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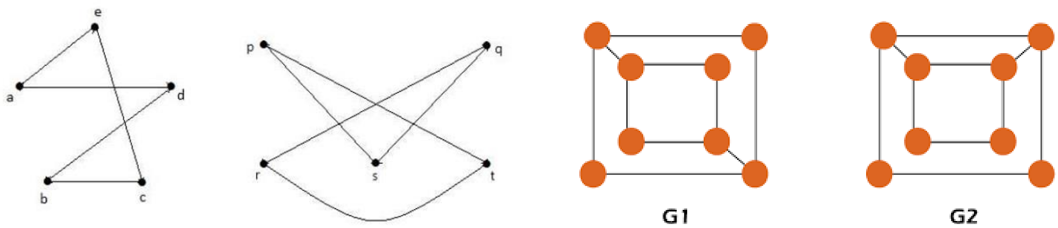
Internal Assessment Test I March 2025

Sub:	**Graph Theory**							Code:	BCS405B
Date:	24/03/2025	Duration:	90 mins	Max Marks:	50	Sem:	IV	Branch:	**CSE**

	Question 1 is compulsory and Answer any 6 from the remaining questions.	Marks	OBE CO	OBE RBT
1	Define isomorphism og graphs. Verify weather the following graphs are isomorphic or not.	[8]	CO1	L1, L3



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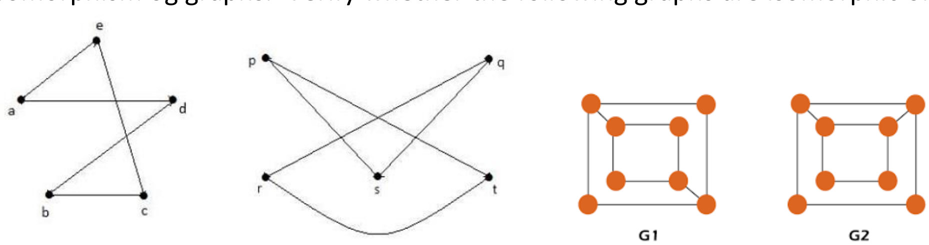

2.	Define the following (a) Complete graph (b) Pendant vertex (c) Regular graph (d) vertex deletion in a graph	[7]	CO1	L1
3	Prove that a simple graph with n vertices and k components can have at most $\frac{(n-k)(n-k+1)}{2}$ $edges.$	[7]	CO1	L3
4	Find the number of edges in a complete graph with n vertices. Show that a simple graph of order 4 and size 7 does not exist. How many edges and how many vertices are there in the complete bipartite graph $K_{4,7}$ and $K_{7,11}$?	[7]	CO1	L2, L3
5	Define Euler circuit, Euler trail, Euler graph. Prove that a connected graph G has an Euler circuit if and only if all vertices of G are of even degree	[7]	CO2	L1, L3
6	Let A={1,2,3,4,5,6} R is a relation on A defined by aRb iff a divides b. Find the relation R, matrix relation and the graph of R. Is R an equivalence relation?.	[7]	CO2	L3
7	In a complete graph with n vertices, where n is an odd number n>=3, prove that there are (n-1)/2 edge disjoint Hamilton cycles.	[7]	CO2	L3
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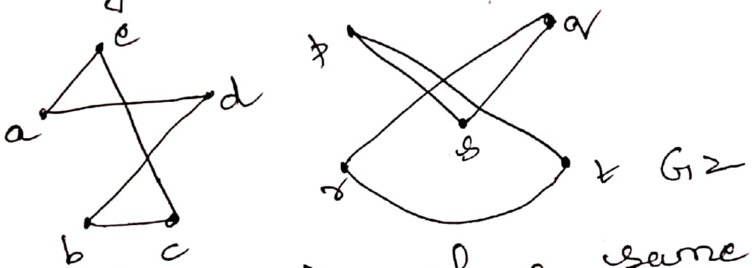
Q1 Consider two graphs $G=(V,E)$ $G'=(V'E')$
Suppose there exists a function $f: V \to V'$ such that 1) f is a 1-1 correspondence
2) for all vertices A, B of G {A, B} is an edge of G iff {f(A), f(B)} is an edge of G'.
Then f is called as an isomorphism between G and G'. We say that G and G' are isomorphic graphs.



