CMR INSTITUTE OF TECHNOLOGY	USN										

Internal Assessment Test – I May 2025

Sub:	Mathematics II for CSE Stream							Code:	BMATS201
Date:	2/05/2025	Duration:	90 mins	Max Marks:	50	Sem:	II	Section:	**I,J,K,L**

Question 1 is compulsory and Answer any 6 from the remaining questions.

		Marks	OBE CO	OBE RBT
1	Evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\ dx$ by changing the order of integration.	[8]	CO1	L3
2	Show that $\int_0^{\pi/2} \sqrt{\tan\theta}\ d\theta = \frac{\pi}{\sqrt{2}}$ using Beta and Gamma functions.	[7]	CO1	L3
3	Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\, dz\ dy\, dx.$	[7]	CO1	L3
4	Using double integration, find the volume of the solid bounded by the planes x=0 ,y=0, z=0 . $x + y + z = 1$	[7]	CO2	L3

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5	If $\emptyset = x^3 + y^3 + z^3 - 3xyz$ at the point $(1,-1,2)$, Find $grad\emptyset$, find Unit Normal Vector	[7]	CO2	L2
6	Prove that the Spherical coordinate system is orthogonal.	[7]	CO2	L2
7	Show that the vector $\vec{F} = \frac{xi+yj}{x^2+y^2}$ is Solenoidal	[7]	CO2	L3
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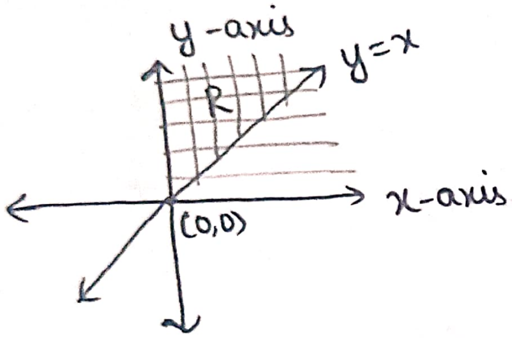
Q1. Given that,

$$I = \int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$$

$$I = \int_{x=0}^{x=\infty} \int_{y=x}^{y=\infty} \frac{e^{-y}}{y}\, dy\, dx.$$



Considering the following figure and changing the order

$$I = \int_{y=0}^{y=\infty} \int_{x=0}^{x=y} \frac{e^{-y}}{y}\, dx\, dy$$

Solving the integral, $I = \int_{y=0}^{y=\infty} \left[\frac{1}{y} e^{-y} . x\right]_0^y dy$

$$I = \int_{y=0}^{y=\infty} \left[\frac{1}{\cancel{y}} . e^{-y} . \cancel{y} - \frac{1}{y} . e^{-y}(0)\right] dy$$

$$I = \int_{y=0}^{y=\infty} \left[e^{-y}\right] dy$$

$[\because e^{-\infty} = 0]$

$$I = \left[\frac{e^{-y}}{(-1)}\right]_0^\infty = -\left[e^{-\infty} - e^{-0}\right]$$

$$I = -[0-1] = 1$$

$\therefore$ The value is $\boxed{I = 1}$

Q2. Given that,

$$I = \int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$$

To prove, $I = \pi/\sqrt{2}$.

Consider LHS,

$$I = \int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$$

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin\theta}}{\sqrt{\cos\theta}}\, d\theta$$

$$I = \int_0^{\pi/2} \sin^{1/2}\theta \, . \cos^{-1/2}\theta \, d\theta$$

Comparing to, $\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

$\therefore p = 1/2 \; , q = -1/2$

$$\therefore I = \frac{1}{2}\beta\left(\frac{\frac{1}{2}+1}{2}, \frac{\frac{-1}{2}+1}{2}\right) = \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{4}\right)$$

We know that, $\beta(m,n) = \frac{\gamma(m).\gamma(n)}{\gamma(m+n)}$

$$\therefore I = \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{4}\right) = \frac{1}{2}\,\frac{\gamma\left(\frac{3}{4}\right).\gamma\left(\frac{1}{4}\right)}{\gamma\left(\frac{3}{4}+\frac{1}{4}\right)}$$

$$I = \frac{1}{2} \cdot \frac{\gamma\left(\frac{3}{4}\right)\gamma\left(\frac{1}{4}\right)}{\gamma(1)}$$

We know that, $\gamma\left(\frac{3}{4}\right)\cdot\gamma\left(\frac{1}{4}\right) = \pi\sqrt{2}$

$\gamma(1) = 1$.

$$\therefore\ I = \frac{1}{2} \cdot \frac{\pi\sqrt{2}}{1} = \frac{\pi\sqrt{2}}{2} = \frac{\pi}{\sqrt{2}}.$$

$$\boxed{I = \frac{\pi}{\sqrt{2}} = RHS}.$$

Hence proved.

