CBCS SCHEME

USN | | | | | | | | | | |

BCS405A

Fourth Semester B.E./B.Tech. Degree Examination, June/July 2025

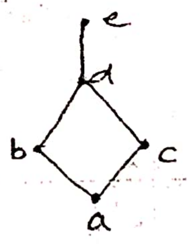
Discrete Mathematical Structures

Time: 3 hrs. Max. Marks: 100

Note: 1. Answer any FIVE full questions, choosing ONE full question from each module.
2. M : Marks , L: Bloom's level , C: Course outcomes.

		Module – 1	M	L	C
Q.1	a.	Define Tautology, show that $[(p \vee q) \wedge \{(p \to r) \wedge (q \to r)\}] \to r$	6	L1	CO1
	b.	Prove the following using the laws of logic : $\neg[\{(p \vee q) \wedge r\} \to \neg q] \Leftrightarrow \neg[\neg\{(p \vee q) \wedge r\} \vee \neg q] \Leftrightarrow q \wedge r$.	7	L2	CO1
	c.	Give i) a direct proof ii) an Indirect proof for the following statement "If n is an odd integer then n + 9 is an even integer".	7	L2	CO1
		OR			
Q.2	a.	Define i) an open statement ii) quantifiers	6	L2	CO1
	b.	Test the validity of the following arguments i) $p \wedge q$ $p \to (q \to r)$ $\therefore r$ ii) p $p \to \sim q$ $\sim q \to \sim r$ $\therefore \sim r$	7	L2	CO1
	c.	For the following statements the universe comprises all non – zero integers. Determine the truth value of each statement. i) $\exists x, \exists y\ [xy = 1]$ ii) $\exists x, \forall y\ [xy = 1]$ iii) $\forall x, \exists y\ [xy = 1]$ iv) $\exists x, \exists y\ [(2x + y = 5) \wedge (x - 3y = -8)]$ v) $\exists x, \exists y\ [(3x - y = 17) \wedge (2x + 4y = 3)]$	7	L2	CO1
		Module – 2			
Q.3	a.	Define the well ordering principle. By Mathematical induction, prove that $1 + 2 + 3 + \ldots + n = \frac{1}{2}n(n+1)$, $n \in z^+$.	6	L2	CO2
	b.	Prove that $F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$. For F_0, F_1, F_2, ... are the Fibonacci numbers.	7	L2	CO2
	c.	Find the number of permutations of the letters of the word 'MASSASAUGA'. In how many of these all four A's are together? How many of them begin with S's?	7	L3	CO2

OR

Q.4	a.	Prove that $4n < n^2 - 7$ for all positive integers $n \geq 6$.	6	L2	CO3
	b.	Find the co-efficients of $x^9 y^3$ in the expansion of $(2x - 3y)^{12}$.	7	L3	CO3
	c.	Let $a_0 = 1$, $a_1 = 2$, $a_3 = 3$ and $a_n = a_{n-1} + a_{n-3}$ for $n \geq 3$, prove that $a_n \leq 3^n$ for all +ve integers n.	7	L2	CO3
		Module – 3			
Q.5	a.	State Pigeon hole principle. Prove that if 30 dictionaries in a library contains a total of 61,327 pages then atleast one of dictionaries must have atleast 2045 pages.	6	L2	CO3
	b.	Define power set. For any sets A, B, $C \subseteq U$, prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.	7	L2	CO3
	c.	Let f and g be functions from R to R defined by $f(x) = ax + b$ and $g(x) = 1 - x + x^2$ if $(gof)(x) = 9x^2 - 9x + 3$, determine a & b.	7	L3	CO3
		OR			
Q.6	a.	Let $f: R \to R$ be defined by $f(x) = \begin{cases} 3x - 5, \text{if } x > 0 \\ 1 - 3x, \text{if } x \leq 0 \end{cases}$ Find $f^{-1}(-5, 5)$ and $f^{-1}(-6, 5)$.	6	L2	CO3
	b.	Let N be the set of Natural numbers. Let a relation R be defined by $R = \{(a, b) / a \in N, b \in N, a - b \text{ is divisible by } 5\}$. Prove that R is an equivalence relation.	7	L2	CO3
	c.	For A = { a, b, c, d, e}, the Hasse diagram for the poset(A, R) is as shown below : i) Determine the relation matrix for R ii) Construct the diagraph for R 	7	L3	CO3
		Module – 4			
Q.7	a.	Determine the number of integers between 1 and 250 that are divisible by 3 and not divisible by 5 and 7	6	L3	CO4
	b.	Solve the recurrence relation $F_{n+2} = F_{n+1} + F_n$, where $n \geq 0$ and $F_0 = 0$, $F_1 = 1$.	7	L2	CO4
	c.	Define Derangement. Find the number of derangement of 1, 2, 3, and 4.	7	L3	CO4