Both the G1 graphs have same number of vertices and same number of edges

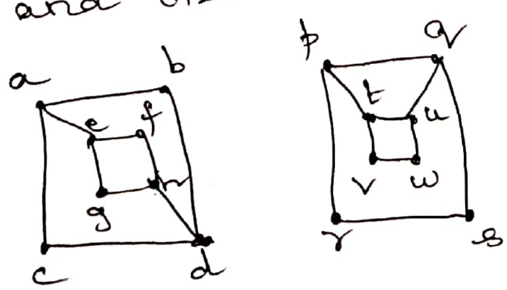
In G1, $d(a) = d(b) = d(c) = d(d) = d(e) = 2$

In G2, $d(p) = d(q) = d(r) = d(s) = d(t) = 2$

a ⟺ s b ⟺ r c ⟺ t d ⟺ q e ⟺ p

ae ⟺ sp ad ⟺ sq bd ⟺ rq bc ⟺ rt ec ⟺ pt

G1 and G2 are isomorphic



In G1, there are 8 vertices and 10 edges.

In G2, there are 8 vertices and 10 edges.

In G1, 4 vertices are of degree 2
other 4 vertices are of degree 3

In G2, 4 vertices are of degree 2
other 4 vertices are of degree 3.

In G1, deg(a) = deg(d) = deg(e) = deg(h) = 3
deg(b) = deg(f) = deg(c) = deg(g) = 2

In G2, deg(p) = deg(q) = deg(t) = deg(u) = 3
deg(v) = deg(w) = deg(r) = deg(s) = 2

In G1, the adjacent vertices (a, e) and (h, d) are of degree 3. But (a, e) are not adjacent to (h, d).

In G2, p, q, t, u are of degree 3 and are adjacent.

a → p
b ⟺ q
t ⟺ e
u ⟺ f

c ⟺ r
d ⟺ s
g ⟺ v
h ⟺ w

We found that edge correspondence is not possible.

G1 $\not\cong$ G2

~~ae ⟺ pt ef ⟺ tu~~

1.8.1 The Königsberg Bridge Problem

In the eighteenth century city named Königsberg in East Prussia (Europe), there flowed a river named Pregel River which divided the city into four parts. Two of these parts were the banks of the river and two were islands. These parts were connected with each other through seven bridges as depicted in Figure 1.122.

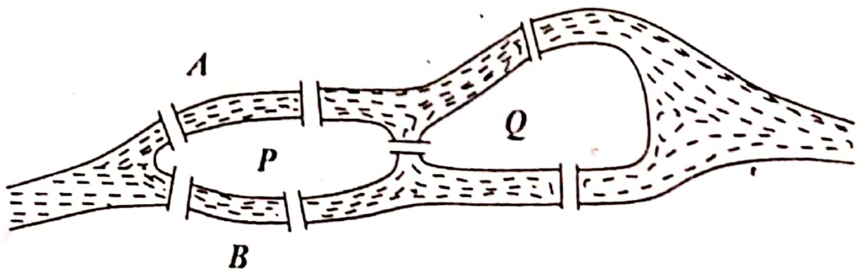


Figure 1.122

The citizens of the city seemed to have posed the following problem: By starting at any of the four land areas, can we return to that area after crossing *each* of the seven bridges *exactly once?*

This problem, now known as the *Königsberg Bridge problem*, remained unsolved for several years. In the year 1736, Euler analyzed the problem with the help of a graph and gave the solution. This was indeed the starting point for the development of graph theory.

Let us see what the solution is (as given by Euler). Denote the land areas of the city by A, B, P, Q, where A, B are the banks of the river and P, Q are the islands (See Figure 1.122). Construct a graph by treating the four land areas as four vertices and the seven bridges connecting them as seven edges. The graph is as shown in Figure 1.123.