Q3. Given that,

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\ dz\ dy\ dx.$$

$$I = \int_{x=0}^{x=1} \int_{y=0}^{y=\sqrt{1-x^2}} \int_{z=0}^{z=\sqrt{1-x^2-y^2}} xyz\ dz\ dy\ dx$$

Integrating,

$$I = \int_{x=0}^{x=1} \int_{y=0}^{y=\sqrt{1-x^2}} \left[\frac{xyz^2}{2}\right]_0^{\sqrt{1-x^2-y^2}} dy\ dx$$

$$I = \int_{x=0}^{x=1} \int_{y=0}^{y=\sqrt{1-x^2}} \left[\frac{xy}{2}(1-x^2-y^2) - 0\right] dy\ dx$$

$$I_1 = \int_{x=0}^{x=1} \int_{y=0}^{y=\sqrt{1-x^2}} \frac{1}{2}\left[xy - x^3y - xy^3\right] dy\ dx.$$

$$I = \frac{1}{2} \int_{x=0}^{x=1} \left[\frac{xy^2}{2} - \frac{x^3y^2}{2} - \frac{xy^4}{4}\right]_0^{\sqrt{1-x^2}} dx$$

$$I = \frac{1}{2}\int_{x=0}^{x=1}\left[\frac{x(1-x^2)}{2} - \frac{x^3(1-x^2)}{2} - \frac{x(1-x^2)^2}{4} - 0\right]dx$$

$$[(1-x^2)^2 = 1+x^4-2x^2$$

$$I = \frac{1}{2}\int_{x=0}^{x=1}\left[\frac{x}{2} - \frac{x^3}{2} - \frac{x^3}{2} + \frac{x^5}{2} - \frac{x}{4} - \frac{x^5}{4} + \frac{2x^3}{4}\right]dx$$

$$I = \frac{1}{2}\int_{x=0}^{x=1}\left[\frac{x}{2} - \frac{x}{4} - \frac{2x^3}{2} + \frac{2x^3}{4} + \frac{x^5}{2} - \frac{x^5}{4}\right]dx$$

$$I = \frac{1}{2}\int_{x=0}^{x=1}\left[\frac{x}{4} - \frac{2x^3}{4} + \frac{x^5}{4}\right]dx$$

$$I = \frac{1}{8}\left[\frac{x^2}{2} - \frac{2x^4}{4} + \frac{x^6}{6}\right]_0^1$$

$$I = \frac{1}{8}\left(\left[\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right] - [0]\right) = \frac{1}{48}$$

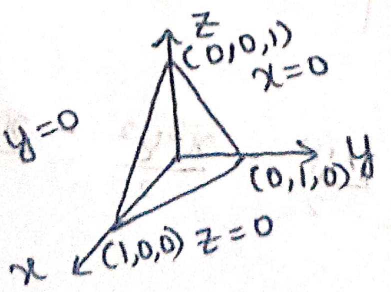
$$\boxed{I = \frac{1}{48}}$$

Q4. given that,

$x=0$, $y=0$, $z=0$

$x+y+z=1$

To find, volume of solid bounded by the planes.



Volume of the solid in cartesian form

$$V = \iint_R z \, dy \, dx$$

when $z=0$, $y \to 0$ to $y = 1-x$

when $y=0, z=0$, $x \to 0$ to $x=1$

$$\therefore V = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} (1-x-y) \, dy \, dx$$

$$V = \int_{x=0}^{x=1} \left[y - xy - \frac{y^2}{2} \right]_0^{1-x} dx$$

$$V = \int_{x=0}^{x=1} \left[(1-x) - x(1-x) - \frac{(1-x)^2}{2} - 0 \right] dx$$

$$\left[(1-x)^2 = 1 + x^2 - 2x \right]$$

$$V = \int_{x=0}^{x=1} \left[1 - x - x + x^2 - \frac{1}{2} - \frac{x^2}{2} + x \right] dx$$

$$V = \int_{x=0}^{x=1} \left[1 - \frac{1}{2} + x^2 - \frac{x^2}{2} - x \right] dx = \int_{x=0}^{x=1} \left[\frac{1}{2} + \frac{x^2}{2} - x \right] dx$$

$$V = \left[\frac{x}{2} + \frac{x^3}{6} - \frac{x^2}{2} \right]_0^1 = \left[\frac{1}{2} + \frac{1}{6} - \frac{1}{2} - 0 \right] = \frac{1}{6}$$