OR

Q.8	a.	Find the Rook polynomial for the chess board contain 4 squares as shown in the Fig.Q8(a). 1 2 3 4 Fig.Q8(a)	6	L3	CO4
	b.	Solve the recurrence relation $a_n = 5a_{n-1} + 6a_{n-2}$, $n \geq 2$, $a_0 = 1$, $a_1 = 3$.	7	L2	CO4
	c.	Find the distinct numbers which are multiples of at least one of 15, 40 and 35 not exceeding 1000.	7	L3	CO4
		Module – 5			
Q.9	a.	Define group and subgroup with example each.	6	L1	CO5
	b.	State and prove Lagrange's theorem.	7	L2	CO5
	c.	Define Klein 4 group. Verify A = {e, a, b, c} is a Klein 4 group.	7	L2	CO5
		OR			
Q.10	a.	Prove that the intersection of two subgroup of a group is a subgroup of the group.	6	L2	CO5
	b.	Prove that the cube roots of unity form a group under the multiplication.	7	L2	CO5
	c.	Let G = S_4, the symmetric group of order 4, for $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$, find the subgroup $H = \langle a \rangle$, determine the number of left cosets of H in G.	7	L3	CO5

Module - 1

Q1 (a)

Tautology:- A compound proposition which is true for all possible truth values of its components is called a tautology.

p	q	r	$p \vee q$ A	$p \to r$ B	$q \to r$ C	$(B \wedge C)$ D	$(A \wedge D)$ E	$E \to r$
0	0	0	0	1	1	1	0	1
0	0	1	0	1	1	1	0	1
0	1	0	1	1	0	0	0	1
0	1	1	1	1	1	1	1	1
1	0	0	1	0	1	0	0	1
1	0	1	1	1	1	1	1	1
1	1	0	1	0	0	0	0	1
1	1	1	1	1	1	1	1	1

Q1 (b)

$$\neg[\{(p\vee q)\wedge r\}\rightarrow\neg q]\Leftrightarrow\neg[\neg[(p\vee q)\wedge r]\vee\neg q]\Leftrightarrow q\wedge r$$

Sol:- Consider,

$$\neg[\{(p\vee q)\wedge r\}\rightarrow\neg q]$$

$\Leftrightarrow\neg[\neg[(p\vee q)\wedge r]\vee\neg q]$ (by demorgan's law and by double negation)

$\Leftrightarrow [(p\vee q)\wedge r]\wedge q$

$\Leftrightarrow [(p\vee q)\wedge(r\wedge q)]$ (Associative law)

$\Leftrightarrow (p\vee q)\wedge(q\wedge r)$ (commutative law)

$\Leftrightarrow [(p\vee q)\wedge q]\wedge r$ (Absorption law)

$\Leftrightarrow \underline{q\wedge r}$

Q1 (c)

Sol: We first form the propositions

p: n is an odd integer

q: $(n+9)$ is an even integer

(i) Direct proof

- Hypothesis :- Assume that p is true
  i.e., n is an odd integer $\Rightarrow n = 2x+1$, $x \in \mathbb{Z}$

- Analysis :- Consider q: $(n+9) = (2x+1)+9$

$$= (2x+10)$$
$$= 2(x+5)$$
$$= 2m, \text{ where } m = x+5 \in \mathbb{Z}$$

Thus, $q: (n+9) = 2m \Rightarrow (n+9)$ is an even integer is true

- Conclusion :- Thus, if n is odd, then $(n+9)$ is an even integer

i.e., $p \rightarrow q$ is true

(ii) Indirect proof

Hypothesis :- Assume that $\sim q$ is true (or) p is true and q is false

i.e., $\sim q$: $(n+9)$ is an odd integer is true

Analysis :- $(n+9) = (2x+1)$

$\Rightarrow n = 2x+1-9$

$\Rightarrow n = 2x-8$

$\Rightarrow n = 2(x-4)$

$\Rightarrow n = 2m$, where $m = x-4 \in \mathbb{Z}$

i.e., $n = 2m$ which implies that n is even this is equivalent to $\sim p$ being true.

Conclusion :- Therefore $\sim q \longrightarrow \sim p$ is true and we have $\sim q \longrightarrow \sim p \iff p \longrightarrow q$, hence $p \longrightarrow q$ is true.

Thus, if n is odd integer, then $(n+9)$ is an even integer.

Q2 (a)

(i) an open statement

- It is a sentence which involves one or more variables which turns out as a statement/ propositions when the variables take permissible values. Obviously these statements will be either true or false.

(ii) quantifiers

open statement involving words of the form "for all" or "for some" with reference to the variables involved are called quantifiers since quantity is involved in it.

Q2 (b)

(i) $p \wedge q$

$p \rightarrow (q \rightarrow r)$

$\therefore r$

Sol:- Since $p \wedge q$ is true, both p and q are true. Since p is true and $p \rightarrow (q \rightarrow r)$ is true, $q \rightarrow r$ has to be true. Since q is true and $q \rightarrow r$ is true, r has to be true. Hence the given argument is valid.

(ii) p

$p \rightarrow \neg q$

$\neg q \rightarrow \neg r$

$\therefore \neg r$

Sol:- The premises $p \rightarrow \neg q$ and $\neg q \rightarrow \neg r$ together yield the premise $p \rightarrow \neg r$. Since p is true, this premise $(p \rightarrow \neg r)$ establishes that $\neg r$ is true. Hence the given argument is valid.

3(C)

For the following statements, the universe comprises all non-zero integers. Determine the truth value of each statement.