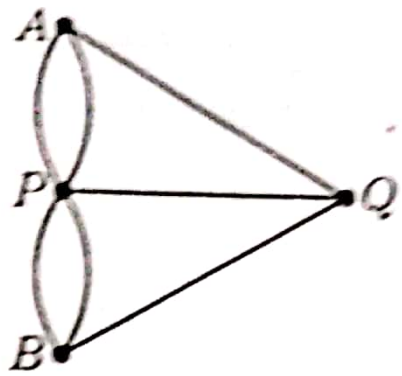


Figure 1.123

We note that, in this graph,

$$\deg(A) = \deg(B) = \deg(Q) = 3, \ \deg(P) = 5.$$

which are not even. Therefore, the graph does not have an Euler circuit*. This means that there does not exist a closed walk that contains all the edges exactly once. This amounts to saying that *it is not possible* to walk over each of the seven bridges exactly once and return to the starting point.

Theorem 2 *A simple graph with n vertices and k components can have at most $(n-k)(n-k+1)/2$ edges.*

Proof: Let n_1 be the number of vertices in the first component, n_2 be the number of vertices in the second component, and so on; and finally n_k be the number of vertices in the kth component, in the given graph G. Then

$$n_1 + n_2 + n_3 + \cdots + n_k = n \qquad \text{(i)}$$

This gives

$$(n_1 - 1) + (n_2 - 1) + \cdots + (n_k - 1) = n - (1 + 1 + 1 + \cdots k \text{ terms})$$
$$= n - k.$$

Squaring both sides, we get

$$(n_1 - 1)^2 + (n_2 - 1)^2 + \cdots + (n_k - 1)^2 + S = (n-k)^2 \qquad \text{(ii)}$$

where S is the sum of products of the form $2(n_i - 1)(n_j - 1)$, $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, k$, $i \neq j$.

Since each of $n_1, n_2, \ldots, n_k$ is greater than or equal to 1, we have $S \geq 0$. Therefore, (ii) yields

$$(n_1 - 1)^2 + (n_2 - 1)^2 + \cdots + (n_k - 1)^2 \leq (n-k)^2$$

or $$n_1^2 + n_2^2 + \cdots + n_k^2 - 2(n_1 + n_2 + \cdots + n_k) + k \leq (n-k)^2$$

or $$n_1^2 + n_2^2 + \cdots + n_k^2 \leq (n-k)^2 + 2n - k, \quad \text{using (i)}$$
$$= n^2 + k^2 - 2nk + 2n - k$$
$$= n^2 - (k-1)(2n-k)$$

i.e., $$\sum_{i=1}^{k} n_i^2 \leq n^2 - (k-1)(2n-k) \qquad \text{(iii)}$$

*See Theorem following the handshaking property, in Section 1.2.1.

Now, since G is a simple graph, each of the components of G is a simple graph. Therefore, the maximum number of edges which the ith component can have is $\frac{1}{2}n_i(n_i - 1)$.* Therefore, the maximum number of edges which G can have is N, where

$$N = \frac{1}{2}\sum_{i=1}^{k} n_i(n_i - 1) \qquad \text{(iv)}$$

From (iv), we find that

$$\begin{aligned}
N &= \frac{1}{2}\sum_{i=1}^{k}(n_i^2 - n_i) = \frac{1}{2}\sum_{i=1}^{k} n_i^2 - \frac{1}{2}\sum_{i=1}^{k} n_i \\
&= \frac{1}{2}\sum_{i=1}^{k} n_i^2 - \frac{1}{2}n, \quad \text{using (i)} \\
&\leq \frac{1}{2}\{n^2 - (k-1)(2n-k)\} - \frac{1}{2}n, \quad \text{using (iii)} \\
&= \frac{1}{2}(n^2 - 2nk + n + k^2 - k) \\
&= \frac{1}{2}(n-k)(n-k+1)
\end{aligned}$$

Thus, the number of edges in G cannot exceed $\frac{1}{2}(n-k)(n-k+1)$. This proves the theorem. ■

1.8 Euler circuits and Euler trails

Consider a connected graph G. If there is a *circuit* in G that contains *all the edges* of G, then that circuit is called an ***Euler circuit*** (or *Eulerian line, or Euler tour*) in G. If there is a *trail* in G that contains *all the edges* of G, then that trail is called an ***Euler trail*** (or *unicursal line*) in G.