$\therefore$ Volume of the solid bounded by the planes,

$$\boxed{V = \frac{1}{6} \text{ cubic units}}$$

Q5. Given that,

$$\phi = x^3 + y^3 + z^3 - 3xyz \quad \text{at } (1,-1,2)$$

To find, grad ϕ and unit normal vector.

$$\text{grad } \phi = \nabla\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot (x^3+y^3+z^3-3xyz)$$

For $\nabla\phi \Rightarrow \frac{\partial\phi}{\partial x} = 3x^2 - 3yz$, $\frac{\partial\phi}{\partial y} = 3y^2 - 3xz$, $\frac{\partial\phi}{\partial z} = 3z^2 - 3xy$.

$$\therefore \text{grad } \phi = \nabla\phi = (3x^2 - 3yz)\hat{i} + (3y^2 - 3xz)\hat{j} + (3z^2 - 3xy)\hat{k}$$

$$[\nabla\phi]_{(1,-1,2)} = (3(1)^2 - 3(-1)(2))\hat{i} + (3(-1)^2 - 3(1)(2))\hat{j} + (3(2)^2 - 3(1)(-1))\hat{k}$$

$$= (3+6)\hat{i} + (3-6)\hat{j} + (12+3)\hat{k}$$

$$[\nabla\phi]_{(1,-1,2)} = 9\hat{i} - 3\hat{j} + 15\hat{k}$$

For unit normal vector, $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$

$$\nabla\phi = 9\hat{i} - 3\hat{j} + 15\hat{k}$$

$$|\nabla\phi| = \sqrt{81+9+225} = \sqrt{315} = 3\sqrt{35}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{9\hat{i} - 3\hat{j} + 15\hat{k}}{3\sqrt{35}}$$

$$\hat{n} = \frac{3\hat{i}}{\sqrt{35}} - \frac{1}{\sqrt{35}}\hat{j} + \frac{5}{\sqrt{35}}\hat{k}$$

Q6. Consider spherical coordinate system,

spherical polar coordinates $= (r, \theta, \phi)$

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$.

Scale factors, $h_1 = 1$, $h_2 = r$, $h_3 = r\sin\theta$.

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{r} = (r\sin\theta\cos\phi)\hat{i} + (r\sin\theta\sin\phi)\hat{j} + r\cos\theta\,\hat{k}$$

$$\frac{\partial\vec{r}}{\partial r} = (\sin\theta\cos\phi)\hat{i} + (\sin\theta\sin\phi)\hat{j} + \cos\theta$$

$$\frac{\partial\vec{r}}{\partial\theta} = (r\cos\theta\cos\phi)\hat{i} + (r\cos\theta\sin\phi)\hat{j} - r\sin\theta\,\hat{k}$$

$$\frac{\partial\vec{r}}{\partial\phi} = (-r\sin\theta\sin\phi)\hat{i} + (r\sin\theta\cos\phi)\hat{j}$$

$$\hat{e}_r = \frac{1}{h_1}\left[\frac{\partial\vec{r}}{\partial r}\right] \quad \hat{e}_\theta = \frac{1}{h_2}\left[\frac{\partial\vec{r}}{\partial\theta}\right] \quad \hat{e}_\phi = \frac{1}{h_3}\left[\frac{\partial\vec{r}}{\partial\phi}\right]$$

$$\therefore \hat{e}_r = \frac{1}{1}\left[\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\right]$$

$$\hat{e}_\theta = \frac{1}{r}\left[r\cos\theta\cos\phi\,\hat{i} + r\cos\theta\sin\phi\,\hat{j} - r\sin\theta\,\hat{k}\right]$$

$$\hat{e}_\theta = \left[\cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}\right]$$