(a) $\exists x \exists y [xy = 1]$ (b) $\exists x \forall y [xy = 1]$ (c) $\forall x \exists y [xy = 1]$

(d) $\exists x \exists y [(2x+y=5) \wedge (x-3y=-8)]$ (e) $\exists x \exists y [(3x-y=7) \wedge (2x+4y=3)]$

Sol:- Let S be the universal set containing all non-zero integers.

(a) $\exists x \exists y [xy = 1]$

This statement is true, since there exist non-zero integers $x=1$ and $y=1$ such that $xy=1$

(b) $\exists x \forall y [xy = 1]$

This statement is false, since for every non-zero integer $x=1$, $y=2$ is also a non-zero integer, but $xy=(2)(1)=2$ which is greater than ~~correspond~~ 1. Hence $xy \neq 1$.

(c) $\forall x \exists y [xy = 1]$

This statement is false, because for $x=2$, there doesn't exist a non-zero integer y such that the statement holds for any non-zero integer y, $xy = 2x > 2.1 = 2 > 1$. Hence $xy \neq 1$.

(d) This statement just states that there exist non-zero integers x and y such that the two equations hold.

Consider, Secon equation $x - 3y = -8$

$$x = -8 + 3y \quad — (1)$$

Then, by inserting it in the first equation, we get

(50)

$$2(-8+3y)+y=5$$

$$y=3$$

Now it follows that $x=-8+3y=-8+3(3)=1$

Statement is true.

(v) Statement is true.

Q3 (a)

Let $S(n): 1+2+3+ \cdots + n = \frac{n(n+1)}{2}$

Step-1 : $S(1)$: LHS $=1$ and RHS $= \frac{1(1+1)}{2} = \frac{1\times2}{2} = 1$

$\therefore S(1)$ is true

Step-2 :- We shall assume that $S(n)$ is true for $n=K$.

That is

$$S(K) = 1+2+3+ \cdots + K = \frac{K(K+1)}{2} \quad \text{--- (1)}$$

We shall add the term $(K+1)$ onto both sides of (1)

i.e., $1+2+3+ \cdots + K + (K+1) = \frac{K(K+1)}{2} + (K+1)$

i.e., $1+2+3+ \cdots + K + (K+1) = \frac{(K+1)(K+2)}{2}$

or $1+2+3+ \cdots + (K+1) = \frac{(K+1)[(K+1)+1]}{2} \quad \text{--- (2)}$

Comparing (1) and (2), we conclude that $S(K+1)$ is true

Thus, by the principle of mathematical induction, $S(n)$ is true for all $n \geq 1$

3 (b) For the Fibonacci sequence $F_0, F_1, F_2, \ldots$ prove that

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

$(\text{anything})^0 = 1$

Sol:- Let, $F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$

Step-1:- when $n=0$, $F_0 = \frac{1}{\sqrt{5}}\left[(1)-(1)\right] = 0$

when $n=1$, $F_1 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)\right] = 1$

But, we have $F_0 = 0$ and $F_1 = 1$ from Fibonacci recursive definition.

$\therefore$ The result is true when $n=0$ and 1.

Step 2:-
Assume that F_n is true for $n = 0, 1, 2, \ldots K$, where $K \geq 1$.

i.e., $F_K = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^K - \left(\frac{1-\sqrt{5}}{2}\right)^K\right]$ —— (1)

Consider, $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$ and $F_0 = 0$, $F_1 = 1$ (by definition) —— (2)

Substitute $n = K+1$ in (2), gives

$$F_{K+1} = F_K + F_{K-1}$$

$$F_{K+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{K} - \left(\frac{1-\sqrt{5}}{2}\right)^{K}\right] + \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{K-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{K-1}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{K} - \left(\frac{1-\sqrt{5}}{2}\right)^{K} + \left(\frac{1+\sqrt{5}}{2}\right)^{K-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{K-1}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{K-1}\left\{\left(\frac{1+\sqrt{5}}{2}\right)+1\right\} - \left(\frac{1-\sqrt{5}}{2}\right)^{K-1}\left\{\left(\frac{1-\sqrt{5}}{2}\right)+1\right\}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{K-1}\left\{\frac{3+\sqrt{5}}{2}\right\} - \left(\frac{1-\sqrt{5}}{2}\right)^{K-1}\left\{\frac{3-\sqrt{5}}{2}\right\}\right]$$

But, $(1+\sqrt{5})^2 = 1+2\sqrt{5}+5 = 6+2\sqrt{5} = 2(3+\sqrt{5})$

$(1-\sqrt{5})^2 = 1-2\sqrt{5}+5 = 6-2\sqrt{5} = 2(3-\sqrt{5})$

$\therefore \frac{3+\sqrt{5}}{2} = \frac{(1+\sqrt{5})^2}{2^2}$ and $\frac{3-\sqrt{5}}{2} = \frac{(1-\sqrt{5})^2}{2^2}$

Hence,

$$F_{K+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{K-1}\left(\frac{1+\sqrt{5}}{2}\right)^{2} - \left(\frac{1-\sqrt{5}}{2}\right)^{K-1}\left(\frac{1-\sqrt{5}}{2}\right)^{2}\right]$$

$$F_{K+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{K+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{K+1}\right] \quad \text{—— (2)}$$

- By comparing (1) and (2), we conclude that the result is true for $n = K+1$.
- Thus by the principle of mathematical induction the result is true for all positive integral values of n.

