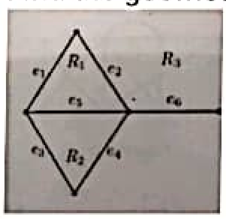
CMR INSTITUTE OF TECHNOLOGY		USN							
		Internal Assessment Test II May 2025							
Sub:	Graph Theory						Code:	BCS405B	
Date:	23/05/2025	Duration:	90 mins	Max Marks:	50	Sem:	IV	Branch:	CSE

	Question 1 is compulsory and Answer any 6 from the remaining questions.	Marks	OBE	
			CO	RBT
1	State and prove Five Colour Theorem.	[8]	CO5	L3
2.	Define the following. Give an example for each one of them. (a) Planar graph (b) Homeomorphism of graphs (c) Chromatic number of a graph (d) Covering of a graph (e) Spanning tree (f) Rooted tree (g) Cut vertex	[7]	CO3, 4, 5	L1
3	Prove that a connected graph is a tree if and only if it is minimally connected.	[7]	CO3	L3
4	(a) Let F be a forest with k components(trees). If n is the number of vertices and m is the number of edges in F, prove that n = m + k. (b) If a tree has 4 vertices of degree 3, 2 vertices of degree 4 and one vertex of degree 5, show that it should have 10 pendant vertices.	[5+2]	CO3	L3
5	Prove that a connected planar graph with n vertices, m edges has exactly m + 2 - n regions in all its diagrams.	[7]	CO4	L3
6	(a) Find the geometric dual of the graph given below.  (b) State the Kuratowski's theorem. Draw the Kuratowski's first and second graphs.	[4+3]	CO4	L3
7	Explain maximal independent sets and finding all maximal independent sets	[7]	CO5	L2, L3
8	Write a short note on Greedy colouring algorithm.	[7]	CO5	L3

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stated result.

Five Color Theorem: *The vertices of every connected simple planar graph can be properly colored with five colors.*

Proof: Let n be the number of vertices in a connected, simple planar graph. If $n \leq 5$, then the theorem is trivially true. Assume that the theorem is true for all graphs with $n \leq k$. Consider a graph G with $k+1$ vertices. Then, by virtue of Euler's theorem, G contains a vertex v of degree at most 5*. If we consider the graph $H = G - v$, obtained by deleting v from G, then H has k vertices. Therefore, by the assumption made, H is 5-colorable.

Since the degree of v is at most 5, v has at most 5 neighbours in G. Suppose v has 4 or less number of neighbours. Then the neighbours can be colored with at most four different colors and v can be colored with the fifth color, all drawn from the colors used in H. Thus, a proper coloring of G can be done by using the five colors with which H can be colored. Thus, G is 5-colorable.

Next, suppose that v has 5 neighbours, say v_1, v_2, v_3, v_4, v_5. Let us arrange them around v in anti-clockwise order as in Figure 2.61. If the vertices $v_1, v_2, v_3, \ldots, v_5$ are all mutually adjacent, then they constitute K_5 which is non-planar. This is not possible, because, being a planar graph, G cannot contain a non-planar graph as a subgraph. Therefore, at least two of $v_1, v_2, \ldots, v_5$, say v_1 and v_3, are non-adjacent.

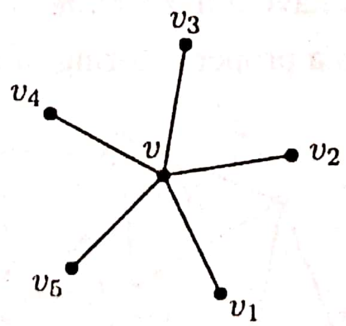

Figure 2.61

Now, construct a graph G' by merging the edges v_3v and vv_1. The graph G' will have $(k+1) - 2 = k - 1$ vertices (with v_3vv_1 as the merged vertex). This graph is, therefore,

*See Section 2.2, Corollary 4.

