CBCS SCHEME

USN 1CR25CS016

1BMATS101

First Semester B.E./B.Tech. Degree Examination, Dec.2025/Jan.2026
Calculus and Linear Algebra : CSE Stream

Time: 3 hrs. Max. Marks: 100

Note: 1. Answer any FIVE full questions, choosing ONE full question from each module.
2. M : Marks , L: Bloom's level , C: Course outcomes.
3. VTU Handbook is permitted.

		Module – 1	M	L	C
Q.1	a.	Find the total derivative when $u = x^3y^2 + x^2y^3$ where $x = at^2, y = 2at$	6	L2	CO1
	b.	If $u = \log(\tan x + \tan y + \tan z)$ then, show that $\sin 2x\frac{\partial u}{\partial x} + \sin 2y\frac{\partial u}{\partial y} + \sin 2z\frac{\partial u}{\partial z} = 2$	7	L2	CO1
	c.	Expand $e^x \sin y$ in the powers of x and y as far as terms of third term using Maclaurin's series.	7	L3	CO1
		OR			
Q.2	a.	If $u = x + y + z,\ v = x^2 + y^2 + z^2, w = xy + yz + zx$. Evaluate the Jacobian of (u, v, w) with respect to (x, y, z).	6	L2	CO1
	b.	If $u = f(\frac{x}{y}, \frac{y}{z}, \frac{z}{x})$ then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$	7	L2	CO1
	c.	Examine the function for the extreme values, given $f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$	7	L3	CO1
		Module – 2			
Q.3	a.	Find the directional derivative of $\emptyset = x^2yz + 4xz^2$ at the point (1,-2,-1) in the direction $2\hat{i} - \hat{j} - 2\hat{k}$.	6	L2	CO1
	b.	Show that $\vec{F} = (y + z)\hat{i} + (z + x)\hat{j} + (x + y)\hat{k}$ is irrotational and find the scalar potential.	7	L2	CO1
	c.	With usual notation, Prove that cylindrical polar coordinate system is orthogonal.	7	L3	CO1
		OR			
Q.4	a.	Given $\vec{F} = \nabla(xy^3z^2)$, Find div$(\vec{F})$ and curl$(\vec{F})$ at (1,-1,1).	6	L2	CO1
	b.	Show that $\vec{F} = \frac{x\hat{i}+y\hat{j}}{x^2+y^2}$ is both solenoidal and irrotaional.	7	L2	CO1
	c.	Express $2y\hat{i} - z\hat{j} + 3x\hat{k}$ in terms of spherical polar coordinates.	7	L3	CO1

1BMATS101

		Module – 3			
Q.5	a.	Find the rank of the matrix A= $\begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & 2 & 4 & 3 \\ 3 & 6 & 12 & 9 \\ 3 & 3 & 3 & 6 \end{bmatrix}$	6	L2	CO2
	b.	Using Gauss Jordan method, solve the system of equations $x + y + z = 9, x - 2y + 3z = 8,\ 2x + y - z = 3$.	7	L2	CO2
	c.	Diagonalize the matrix $\begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}$.	7	L3	CO2
		OR			
Q.6	a.	Investigate the value of μ and λ such that the equations, $x + y + z = 6, x + 2y + 3z = 10, x + 2y + \lambda z = \mu$, may have (i) unique solution (ii) infinite solution (iii) no solution	6	L2	CO2
	b.	Using Gauss Elimination method, solve the system of equations $x + 2y + z = 3, 2x + 3y + 3z = 10,\ 3x - y + 2z = 13$	7	L2	CO2
	c.	Find the eigen value and eigen vectors of the matrix A = $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$	7	L3	CO2
		Module – 4			
Q.7	a.	Determine whether the vectors (8,0,5) is a linear combination of the vectors (1,2,3), (0,1,4), (2, −1,1)	6	L2	CO3
	b.	Find the basis and dimension of the subspace spanned by the vectors (2,4,2), (1, −1,0), (1,2,1), (0,3,1) in $V_3(R)$.	7	L2	CO3
	c.	Find the basis and dimension of the row space, column space and null space of the matrix $\begin{bmatrix} 1 & -1 & 1 & 3 & 2 \\ 2 & -1 & 1 & 5 & 1 \\ 0 & 1 & -1 & -1 & -3 \end{bmatrix}$.	7	L3	CO3
		OR			
Q.8	a.	Find the coordinates of the vector v = (0,1,3) with respect to the basis B = {(1,1,0), (0,1,1), (1,0,1)}.	6	L2	CO3
	b.	Define inner product space. Given $u = (1,2,4), v = (2, -3,5), w = (4,2, -3)$ in R^3. Find (i) $\langle u, v\rangle$ (ii) $\langle v, w\rangle$ (iii) $\langle u, w\rangle$ (iv) $\lVert u\rVert$ (v) $\lVert v\rVert$	7	L2	CO3
	c.	What is a subspace? Prove that the subset W = {(x, y, z)/x − 3y + 4z = 0} of the vector space R^3 is a subspace.	7	L3	CO3
		Module – 5			
Q.9	a.	Show that the transformation $T: R^2 \rightarrow R^3$ defined by $T(x, y) = (x + y,\ x - y,\ y)$ is a linear transformation.	6	L2	CO3