Q3 c

- The given word has 10 letters of which 4 are A, 3 are S and 1 each are M, U and G. Therefore, the required number of permutations is

$$\frac{10!}{4!\,3!\,1!\,1!\,1!} = 25{,}200$$

- If, in a permutation, all A's are to be together, we treat all of A's as one single letter. Then the letters to be permuted read (AAAA), S, S, S, M, U, G, (which are 7 in number), and the number of permutation is.

$$\frac{7!}{1!\,3!\,1!\,1!\,1!} = 840$$

- For permutations beginning with S, there occur nine open positions to fill, where two are S, four are A, and one each are M, U, G. The number of such permutations is

$$\frac{9!}{2!\,4!\,1!\,1!\,1!} = 7560.$$

4 (a)

P.T $4n < (n^2 - 7)$ for all positive integers $n \geq 6$.

Sol:- Let $S(n): 4n < (n^2 - 7)$, $n \in Z^+$, $n \geq 6$

(1) Basis step

Since $n \geq 6$, we shall take $n = 6$ initially.

Hence, $S(6) = (4)(6) < (6^2 - 7)$ or $24 < 29$ is true.

$\therefore$ $S(6)$ is true.

(2) Induction step

- Assume that $S(n)$ is true for $n = K$

i.e., $4K < (K^2 - 7)$, $K \in Z^+$, $K \geq 6$ ——— (1)

- Consider, $4(K+1) < ((K+1)^2 - 7)$

i.e., $4(K+1) < (K^2 + 1 + 2K - 7)$

i.e., $4(K+1) < \{(K^2 - 7) + (2K+1)\}$

i.e., $4K + 4 < (K^2 - 7) + (2K+1)$

- from (1), we have $4K < (K^2 - 7)$

~~Hence~~, Since $K \geq 6$, $2K + 1 = 2(6) + 1 = 7 > 4$ or $4 < 7$

i.e., $4 < (2K+1)$ is valid.

Hence, $4(K+1) < (K+1)^2 - 7$ is true ——— (2)

Comparing (1) and (2), we conclude that $S(K+1)$ is true.

Thus by the principle of mathematical induction $S(n)$ is true for all $n \geq 6$.

Q4 (b) By Binomial theorem, we have

$$(x+y)^n = \sum_{r=0}^{n} {}^nC_r \, x^r y^{n-r} \quad \text{---} (1)$$

ie. $$(2x-3y)^{12} = \sum_{r=0}^{12} {}^{12}C_r (2x)^r (-3y)^{12-r}$$

$$(2x-3y)^{12} = \sum_{r=0}^{12} 2^r (-3)^{12-r} x^r y^{12-r} \, {}^{12}C_r$$

By taking, $r=9$, we have

$$(2x-3y)^{12} = \sum_{r=9}^{12} 2^9 (-3)^3 x^9 y^3 \, {}^{12}C_r$$

$\therefore$ The coefficient of $x^9y^3 = {}^{12}C_r \, 2^9 (-3)^3$

$$= -(2^9 \times 3^3) \times \frac{12!}{9!\,3!}$$

$$= -2^9 \times 3^3 \times \frac{12 \times 11 \times 10}{6}$$

$$= -2^{10} \times 3^3 \times 11 \times 10.$$

ie. the coefficient of x^9y^3 is $-2^{10} \times 3^3 \times 11 \times 10$.

Q4 (c) Consider $S(n) = a_n \leq 3^n$

Step-I $a_0 = 1 \leq 3^0$; $a_1 = 2^1 < 3^1$; $a_2 = 3 < 3^2$

$\therefore$ $S(0)$, $S(1)$, $S(2)$ are all true.

Step-II We shall assume that $S(n)$ is true for where $n = 0, 1, 2, 3, \ldots, k$, $k \geq 2$.

i.e. $S_k = a_k \leq 3^k$ —— (1)

Now, $a_{k+1} = a_k + a_{k-1} + a_{k-2}$, using the definition of a_n.

i.e. $a_{k+1} \leq 3^k + 3^{k-1} + 3^{k-2}$

$\leq 3^k + 3^k + 3^k$ because $3^{k-1} \leq 3^k$ and $3^{k-2} \leq 3^k$

$\Rightarrow$ $a_{k+1} = 3 \times 3^k$

$= 3^{k+1}$ —— (2).

Thus, $S(k+1)$ is true.

comparing (1) and (2), we conclude that $S(k+1)$ is true. Thus by the principale of mathematical induction, $S(n)$ is true for all positive integers.

Q5 (a) Here $m = 61,327$.

$$p = \left\lfloor \frac{m-1}{n} \right\rfloor = \left\lfloor \frac{61327-1}{30} \right\rfloor$$

$$\Rightarrow p = \lfloor 2044.2 \rfloor$$

$$\Rightarrow p = 2044$$

Generalization $= p+1$
$= 2044+1$
$= 2045.$

Hence proved.

Q5 (b) Power Set:

A power set is the set of all possible subsets of a given set, including the empty set and the set itself.