$$\hat{e}_\phi = \frac{1}{r\sin\theta}\left[-r\sin\theta\sin\phi\,\hat{i} + r\sin\theta\cos\phi\,\hat{j}\right]$$

$$\hat{e}_\phi = \left[-\sin\phi\,\hat{i} + \cos\phi\,\hat{j}\right]$$

To check orthogonality,

$$\hat{e}_r \cdot \hat{e}_\theta = [\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}] \cdot (\cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k})$$

$$\hat{e}_r \cdot \hat{e}_\theta = (\sin\theta\cos\theta\cos^2\phi + \sin\theta\cos\theta\sin^2\phi - \sin\theta\cos\theta)$$

$$= (\sin\theta\cos\theta(\sin^2\phi + \cos^2\phi) - \sin\theta\cos\theta)$$

$$= (\sin\theta\cos\theta - \sin\theta\cos\theta) = 0$$

$$\hat{e}_\theta \cdot \hat{e}_\phi = (\cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}) \cdot (-\sin\phi\,\hat{i} + \cos\phi\,\hat{j})$$

$$= (-\sin\phi\cos\theta\cos\phi + \cos\phi\sin\phi\cos\theta + 0) = 0$$

$$\hat{e}_\phi \cdot \hat{e}_r = (-\sin\phi\,\hat{i} + \cos\phi\,\hat{j}) \cdot (\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k})$$

$$= (-\sin\theta\sin\phi\cos\phi + \sin\phi\cos\phi\sin\theta + 0) = 0$$

$$\boxed{\therefore\ \hat{e}_r \cdot \hat{e}_\theta = 0,\quad \hat{e}_\theta \cdot \hat{e}_\phi = 0\quad \hat{e}_\phi \cdot \hat{e}_r = 0}$$

$\therefore$ Spherical coordinate system is orthogonal.

Q7. Given that,

$$\vec{F} = \frac{x\hat{i}+y\hat{j}}{x^2+y^2}$$

$$\vec{F} = \frac{x\hat{i}}{x^2+y^2} + \frac{y\hat{j}}{x^2+y^2}$$

To prove, $\vec{F}$ is solenoidal $(\nabla.\vec{F}=0)$.

$$\nabla.\vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}\right).\left(\frac{x\hat{i}}{x^2+y^2} + \frac{y\hat{j}}{x^2+y^2}\right)$$

$$\nabla.\vec{F} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right).$$

Partial differentiation

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{1(x^2+y^2)-x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\left[\frac{\partial}{\partial x}(u/v) = \frac{u'v-uv'}{v^2}\right]$$

$$\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = \frac{1(x^2+y^2)-y(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}.$$

$$\therefore \nabla.\vec{F} = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} = \frac{x^2-x^2+y^2-y^2}{(x^2+y^2)^2} = 0$$

$$\therefore \boxed{\nabla.\vec{F}=0}$$

$\therefore$ The Vector $\vec{F}$ is solenoidal.

Q8. Given that,

$$\vec{F} = (axy - z^3)\,\hat{i} + (bx^2 + z)\,\hat{j} + (bxz^2 + cy)\,\hat{k}$$

$\vec{F}$ is irrotational $(\nabla \times \vec{F} = 0)$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ axy - z^3 & bx^2 + z & bxz^2 + cy \end{vmatrix} = 0$$

$$\nabla \times \vec{F} = \hat{i} \begin{vmatrix} \partial/\partial y & \partial/\partial z \\ bx^2 + z & bxz^2 + cy \end{vmatrix} - \hat{j} \begin{vmatrix} \partial/\partial x & \partial/\partial z \\ axy - z^3 & bxz^2 + cy \end{vmatrix}$$

$$+ \hat{k} \begin{vmatrix} \partial/\partial x & \partial/\partial y \\ axy - z^3 & bx^2 + z \end{vmatrix} = 0$$

$$\nabla \times \vec{F} = \hat{i}\,(c - 1) - \hat{j}\,(bz^2 + 3z^2) + \hat{k}\,(2bx - ax) = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$c - 1 = 0$

$c = 1$

$-(bz^2 + 3z^2) = 0$

$-bz^2 = +3z^2$

$b = -3$

$2bx - ax = 0$

$2bx = ax$

$a = -6$

$\therefore$ The values of a, b, c are

$$\boxed{\begin{matrix} a = -6 \\ b = -3 \\ c = 1 \end{matrix}}$$