = 1/\sqrt{2}}$$

$$1 = (\sqrt{2})^0 \left(A\cos\left(\frac{0\times\pi}{4}\right) + B\sin\left(\frac{0\times\pi}{4}\right)\right)$$

$$1 = 1\left(A\cos(0) + B\sin(0)\right)$$

$$1 = 1((A) + 0)$$

$$\boxed{A = 1}$$

At $n=1$, eq ② becomes

$2 = (\sqrt{2})^1 \left(A\cos\left(\frac{1\times\pi}{4}\right) + B\sin\left(\frac{\cancel{2}^{1}\times\pi}{\cancel{4}\,2}\right)\right)$

$2 = \sqrt{2}\left(A\cos\left(\frac{\pi}{2}\right) + B\sin\left(\frac{\pi}{2}\right)\right)$

$2 = \sqrt{2}\left(A\times(0) + B\times(1)\right)$

$2 = \sqrt{2}\,(0+B)$

$2 = \sqrt{2}\,B$

$B = \frac{2}{\sqrt{2}} = \frac{\cancel{\sqrt{2}}}{\sqrt{2}\times\cancel{\sqrt{2}}} = \frac{1}{\sqrt{2}}$

$= \frac{\sqrt{2}\times\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}} = \sqrt{2}$

$2 = \sqrt{2}\left(A\cos\left(\frac{\pi}{4}\right) + B\sin\left(\frac{\pi}{4}\right)\right)$

$2 = \sqrt{2}\left(\frac{A}{\sqrt{2}} + \frac{B}{\sqrt{2}}\right)$

$2 = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{B}{\sqrt{2}}\right)$

$2 = 1 + B$

$\boxed{B = 1}$

So,

$\boxed{\cancel{a_n = \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos\left(\frac{n\pi}{4}\right) + \sqrt{2}\sin\left(\frac{n\pi}{4}\right)\right)}}$

$\boxed{\cancel{a_n = \sqrt{2}^{\,n}\left(1\times\cos\left(\frac{n\pi}{4}\right) + \sqrt{2}\sin\left(\frac{n\pi}{4}\right)\right)}}$

$\boxed{a_n = \sqrt{2}^{\,n}\left(\cos\left(\frac{n\pi}{4}\right) + \sin\left(\frac{n\pi}{4}\right)\right)}$

Fourth roots of unity are = {1, -1, i, -i} (Q7)

Let G = {1, -1, i, -i}

Consider the following table for the following (G, x) where G is a group representing fourth roots of unity and 'x' is a multiplication operator.

x	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

(i) Closure

From the table, we can see that each row and each colum have entries which belong to G.

Thus (G, x) satisfy the closure property.

(ii) Associativity

$a * (b * c) = (a * b) * c$ where $\forall\, a, b, c \in G$

Here

$1 * (-1 * i) = (1 * -1) * i$

$1 \times (-1 \times i) = (1 \times (-1)) \times i$

$1 \times (-i) = -1 \times i$

$-i = -i$

Thus associativity property also satisfied.

(iii) Existence of identity $(a * e = e * a = a)$

From the table

$1 \times 1 = 1$

$1 \times (-1) = -1$

$1 \times i = i$

$1 \times (-i) = -i$

$\therefore$ We can say that '1' is the identity and $1 \in G$

(iv) Existence of inverse ~~(a * e = e * a =~~ $(a * a^{-1} = a^{-1} * a = e)$

From the table

~~$1 * 1 = 1 = 1 * 1$~~

~~$-1 * -1 = 1 = -1 * -1$~~

~~$i * (-i)$~~

$1 \times 1 = 1 = 1 \times 1$

$-1 \times (-1) = 1 = -1 \times (-1)$

$i \times (-i) = 1 = (-i) \times i$

$-i \times i = 1 = i \times (-i)$

Thus inverse for each element exists.

(v) Commutative

$\forall a, b \in G \Rightarrow a * b = b * a$

$1 \times (-1) = -1 \times 1 = -1$

$i \times (-i) = (-i) \times i = 1$

$-i \times 1 = 1 \times (-i) = -i$

Similarly we can show for others and commutative property is satisfied.

Thus $\boxed{(G, \times) \text{ is an abelian group}}$

Lagrange's theorem (Q8)

If G is a finite group and H is a subgroup of G then order of H divides order of G. i.e $K = \frac{O(G)}{O(H)}$

Proof :-

Since G is finite then H is also finite.

So there are finite cosets of H

Let $H_1 a$, $H_2 a$, $H_3 a$ .... be the distinct finite right cosets of ~~H~~ G

So, By decomposition of right cosets, we have

$$G = H_1 a \cup H_2 a \cup H_3 a \text{ - - - - } \cup H_n a$$

So,

$$O(G) = O(H_1 a) \cup O(H_2 a) \cup O(H_3 a) \text{ - - - - } \cup O(H_n a)$$

But

$$O(H_1 a) = O(H_2 a) = O(H_3 a) = \text{ - - - } O(H_n a)$$

Let $Ha = H$

So,

$$O(G) = O(Ha) \cup O(H_a) \cup O(H_a) \cup O(H_a) \text{ . . . } O(H_a$$

$$O(G) = K \times O(H)$$

$$\boxed{K = \frac{O(G)}{O(H)}}$$

18

24/5/25