Consider LHS $= A \times (B \cup C)$

Let $(x,y) \in A \times (B \cup C)$

$\Leftrightarrow x \in A$ and $y \in B \cup C$

$\Leftrightarrow x \in A$ and $(y \in B$ or $y \in C)$

$\Leftrightarrow (x \in A$ and $y \in B)$ or $(x \in A$ and $y \in C)$

$\Leftrightarrow (x \in A \times B)$ or $(x \in A \times C)$

$\Leftrightarrow x \in (A \times B) \cup (A \times C)$

$\Rightarrow$ RHS $= (A \times B) \cup (A \times C)$.

Thus, $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Q5 (c) $gof(x) = 9x^2 - 9x + 3$

$\Rightarrow g(f(x)) = 9x^2 - 9x + 3$

$\Rightarrow g(ax+b) = 9x^2 - 9x + 3$

$\Rightarrow 1 - (ax+b) + (ax+b)^2 = 9x^2 - 9x + 3$

$\Rightarrow 1 - ax - b + a^2x^2 + b^2 + 2abx = 9x^2 - 9x + 3$

$\Rightarrow a^2x^2 + (2ab - a)x + (1 - b + b^2) = 9x^2 - 9x + 3$

Equating the coefficients of x, x^2 & constants

$x^2 : \quad a^2 = 9 \Rightarrow a = \pm 3$

$x : \quad 2ab - a = -9 \Rightarrow$ $\underline{\text{when } a = 3}$

$6b - 3 = -9$

$\Rightarrow \boxed{b = -1}$

$\Rightarrow$ $\underline{\text{when } a = -3}$

$-6b + 3 = -9$

$\Rightarrow \boxed{b = 2}$

Q6 (a) Let f from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$f(x) = \begin{cases} 3x-5 & \text{for } x>0 \\ -3x+1 & \text{for } x\leq 0 \end{cases}$$

i) Let $f^{-1}([-5,5]) = x$

$\Rightarrow f(x) = [-5,5]$

$\Rightarrow -5 \leq f(x) \leq 5$

Case I $\quad -5 \leq 3x-5 \leq 5$

$-5+5 \leq 3x-5+5 \leq 5+5$

$0 \leq 3x \leq 10$

$\Rightarrow 0 \leq x \leq \frac{10}{3}$

Case-II $\quad -5 \leq -3x+1 \leq 5$

$-1-5 \leq -3x+1-1 \leq 5-1$

$\Rightarrow -6 \leq -3x \leq 4$

$\Rightarrow 6 \geq 3x \geq -4$

$\Rightarrow 2 \geq x \geq -4/3$

or $\quad -4/3 \leq x \leq 2$

one interval is $[-4/3, 10/3]$.

Again let $f^{-1}([-6,5]) = x$

$\Rightarrow f(x) = [-6,5]$

$\Rightarrow -6 \leq f(x) \leq 5.$

Case-I $\quad -6 \leq 3x-5 \leq 5$

$\Rightarrow -6+5 \leq 3x-5+5 \leq 5+5$

$\Rightarrow -1 \leq 3x \leq 10$

$$-1 \leq 3x \leq 10$$

$$\Rightarrow -\frac{1}{3} \leq x \leq \frac{10}{3}.$$

<u>Case II</u>

$$-6 \leq -3x+1 \leq 5$$

$$-6-1 \leq -3x \leq 5-1$$

$$\Rightarrow -7 \leq -3x \leq 4$$

$$\Rightarrow 7 \geq 3x \geq -4$$

$$\Rightarrow \frac{7}{3} \geq x \geq \frac{-4}{3}$$

$\therefore$ the interval is $[-4/3, 10/3]$.

Q6 (b) Given for $a, b \in \mathbb{Z}$ then $a \equiv b \pmod 5$

ie. $5 \mid a-b$

or $a-b$ is a multiple of 5.

(i) $a \equiv a \pmod 5$ ie. $a-a=0$ is a multiple of 5.

$\therefore$ R is reflexive.

(ii) $\forall\, a, b \in \mathbb{Z}$, $a \equiv b \pmod 5 \Rightarrow a-b$ is a multiple of 5.

$\Rightarrow b-a$ is a multiple of 5.

$\Rightarrow b \equiv a \pmod 5$

$\therefore$ R is symmetric.

(iii) $\forall\, a, b, c \in \mathbb{Z}$, $a \equiv b \pmod 5$ & $b \equiv c \pmod 5$

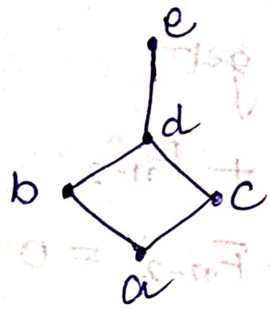
$\Rightarrow a-b$ & $b-c$ is a multiple of 5.

$\Rightarrow (a-b)+(b-c) = a-c$ is a multiple of 5.

$\therefore$ R is transitive.