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5-colorable. Let us assign a color α_1 to the merged vertex v_3vv_1, a color α_2 to v_2, a color α_4 to v_4 and a color α_5 to v_5. With this scheme of coloring of v_1, v_2, v_3, v_4, v_5 and with the use of just one more color α_3 assigned to other appropriate vertices, the graph G' gets properly colored. Now, unravel the merged vertex v_3vv_1 and assign the color α_1 to both v_3 and v_1 and the color α_3 to v, without disturbing the colors of other vertices. This will produce a proper coloring of G with colors α_1, α_2, α_3, α_4, α_5. Thus, G is 5-colorable in this case also (where the degree of v is 5).

We have proved that a graph with $n = k + 1$ vertices is 5-colorable if a graph with $n \leq k$ vertices is 5-colorable. Hence, by induction, it follows that a graph with n vertices, where n is any positive integer, is 5-colorable. •

This completes the proof of the theorem.

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Theorem 6 *A connected graph is a tree if and only if it is minimally connected.*

Proof: Suppose G is a connected graph which is not a tree. Then G contains a cycle C. The removal of any one edge e from this cycle will not make the graph disconnected. Therefore, G is not minimally connected. Thus, if a connected graph is not a tree then it is not minimally connected. This is equivalent to saying that if a connected graph is minimally connected then it is a tree (contrapositive).

Conversely, suppose G is a connected graph which is not minimally connected. Then there exists an edge e in G such that $G - e$ is connected. Therefore, e must be in some cycle in G. This implies that G is not a tree. Thus, if a connected graph is not minimally connected then it is not a tree. This is equivalent to saying that if a connected graph is a tree, then it is minimally connected (contrapositive).

This completes the proof of the theorem. ●

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Example 3 *Let F be a forest with k components (trees). If n is the number of vertices and m is the number of edges in F, prove that $n = m + k$.*

▶ Let $H_1, H_2, \ldots, H_k$ be the components of F. Since each of these is a tree, if n_i is the number of vertices in H_i and m_i is the number of edges in H_i, we have

$$m_i = n_i - 1, \quad \text{for} \quad i = 1, 2, \ldots, k.$$

This gives

$$\begin{aligned} m_1 + m_2 + \cdots + m_k &= (n_1 - 1) + (n_2 - 1) + \cdots + (n_k - 1) \\ &= n_1 + n_2 + \cdots + n_k - k. \end{aligned}$$

But $m_1 + m_2 + \cdots + m_k = m$ and $n_1 + n_2 + \cdots + n_k = n$. Therefore, $m = n - k$, or $n = m + k$. ∎

Theorem : *A connected planar graph G with n vertices and m edges has exactly m − n + 2 regions in all of its diagrams.*

Proof: Let r denote the number of regions in a diagram of G. The theorem states that

$$r = m - n + 2, \quad \text{or} \quad n - m + r = 2 \tag{1}$$ [§]

We give the proof by induction on m.

If $m = 0$, then n must be equal to 1. Because, if $n > 1$, then G will have at least two vertices and there must be an edge connecting them (because G is connected), so that $m \neq 0$, which is a contradiction.

If $n = 1$, a diagram of G determines only one region — the entire plane region (as shown in Figure 2.21(a)).

[§] This formula is known as the ***Euler's formula***.

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Thus, if $m = 0$, then $n = 1$ and $r = 1$, so that $n - m + r = 2$. This verifies the theorem for $m = 0$.

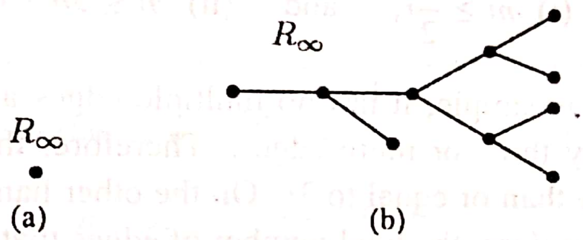

Figure 2.21

Now, assume that the theorem holds for all graphs with $m = k$ number of edges, where k is a non-negative integer.