Recall that in a trail and a circuit no edge can appear more than once but a vertex can appear more than once. This property is carried to Euler trails and Euler circuits also.

Since Euler circuits and Euler trails include all edges, they automatically should include all vertices as well.

A connected graph that contains an Euler circuit is called an ***Euler graph*** (*or Eulerian graph*).

A connected graph that contains an Euler trail is called a ***semi-Euler graph*** (or a *semi-Eulerian graph* or *unicursal graph*).

For example, in the graph shown in Figure 1.116, the closed walk

$$Pe_1Qe_2Re_3Pe_4Se_5Re_6Te_7P$$

is an Euler circuit. Therefore, this graph is an Euler graph.

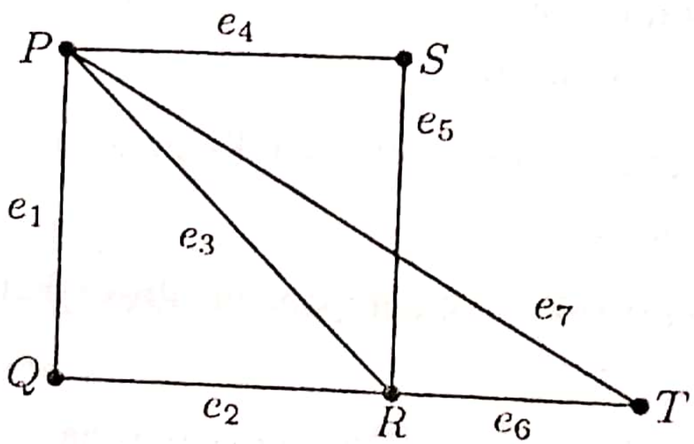


Figure 1.116

Theorem 1 *A connected graph G has an Euler circuit (that is, G is an Euler graph) if and only if all vertices of G are of even degree.*

Proof: First, suppose that G has an Euler circuit. While tracing this circuit we observe that every time the circuit meets a vertex v it goes through two edges incident on v (— with the one

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through which we enter v and the other through which we depart from v). This is true for all vertices that belong to the circuit. Since the circuit contains all edges, it meets all the vertices at least once. Therefore, the degree of every vertex is a multiple of two (i.e. every vertex is of even degree).

Conversely, suppose that all the vertices of G are of even degree. Now, we construct a circuit starting at an arbitrary vertex v and going through the edges of G such that no edge is traced more than once. Since every vertex is of even degree, we can depart from every vertex we enter, and the tracing cannot stop at any vertex other than v. In this way, we obtain a circuit q having v as the initial and final vertex. If this circuit contains all the edges in G, then the circuit is an Euler circuit. If not, let us consider the subgraph H obtained by removing from G all edges that belong to q. The degrees of vertices in this subgraph are also even. Since G is connected, the circuit q and the subgraph H must have at least one vertex, say v', in common. Starting from v', we can construct a circuit q' in H as was done in G. The two circuits q and q' together constitute a circuit which starts and ends at the vertex v and has more edges than q. If this circuit contains all the edges in G, then the circuit is an Euler circuit. Otherwise, we repeat the process until we get a circuit that starts from v and ends at v and which contains all edges in G. In this way, we obtain an Euler circuit in G.

This completes the proof of the theorem. ●

Example 5 *Show that a complete graph with n vertices, namely K_n, has $\frac{1}{2}n(n-1)$ edges.*

▶ In a complete graph, there exists exactly one edge between every pair of vertices. As such, the number of edges in a complete graph is equal to the number of pairs of vertices. If the number of vertices is n, then the number of pairs of vertices is

$$^nC_2 = \frac{n!}{(n-2)!2!} = \frac{1}{2}n(n-1)$$

Thus, the number of edges in a complete graph with n vertices is $\frac{1}{2}n(n-1)$. ■

Note: For another proof of this result, see Example 12, Section 1.2.1.