Q6 (a) Given $A = \{a, b, c, d, e\}$

The Hasse diagram is given by



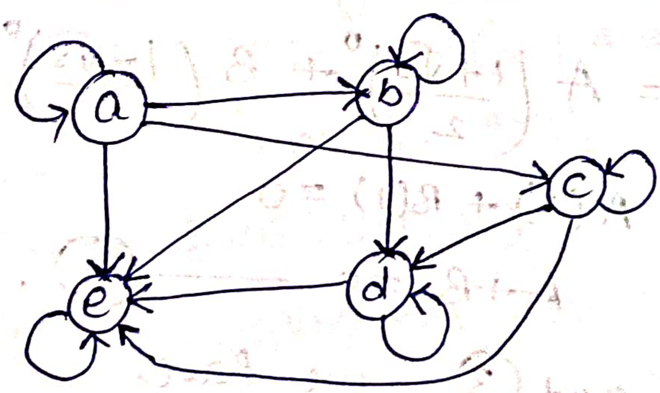
The relation matrix for R is given as

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

By examining the Hasse diagram, we have

$R = \{(a,a), (a,b), (a,c), (a,d), (a,e), (b,b), (b,d), (b,e), (c,c), (c,d), (c,e), (d,d), (d,e), (e,e)\}$

The digraph of R is as:



Hence, it is equivalence relation.

Q7 (a) Let A be the set of numbers, $1 \leq n \leq 250$ that are divisible by 3.

Let B be the set of numbers, $1 \leq n \leq 250$ that are divisible by 5.

Let C be the set of numbers (integers), $1 \leq n \leq 250$ that are divisible by 7.

$$\therefore |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

But $|A| = \left\lfloor \frac{250}{3} \right\rfloor = \lfloor 83.3 \rfloor = 83$

$|B| = \left\lfloor \frac{250}{5} \right\rfloor = \lfloor 50 \rfloor = 50$

& $|C| = \left\lfloor \frac{250}{7} \right\rfloor = \lfloor 35.7 \rfloor = 35$

Also $|A \cap B| = \left\lfloor \frac{250}{3 \times 5} \right\rfloor = \lfloor 41.6 \rfloor = 41$

$|A \cap C| = \left\lfloor \frac{250}{3 \times 7} \right\rfloor = \lfloor 11.90 \rfloor = 11$

$|B \cap C| = \left\lfloor \frac{250}{5 \times 7} \right\rfloor = \lfloor 7.14 \rfloor = 7$

& $|A \cap B \cap C| = \left\lfloor \frac{250}{3 \times 5 \times 7} \right\rfloor = \lfloor 2.38 \rfloor = 2$

$\therefore |A \cup B \cup C| = 83 + 50 + 35 - 41 - 11 - 7 + 2$
$= 111$

$\therefore$ the number of integers not divisible by 3, 5 and 7 are (for $1 \le n \le 250$) $= 250 - 111$
$= 139$

Now, $|B \cup C| = |B| + |C| - |B \cap C|$
$= 50 + 35 - 7$
$= 78$

$\therefore$ the number of integers ($1 \le n \le 250$) not divisible by 5 and 7 are $= 250 - 78$
$= 172$

Now, the number of integers ($1 \le n \le 250$) not divisible by 5 and 7 but divisible by 3 are =

(No. of integers not divisible by 5 & 7) − (No. of integers not divisible by 3, 5 & 7)

$= 172 - 111$

$= \underline{\underline{61}}$.

Q7 (b) Given $F_{n+2} = F_{n+1} + F_n$ where $n \geq 0$.

The given recurrence relation is not in the standard form of second order Homogeneous.

So take $n = n-2$, we get

$$F_n = F_{n-1} + F_{n-2}$$

$$\text{or } F_n - F_{n-1} - F_{n-2} = 0, \quad n \geq 2.$$

The characteristic Equation is

$$k^2 - k + 1 = 0$$

$$\Rightarrow \quad k = \frac{1 \pm \sqrt{5}}{2}.$$

ie, $k_1 = \dfrac{1+\sqrt{5}}{2}$ and $k_2 = \dfrac{1-\sqrt{5}}{2}$.

$\therefore$ The general solution is given as

$$F_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n \quad \text{——— } (1).$$

But $F_0 = 0$ and $F_1 = 1$ (Given).

$\quad$ └ (2) $\qquad$ └ (3).

from (1) and (2), we have

$$F_0 = A\left(\frac{1+\sqrt{5}}{2}\right)^0 + B\left(\frac{1-\sqrt{5}}{2}\right)^0 = 0.$$

$$\Rightarrow \quad A(1) + B(1) = 0$$

$$\Rightarrow \quad A + B = 0 \quad \text{——— } (4).$$

from (1) and (3), we have

CMR

$$F_1 = A\left(\frac{1+\sqrt{5}}{2}\right)^1 + B\left(\frac{1-\sqrt{5}}{2}\right)^1 = 1$$

$$\Rightarrow \quad A(1+\sqrt{5}) + B(1-\sqrt{5}) = 2 \quad \text{—— (5)}$$

Solving (4) and (5), we get

$$\boxed{A = \frac{1}{\sqrt{5}}} \text{ and } \boxed{B = -\frac{1}{\sqrt{5}}}$$

Hence, the particular solution is

$$\boxed{F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n}$$

A permutation of n distinct objects in which *none* of the objects is in its natural (original) place is called a ***derangement***. For example, a permutation of the integers $1, 2, 3, 4, \ldots n$, in which 1 is not in the first place, 2 is not in the second place, 3 is not in the third place, and so on, and n is not in the nth place is a derangement.