Consider a graph G_{k+1} with $k + 1$ edges and n vertices. First, suppose that G_{k+1} has no cycles in it. Then a diagram of G_{k+1} will be of the form shown in Figure 2.21(b)[†] in which the number of vertices will be exactly one more than the number of edges, and the diagram will determine only one region — the entire plane region (as in Figure 2.21(b)). Thus, for G_{k+1}, we have, in this case, $n = (k + 1) + 1$ and $r = 1$, so that

$$n - (k + 1) + r = 2.$$

This means that the result (1) is the true when $m = k + 1$ as well, if G_{k+1} contains no cycles in it.

Next, suppose G_{k+1} contains at least one cycle. Let r be the number of regions which a diagram of G_{k+1} determine. Consider an edge e in a cycle and remove it from G_{k+1}. The resulting graph, $G_{k+1} - e$, will have n vertices and $(k + 1) - 1 = k$ edges, and its diagram will determine $r - 1$ regions. Since $G_{k+1} - e$ has k edges, the theorem holds for this graph (by the induction assumption made). That is, we have

$$r - 1 = k - n + 2, \qquad \text{or} \qquad n - (k + 1) + r = 2.$$

This means that in this case also the result (1) is true when $m = k + 1$ as well.

Hence, by induction, it follows that the result (1) is true for all non-negative integers m. This completes the proof of the theorem.

Theorem : ‡ *A necessary and sufficient condition for a graph G to be planar is that G does not contain K_5 or $K_{3,3}$ as a subgraph or any subgraph homeomorphic to either of these.*

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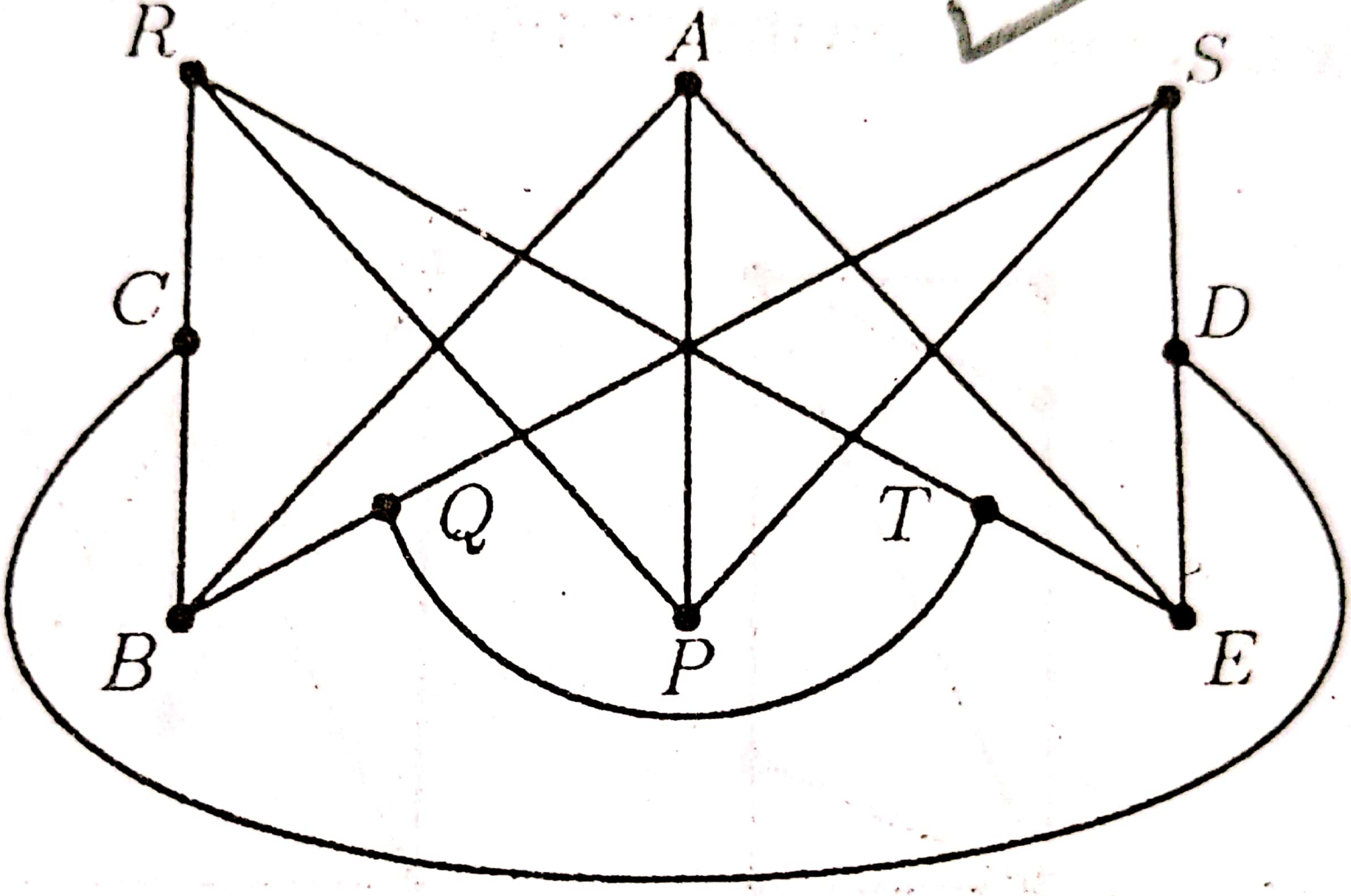