	b.	Prove that the transformation $T: R^2 \to R^2$ is singular and find its kernel, if the transformation is given by $T(x,y) = (2x+4y, x+2y)$.	7	L2	CO3
	c.	Verify Rank- Nullity theorem for the transformation $T: R^3 \to R^3$ defined by $T(x,y,z) = (x+2y, y-z, x+2z)$	7	L3	CO3
		OR			
Q.10	a.	Find the matrix representing the linear transformation $T: R^3 \to R^4$ defined by $T(x,y,z) = (x+y+z,\ 2x+z,\ 2y-z,\ 6y)$	6	L2	CO3
	b.	If $T: R^3 \to R^3$ is a linear transformation defined by $T(x,y) = (x+z, x-z, y)$Show that T in invertible and find T^{-1}	7	L2	CO3
	c.	Verify Rank- Nullity theorem for the transformation $T: R^3 \to R^2$ defined by $T(x,y,z) = (x+y, z)$	7	L3	CO3

*****

-1

a. $u = x^3y^2 + x^2y^3$ $\quad x = at^2 \quad y = 2at$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$u = x^3y^2 + x^2y^3$

$\frac{\partial u}{\partial x} = 3x^2y^2 + 2xy^3 \quad \frac{\partial u}{\partial y} = 2x^3y + 3x^2y^2$

$x = at^2 \qquad y = 2at$

$\frac{dx}{dt} = 2at$ $\qquad \frac{dy}{dt} = 2a$

$$\frac{du}{dt} = (3x^2y^2 + 2xy^3)2at + (2x^3y + 3x^2y^2)2a$$

$$\frac{du}{dt} = \left[3(a^2t^4)(4a^2t^2) + 2(at^2)\,8a^3t^3\right]2at + \left[2(a^3t^6)(2at) + 3(a^2t^4)(4a^2t^2)\right]2a$$

$$= \left[12a^4t^6 + 16a^4t^5\right]2at + \left[4a^4t^7 + 12a^4t^6\right]2a$$

$$= 24a^5t^7 + 32a^5t^6 + 8a^5t^7 + 24a^5t^6$$

$$\frac{du}{dt} = 32a^5t^7 + 56a^5t^6$$

$\sec\theta = \frac{1}{\cos\theta}$

$\sin 2\theta = 2\sin\theta\cos\theta$

Q. $u = \log(\tan x + \tan y + \tan z)$, ST

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2$$

Partial differentiation w.r.t x

$$\frac{\partial u}{\partial x} = \frac{1}{\tan x + \tan y + \tan z} \times \sec^2 x$$

Partial diff w.r.t y

$$\frac{\partial u}{\partial y} = \frac{1}{\tan x + \tan y + \tan z} \times \sec^2 y$$

Partial diff w.r.t z

$$\frac{\partial u}{\partial z} = \frac{1}{\tan x + \tan y + \tan z} \times \sec^2 z$$

$$LHS \Rightarrow \sin 2x \left[\frac{\sec^2 x}{\tan x + \tan y + \tan z}\right] + \sin 2y \left[\frac{\sec^2 y}{\tan x + \tan y + \tan z}\right] + \sin 2z \left[\frac{\sec^2 z}{\tan x + \tan y + \tan z}\right]$$

$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

$$\frac{1}{\tan x+\tan y+\tan z}\left[\sin 2x\sec^2 x+\sin 2y\sec^2 y+\sin 2z\sec^2 z\right]$$

$$\frac{1}{\tan x+\tan y+\tan z}\left[2\sin x\cancel{\cos x}\times\frac{1}{\cos^2 x}+2\sin y\cancel{\cos y}\times\frac{1}{\cos^2 y}+2\sin z\cancel{\cos z}\times\frac{1}{\cos^2 z}\right]$$

$$\frac{1}{\tan x+\tan y+\tan z}\left[2\frac{\sin x}{\cos x}+2\frac{\sin y}{\cos y}+2\frac{\sin z}{\cos z}\right]$$

$$\frac{1}{\tan x+\tan y+\tan z}\left[2\tan x+2\tan y+2\tan z\right]$$

$$\frac{2}{\cancel{\tan x+\tan y+\tan z}}\cancel{(\tan x+\tan y+\tan z)}$$

$$2=RHS$$

Hence showed

$$x^3+y^3+z^3=(x+y+z)(x^2+y^2+z^2-xy-yz-xz)$$

1
c

$f(x,y) = e^x \sin y$

$f(x,y) = f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right] + \frac{1}{3!}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$