The number of possible derangements of n distinct objects $1, 2, 3, \ldots, n$ is denoted by d_n. If there is only one object, it continues to be in its original place in every arrangement; therefore $d_1 = 0$. If there are two objects, a derangement can be done in only one way – by interchanging their places; therefore $d_2 = 1$. For three objects 1, 2, 3, the possible derangements are 231 and 312; therefore $d_3 = 2$.

Formula for d_n

The following is the formula for d_n for $n \geq 1$:

$$d_n = n!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right\}$$

$$= n! \times \sum_{k=0}^{n} \frac{(-1)^k}{k!} \qquad (1)$$

► Here, there are 4 objects. Therefore, the number of derangements is

$$d_4 = 4! \times \left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right\}$$

$$= 24 \times \left\{1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24}\right\}$$

$$= 12 - 4 + 1 = 9.$$

We can check that the nine derangements of 1, 2, 3, 4 are:

2143	2341	2413
3142	3412	3421
4123	4312	4321

$$r_1 = n = \text{number of squares in the board.}$$

Example 1. Consider the board shown in Figure 7.1, which contains 4 squares.

8a

1	2
3	4

Figure 7.1

For this board, $r_1 = 4$. The number of ways in which two rooks can be placed on this board such that no two of them capture each other is 2; the two possible positions are (1,4) and (2,3). Thus, $r_2 = 2$. Three rooks cannot be placed on the board such that no two pawns capture each other. Thus, $r_3 = 0$. Similarly, $r_4 = 0$.

Accordingly, the rook polynomial for the board is

$$r(C, x) = 1 + r_1 x + r_2 x^2 = 1 + 4x + 2x^2.$$

8a

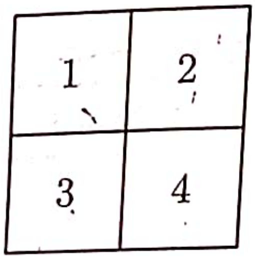


Figure 7.1

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Accordingly, the rook polynomial for the board is

$$r(C, x) = 1 + r_1x + r_2x^2 = 1 + 4x + 2x^2.$$

8 (b) $a_n = 5a_{n-1} + 6a_{n-2} \quad n \geq 2$

$a_0 = 1 \quad a_1 = 3$

Sol The characteristic equation is

$C_n k^2 + C_{n-1} K + C_{n-2} = 0$

$K^2 - 5K + 6 = 0$

$(K-3)(K-2) = 0$

$K = 3, 2 \qquad K_1 = 3 \qquad K_2 = 2$

General solution is

$a_n = A\, 3^n + B\, 2^n$.

$a_0 = 1 \Rightarrow n = 0$

$a_0 = A 3^0 + B 2^0$

$A + B = 1$ —— (1)

$a_1 = 3 \qquad n = 1$

$a_1 = A(3) + B(2)$

~~2A+2B=2~~.

$3A + 2B = 3$ —— (2)

Solving 1 and 2

~~A+B=1~~

$$\cancel{3A} + 3B = 1$$ Multiply eq (1) by 3

$$\cancel{3A} + 2B = 3$$

$$- \quad - \quad -$$

$$\boxed{B = -2}$$

$$B = -2$$

$$\Rightarrow A + B = 1$$

$$A - 2 = 1$$

$$A = 1 + 2 = 3.$$

$$\therefore \quad a_n = 3 \times 3^n + (-2)\, 2^n.$$

9(a) Group Definition:

Definition:- Let G be a non-empty set together with a binary operation $*$ defined on it. Then the algebraic structure $(G, *)$ is said to be a group, if the binary operation satisfies the following properties.

1. Closure property

$$a, b \in G, \ (a * b) \in G$$

2. Associative property

(For any 3 elements $a, b, c \in G$

$$a * (b * c) = (a * b) * c$$

3. Existence of identity.

$$a * e = a = e * a \quad \forall a \in G.$$

4. Existence of inverse

(For each element $a \in G$, there exists an element $b \in G$ such that)

$$b * a = e = a * b$$

(Here b is called an inverse of a and is denoted by $b = a^{-1}$. Thus $a^{-1} \in G$ such that

$$a^{-1} * a = e = a * a^{-1}$$

3

Abelian (or commutative) Group

A group G is said to be an abelian group if in addition to the above 4 properties the following property is also satisfied.

$\boxed{5.}$ Commutative property:

$$a * b = b * a, \quad \forall \; a, b \in G.$$

Problem - 1

Example

① Show that the set of integers is a group under the operation of addition.

Soln:- (i) Closure property

Since sum of 2 integers is an integer,

$\therefore$ I is closed with respect to addition.

(ii) Associative property.

If a, b, c are any arbitrary elements in I, then

$(a+b)+c = a+(b+c)$ is true.

iii) Existence of identity.

The number $0 \in I$, such that

$a + 0 = a = 0 + a, \forall a \in I$

Thus the integer 0 is the additive identity.

iv) Existence of inverse.

If $a \in I$, then $-a \in I$ such that

$a + (-a) = 0 = (-a) + a$

Thus every element in I has its additive inverse.

$\therefore$ I is a group with respect to addition.

Again, it is noted that addition of integers is commutative.

ie) $a + b = b + a, \forall a, b \in I$

Hence $(I, +)$ is an abelian group.