(b)

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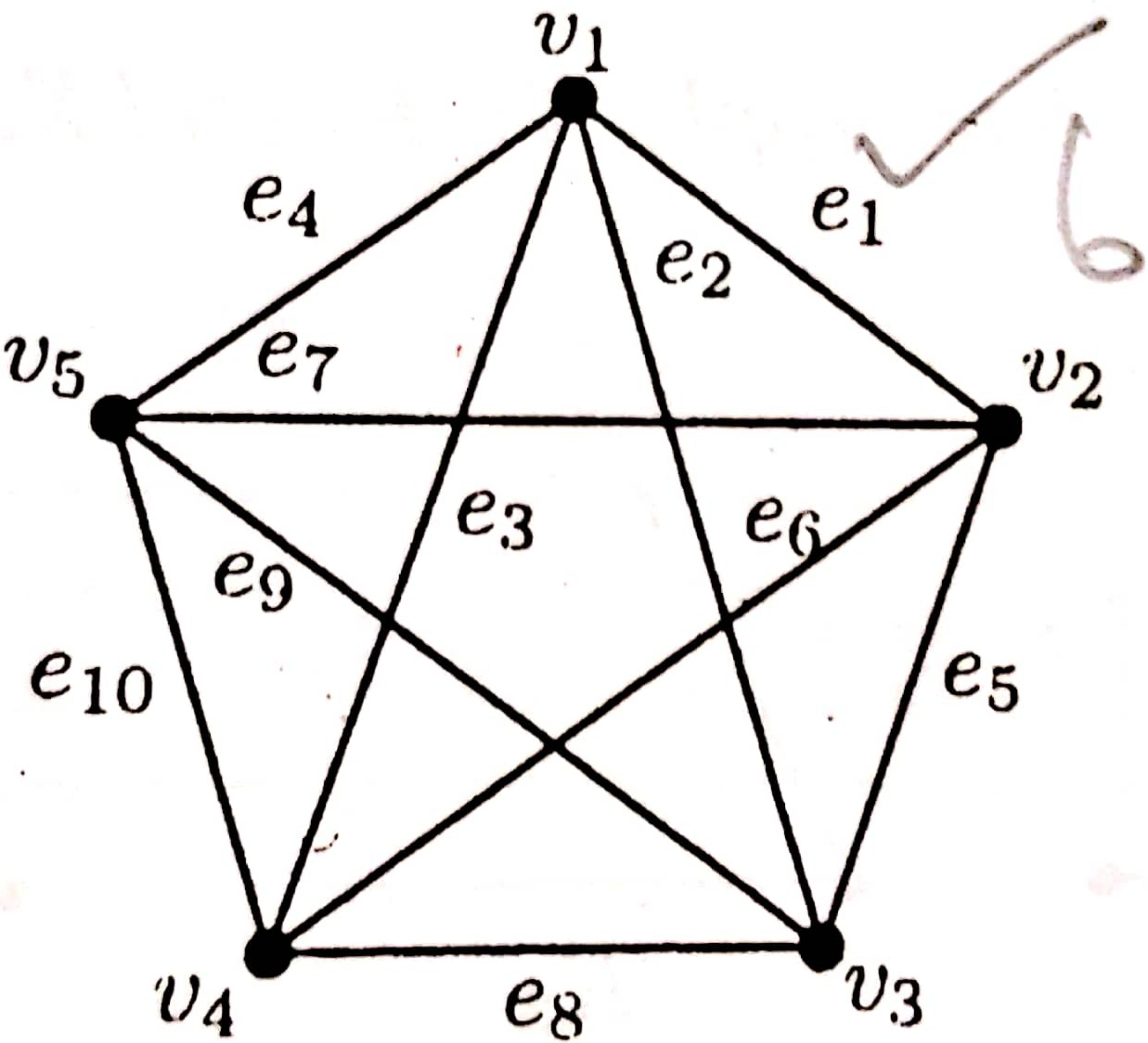