Example 6 *Show that a simple graph of order $n = 4$ and size $m = 7$ and a complete graph of order $n = 4$ and size $m = 5$ do not exist.*

▶ For $n = 4$, we have

$$\frac{1}{2}n(n-1) = \frac{1}{2} \times 4 \times 3 = 6$$

Since $m = 7$ exceeds this number, a simple graph of order $n = 4$ and size $m = 7$ does not exist.

Similarly, since $m = 5$ is not equal to $\frac{1}{2}n(n-1) = 6$, a complete graph of order 4 and size $m = 5$ does not exist. ■

Example 8 (a) *How many vertices and how many edges are there in the complete bipartite graphs* $K_{4,7}$ *and* $K_{7,11}$?

(b) *If the graph* $K_{r,12}$ *has 72 edges, what is r?*

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► Recall that the complete bipartite graph $K_{r,s}$ has $r + s$ vertices and rs edges. Accordingly:

(a) The graph $K_{4,7}$ has $4 + 7 = 11$ vertices and $4 \times 7 = 28$ edges, and the graph $K_{7,11}$ has 18 vertices and 77 edges.

(b) If the graph $K_{r,12}$ has 72 edges, we have $12r = 72$ so that $r = 6$. ■

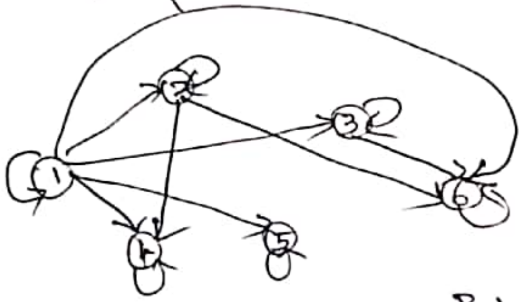
Q6 $A = \{1, 2, 3, 4, 5, 6\}$

aRb iff $a \mid b$

$R = \{(1,1)\ (2,2)\ (3,3)\ (4,4)\ (5,5)\ (6,6)\ (1,2)\ (1,3)$
$(1,4)\ .\ (1,5)\ (1,6)\ \ (2,4)\ (2,6)\ (3,6)\}$

$$M_R = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 0 & 1 & 0 & 1 \\ 3 & 0 & 0 & 1 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & 1 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 1 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

Graph



R is reflexive as 1R1, 2R2, 3R3, 4R4, 5R5, 6R6

R is not symmetric bcz 2 R̸ 1, 3 R̸ 1, 4 R̸ 1, 5 R̸ 1, 6 R̸ 1 . . .

R is transitive bcz $(a, b)\ (b, c) \in R \Rightarrow (a, c) \in R$

R is not an equivalence relation bcz R is not transitive.

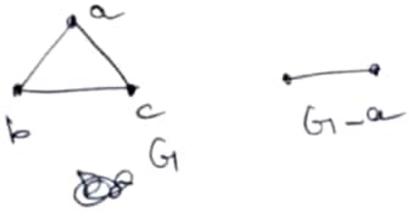
Q2 a) A simple graph is said to be a complete graph if there is an edge between every pair of vertices. $(n \geq 2)$

b) A vertex of degree one is called as the pendant vertex. An edge incident with the pendant vertex is called as the pendant edge.

A subgraph G1 is called as the spanning subgraph if the vertex set of G1 is same as that of the vertex set of G.

Given a graph $G = (V, E)$ a subgraph $G1 = (V1, E1)$ of G such that every edge $\{A, B\}$ of G where $A, B \in V1$ is an edge of G1 also. G1 is called as an induced subgraph of G.

Deletion of a vertex in G means deletion of all edges incident on the vertex.



a, c pendant vertices

bc, ac are pendant edges