9(b) Lagrange's Theorem

Statement :- If G is a finite group and H is a subgroup of G, then the order of H divides the order of G.

30

Proof :- Since G is a finite group,

2.

H is finite,

20 ∴ the number of cosets of H in G is finite. Let $a \in G$, then Ha is right coset of H in G

~~Let~~ ∴ $Ha_1, Ha_2, \ldots Ha_k$ be the distinct right cosets of H in G.

Then, by the right coset decomposition of G we have.

$$G = Ha_1 \cup Ha_2 \cup \ldots \cup Ha_k$$

~~So~~ So that.

$$o(G) = o(Ha_1) + o(Ha_2) + \ldots + o(Ha_k)$$

But $o(Ha_1) = o(Ha_2) = \ldots = o(Ha_k) = o(H)$

$$\therefore o(G) = o(H) + (o(H)) + o(H) + \ldots + o(H)$$

$$= k\, o(H)$$

$$k = \frac{o(G)}{o(H)}$$

This shows that $o(H)$ divides $o(G)$.

$(a*b)*c$ ac

ab'

operation.

Klein - 4 group: 9(c)

The Klein four group is an abelian group with 4 elements in which each element is self-inverse and in which composing of any two of the 3 non-identity elements produces the third one.

Eg: $A = \{e, a, b, c\}$.

*	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

(1) Closure Law:-

Each row and Column Contains all the elements of the Set.

(2) Associative:-

$$(a*b)*c = a*(b*c)$$

$$c*c = a*a.$$

$$e = e$$

$\therefore (a*b)*c = a*(b*c).$

(3) Identity:-

$$a*e = a = e*a.$$

$$b*e = b = e*b.$$

$$c*e = c = e*c.$$

Hence 'e' is the Identity element.

(4) Inverse:-

$a*a = e$

$b*b = e$ So each elem is its own

$c*c = e$ inverse.

(5) Commutative Law:-

$a*b = b*a.$ $a*c = c*a.$

$c = c.$ $b = b.$

$b*c = c*b$ we Conclude A is an Abelian

$K-4$

group.

10(a) (4) If H and K are two subgroups of G, Then $H \cap K$ is also a subgroup of G.

Proof:- Let e be an identity of G.

Then e will also be the identity element in H as well as K.

2) i.e) $e \in H \cap K$

Thus $e \in H \cap K$ and $H \cap K \neq \phi$

If $a, b \in H$, then $a * b^{-1} \in H$

Illy if $a, b \in K$ and $a * b^{-1} \in K$

then $a * b^{-1} \in H \cap K$

Hence, $H \cap K$ is subgroup of G.

3) Show that the set, $\{1, \omega, \omega^2\}$ when $1, \omega, \omega^2$ are cube roots of unity, form an abelian group under the operation of ordinary multiplication.

(or)

show that ~~fourth~~ cube roots of unity is abelian group.

Soln:- we prepare a composition table as follows:

$\times$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

$$\boxed{\omega = \frac{-1+i\sqrt{3}}{2} \quad \omega^2 = 1/\omega}$$

$\omega^3 = 1$

The following properties can be verified from the composition table.

(i) closure property.

Since all the entries in the composition table are also elements of the set G.

$\therefore$ it is closed for multiplication

(ii) Associative property.

$$1\cdot(\omega\cdot\omega^2) = (1\cdot\omega)\cdot\omega^2.$$

ordinary multiplication is always associative.

(iii) ~~Inverse~~ Identity element.

From the above table.

$1\cdot 1 = 1 \quad . \quad 1\cdot\omega = \omega = \omega\cdot 1$

$1\cdot\omega^2 = \omega^2 = \omega^2\cdot 1$

iv) Inverse element:

$1 \cdot 1 = 1$ and $\omega^2 \cdot \omega = \omega \cdot \omega^2 = \omega^3 = 1$

$\therefore$ inverse of $1, \omega, \omega^2$ are $1, \omega^2, \omega$ respectively.

5

v) Commutative.

$1 \cdot \omega = \omega = \omega \cdot 1.$

$\omega \cdot \omega^2 = \omega^3 = \omega^2 \cdot \omega$

$\omega^2 \cdot 1 = \omega^2 = 1 \cdot \omega^2$

(1) Let $G = S_4$, the symmetric group of order (degree) 4.

10C

For $\alpha = \begin{pmatrix} 1 & 2 & 2 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$, find the subgroup $H = \langle \alpha \rangle$. Also, determine the no. of left cosets of H in G.

Soln:- given $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$

In S_4, the identity element.

$$P_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

$$H = \langle \alpha \rangle = \{\alpha^1, \alpha^2, \alpha^3, \alpha^4 \dots \alpha^n = p^n\}$$

$$\alpha^2 = \alpha \times \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

$$\alpha^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

$$\alpha^3 = \alpha^2 \times \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

$$\alpha^3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

$$\alpha^4 = \alpha^3 \times \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

$$\alpha^4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = P_1$$

$\therefore H = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \right\}$

The number of elements of the group is called order of the group and it is denoted by $o(G)$

$o(H) = 4$

$o(G) = o(S_4) = 4! = 24$

$\therefore$ number of distinct left cosets of H in G is $[G:H] = \frac{24}{4} = 6.$