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Greedy Coloring Algorithm

Input : a graph $G = (V, E)$ connected and planar

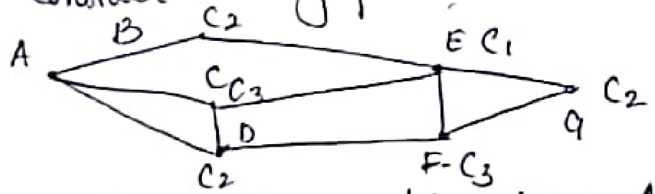
Step 1 : Label the vertices in an order say $V_1, V_2, \ldots V_n$

Step 2 : Order the colours available say $C_1, C_2 \ldots C_n$.

Step 3 : In the order decided for vertices and colours, each vertex with the first available colour on the list. Such that no two adjacent colour on the list. Such that no two vertices have the same colour.

Step 4 : Continue in this way till each vertex is coloured.

Example: Consider a graph with 7 vertices as below:



Vertices are given an order from A to G. Colours are listed in order C_1, C_2, C_3. Let A be coloured with C_1

Since B C & D are adjacent to A, they can't be coloured with C_1. So colour B with C_2, C with C_3.

(∵ B & C are adjacent) colour D with C_2

∵ E is adjacent to B & C it can't be coloured with C_2 & C_3 but can be coloured with C_1.

∵ F is adjacent to D & E it can't be coloured with C_1 and C_2, so colour F with C_3.

∵ G is adjacent to E & F, it can't be coloured with C_1 & C_3. So colour G with C_2. Hence G is properly coloured with 3 colours.

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8-2. CHROMATIC PARTITIONING

A proper coloring of a graph naturally induces a partitioning of the vertices into different subsets. For example, the coloring in Fig. 8-1(c) produces the partitioning

$$\{v_1, v_4\}, \{v_2\}, \text{ and } \{v_3, v_5\}.$$

No two vertices in any of these three subsets are adjacent. Such a subset of vertices is called an independent set; more formally:

A set of vertices in a graph is said to be an *independent set* of vertices or simply an *independent set* (or an *internally stable set*) if no two vertices in the set are adjacent. For example;* in Fig. 8-3, {*a*, *c*, *d*} is an independent set. A single vertex in any graph constitutes an independent set.

A *maximal independent set* (or *maximal internally stable set*) is an independent set to which no other vertex can be added without destroying its independence property. The set {*a*, *c*, *d*, *f*} in Fig. 8-3 is a maximal independent set. The set {*b*, *f*} is another maximal independent set. The set {*b*, *g*} is a third one. From the preceding example, it is clear that a graph, in general, has many maximal independent sets; and they may be of different sizes. Among all maximal independent sets, one with the largest number of vertices is often of particular interest.

Suppose that the graph in Fig. 8-3 describes the following problem. Each of the seven vertices of the graph is a possible code word to be used in some communication. Some words are so close (say, in sound) to others that they might be confused for each other. Pairs of such words that may be mistaken for one another are joined by edges. Find a largest set of code words for a reliable communication. This is a problem of finding a maximal independent set with largest number of vertices. In this simple example, {*a*, *c*, *d*, *f*} is an answer.

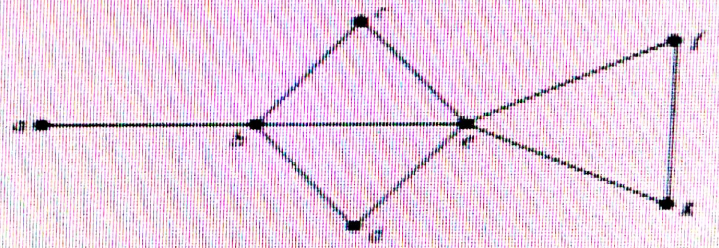

Fig. 8-3

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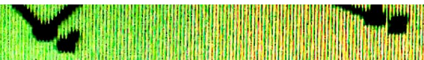

Fig. 8-3

The number of vertices in the largest independent set of a graph G is called the independence number (or *coefficient of internal stability*), $\beta(G)$.

Consider a κ-chromatic graph G of n vertices properly colored with κ different colors. Since the largest number of vertices in G with the same color cannot exceed the independence number $\beta(G)$, we have the inequality

$$\beta(G) \geq \frac{n}{\kappa}.$$

Finding a Maximal Independent Set: **A reasonable method of finding a maximal independent set in a graph G will be to start with any vertex v of G in the set. Add more vertices to the set, selecting at each stage a vertex that is not adjacent to any of those already selected. This procedure will ultimately produce a maximal independent set. This set, however, is not necessarily a maximal independent set with a largest number of vertices.**

Finding All Maximal Independent Sets: A reasonable (but not very efficient for large graphs) method for obtaining all maximal independent sets in any graph can be developed using Boolean arithmetic on the vertices. Let each vertex in the graph be treated as a Boolean variable. Let the logical (or Boolean) sum $a + b$ denote the operation of including vertex a or b or both; let the logical multiplication ab denote the operation of including both vertices a and 6, and let the Boolean complement a' denote that vertex a is not included.

For a given graph G we must find a maximal subset of vertices that does not include the two end vertices of any edge in G. Let us express an edge (x, y) as a Boolean product, xy, of its end vertices x and y, and let us sum all such products in G to get a Boolean expression

$$\varphi = \Sigma\, xy \text{ for all } (x, y) \text{ in } G.$$

Let us further take the Boolean complement φ' of this expression, and express it as a sum of Boolean products :

$$\varphi' = f_1 + f_2 + \dots + f_k.$$

A vertex set is a maximal independent set if and only if $\varphi = 0$ (logically false), which is possible if and only if $\varphi' = 1$ (true), which is possible if and only if at least one $f_i = 1$, which is possible if and only if each vertex appearing in f_i, (in complemented form) is excluded from the vertex set of G. Thus each f_i will yield

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$$\varphi = ab + bc + bd + be + ce + de + ef + eg + fg.$$
$$\varphi' = (a' + b')(b' + c')(b' + d')(b' + e')(c' + e')(d' + e')$$
$$(e' + f')(e' + g')(f' + g').$$

Multiplying these out and employing the usual identities of Boolean arithmetic, such as

$$aa = a,$$
$$a + a = a,$$
$$a + ab = a,$$

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we get

$$\varphi' = b'e'f + b'e'g' + a'c'd'e'f + a'c'd'e'g + b'c'd'f'g'.$$

Now if we exclude from the vertex set of G vertices appearing in any one of these five terms, we get a maximal independent set. The five maximal independent sets are

$$acdf, acdg, bg, bf, \text{ and } ae.$$

These are all the maximal independent sets of the graph.

Finding Independence and Chromatic Numbers: Once all the maximal independent sets of G have been obtained, we find the size of the one with the largest number of vertices to get the independence number $\beta(G)$. The independence number of the graph in Fig. 8-3 is four.

To find the chromatic number of G, we must find the minimum number of these (maximal independent) sets, which collectively include all the vertices of G. For the graph in Fig. 8-3, sets $\{a, c, d, f\}$, $\{b,g\}$, and $\{a, e\}$, for example, satisfy this condition. Thus the graph is 3-chromatic.

Chromatic Partitioning: Given a simple, connected graph G, partition all vertices of G into the smallest possible number of disjoint, independent sets. This problem, known as the *chromatic partitioning* of graphs, is perhaps the most important problem in partitioning of graphs.

By enumerating all maximal independent sets and then selecting the smallest number of sets that include all vertices of the graph, we just solved this problem. The following four are some chromatic partitions of the graph in Fig. 8-3, for example.

$$\{(a, c, d, f), (b, g), (e)\},$$
$$\{(a, c, d, g), (b, f), (e)\},$$
$$\{(c, d, f), (b, g), (a, e)\},$$
$$\{(c, d, g), (b, f), (a, e)\}.$$

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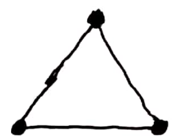
2.(a) A graph which can be represented by at least one plane drawing (drawing done on a plane surface) in which the edges meet only at the vertices is called a planar graph



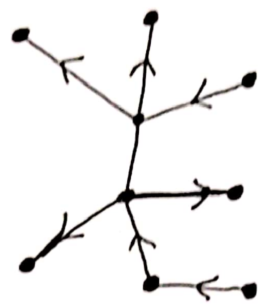
K_3

b) Two planar graphs G_1 & G_2 are said to be homeomorphic if one of these can be obtained from the other by the process of (i) insertion of ~~new~~ new vertices of degree two into its edges, or (ii) merger of adjacent edges

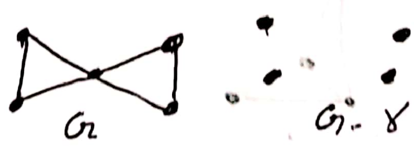
~~c) A K-chromatic, than k is called the chromatic number of G~~

c) A K-Chromatic graph is a graph that can be properly colored with k colors but not with less than K colors

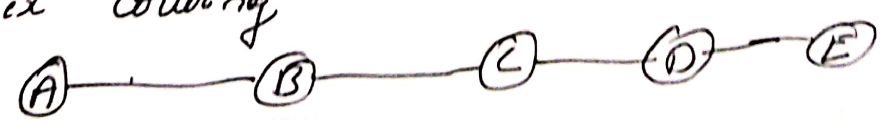
d) A directed tree T is called a rooted tree if (i) T contains a unique vertex, called the root whose in degree is equal to 0, and (ii) the in degrees of all other vertices of T are equal to 1.



Let G be a connected graph. A vertex $V \in G$ is called a cut vertex of 'G', if 'G-V' (Delete 'V' from 'G') results in a disconnected graph. Removing a cut vertex from a graph breaks it into two or more graphs



A covering graph is a sub-graph which contains either all the vertices of all the edges corresponding to some other graph. A sub graph which contain all the vertices is called a line/edge covering. A sub graph which contains all the edges is called vertex covering



- 4 vertices of degree 3
- 2 vertices of degree 4
- 1 vertex of degree 5

Let x = number of pendant (of degree 1) vertices

Total number of vertices $n = x + 4 + 2 + 1 = x + 7$

Step 1: Sum of degrees

Sum of degrees $= 1 \cdot x + 4 \cdot 3 + 2 \cdot 4 + 1 \cdot 5 = x + 12 + 8 + 5 = x + 25$

Also, since its a tree

$\Sigma \deg(v_i) = 2(n-1) = 2((x+7)-1) = 2(x+6) = 2x+12$

Step 2: Set up equation

$x + 25 = 2x + 12$

Subtract x from both sides

$25 = x + 12$

$x = 13$

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