$f(x,y) = e^x \sin y \Rightarrow f(0,0) = 0$

$f_x = e^x \sin y \Rightarrow f(0,0) = 0$

$f_y = e^x \cos y \Rightarrow f(0,0) = 1$

$f_{xx} = e^x \sin y \Rightarrow f(0,0) = 0$

$f_{yy} = -e^x \sin y \Rightarrow f(0,0) = 0$

$f_{xy} = e^x \cos y \Rightarrow f(0,0) = 1$

$f_{xxx} = e^x \sin y \Rightarrow f(0,0) = 0$

$f_{xyy} = -e^x \sin y \Rightarrow f(0,0) = 0$

$f_{yyy} = -e^x \cos y \Rightarrow f(0,0) = -1$

$f_{xxy} = e^x \cos y \Rightarrow f(0,0) = 1$

Experiment No. Page No.:

$$\therefore f(x,y) = y + \frac{1}{2!}\left[2xy(1)\right] + \frac{1}{3!}\left[3x^2y - y^3\right]$$

$$f(x,y) = y + xy + \frac{1}{6}\left[3x^2y - y^3\right]$$

① $u = x^2 - 2y^2$, $v = 2x^2 - y^2$, find $\frac{\partial(u,v)}{\partial(x,y)}$?

② $J = \frac{\partial(x,y,z)}{\partial(\rho,\phi,z)}$, $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$?

$$J = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$J = 1$ //

7M IAT2

10/30

Find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ where $x^2+y^2+z^2 = u$, $v = xy+yz+xz$, $w = x+y+z$?

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$u = x^2+y^2+z^2$

$\frac{\partial u}{\partial x} = 2x+0+0 = 2x$, $\frac{\partial u}{\partial y} = 0+2y+0 = 2y$, $\frac{\partial u}{\partial z} = 0+0+2z$

$v = xy+yz+xz$

$\frac{\partial v}{\partial x} = y+0+z = y+z$, $\frac{\partial v}{\partial y} = x+z+0 = x+z$, $\frac{\partial v}{\partial z} = 0+y+x = x+y$

$w = x+y+z$

$\frac{\partial w}{\partial x} = 1+0+0 = 1$, $\frac{\partial w}{\partial y} = 0+1+0 = 1$, $\frac{\partial w}{\partial z} = 0+0+1 = 1$

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 2x & 2y & 2z \\ y+z & x+z & x+y \\ 1 & 1 & 1 \end{vmatrix}$$

$J = 2x[(x+z)-(x+y)] - 2y[(y+z)-(x+y)] + 2z[(y+z)-(x+z)]$

$J = 2x[x+z-x-y] - 2y[y+z-x-y] + 2z[y+z-x-z]$ $+ 2xz - 2xy - 2yz + 2xy + 2zy - 2xz$

$J = 0$ //

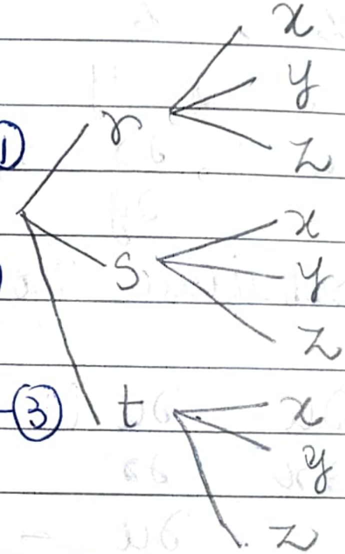
Hence, Proved

Type 2

4. If $V = f(r, s, t)$ if $r = \frac{x}{y}$, $s = \frac{y}{z}$, $t = \frac{z}{x}$. ST

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z} = 0$$

Ans:



$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial V}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial V}{\partial t}\frac{\partial t}{\partial x} \quad -(1)$$

$$\frac{\partial V}{\partial y} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial V}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial V}{\partial t}\frac{\partial t}{\partial y} \quad -(2)$$

$$\frac{\partial V}{\partial z} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial V}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial V}{\partial t}\frac{\partial t}{\partial z} \quad -(3)$$

$$r = \frac{x}{y}$$

$$\frac{\partial r}{\partial x} = \frac{1}{y}, \quad \frac{\partial r}{\partial y} = \frac{-x}{y^2}, \quad \frac{\partial r}{\partial z} = 0$$

$$s = \frac{y}{z}$$

$$\frac{\partial s}{\partial x} = 0, \quad \frac{\partial s}{\partial y} = \frac{1}{z}, \quad \frac{\partial s}{\partial z} = \frac{-y}{z^2}$$

$t = \frac{z}{x}$

$\frac{\partial t}{\partial x} = -\frac{z}{x^2}$, $\frac{\partial t}{\partial y} = 0$, $\frac{\partial t}{\partial z} = \frac{1}{x}$

subtitute all values in eq ①, ②, ③

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\left(\frac{1}{y}\right) + \frac{\partial v}{\partial s}(0) + \frac{\partial v}{\partial t}\left(\frac{-z}{x^2}\right)$$

$$= \frac{1}{y}\frac{\partial v}{\partial r} - \frac{z}{x^2}\frac{\partial v}{\partial t}$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\left(\frac{-x}{y^2}\right) + \frac{\partial v}{\partial s}\left(\frac{1}{z}\right) + \frac{\partial v}{\partial t}(0)$$

$$= -\frac{x}{y^2}\frac{\partial v}{\partial r} + \frac{1}{z}\frac{\partial v}{\partial s}$$

$$\frac{\partial v}{\partial z} = \frac{\partial v}{\partial r}(0) + \frac{\partial v}{\partial s}\left(\frac{-y}{z^2}\right) + \frac{\partial v}{\partial t}\left(\frac{1}{x}\right)$$

$$= -\frac{y}{z^2}\frac{\partial v}{\partial s} + \frac{1}{x}\frac{\partial v}{\partial t}$$

LHS [Sub $\partial v/\partial x$, $\partial v/\partial y$, $\partial v/\partial z$ in LHS]

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z}$$

$$x\left[\frac{1}{y}\frac{\partial v}{\partial r} - \frac{z}{x^2}\frac{\partial v}{\partial t}\right] + y\left[-\frac{x}{y^2}\frac{\partial v}{\partial r} + \frac{1}{z}\frac{\partial v}{\partial s}\right] + z\left[-\frac{y}{z^2}\frac{\partial v}{\partial s} + \frac{1}{x}\frac{\partial v}{\partial t}\right]$$

$$\cancel{\frac{x}{y}\frac{\partial v}{\partial r}} - \cancel{\frac{z}{x}\frac{\partial v}{\partial t}} - \cancel{\frac{x}{y}\frac{\partial v}{\partial r}} + \cancel{\frac{y}{z}\frac{\partial v}{\partial s}} - \cancel{\frac{y}{z}\frac{\partial v}{\partial s}} + \cancel{\frac{z}{x}\frac{\partial v}{\partial t}}$$

0 //

Minimum value of $f(x, y)$ is $f(1, 2) = 1 + 8 - 3 - 24 + 20 = 2$

Thus, **Maximum value is 38 and Minimum value is 2.**

[50] *Find the maximum and minimum values of the function*

$x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$ **[Sep 20, Apr 23]**

For the given function $f(x, y)$ we have,

$f_x = 3x^2 + 3y^2 - 30x + 72, \quad f_y = 6xy - 30y$

We shall find points (x, y) such that $f_x = 0$ and $f_y = 0$.

$3x^2 + 3y^2 - 30x + 72 = 0$ or $x^2 + y^2 - 10x + 24 = 0$ ... (1)

$6xy - 30y = 0$ or $y(x - 5) = 0$ ... (2)

(2) gives us $y = 0$ and $x = 5$.

Putting $y = 0$ in (1) we get $x^2 - 10x + 24 = 0$ or

$(x - 4)(x - 6) = 0$ or $x = 4, 6$

$\therefore$ $(4, 0), (6, 0)$ are stationary points

Putting $x = 5$ in (1), we get $y^2 - 1 = 0$ or $y = \pm 1$

$\therefore$ $(5, 1), (5, -1)$ are also stationary points.

Let us examine these points for maxima and minima.

Let $A = f_{xx}$, $B = f_{xy}$, $C = f_{yy}$

	$(4, 0)$	$(6, 0)$	$(5, 1)$	$(5, -1)$
$A = 6x - 30$	$-6 < 0$	$6 > 0$	0	0
$B = 6y$	0	0	6	-6
$C = 6x - 30$	-6	6	0	0
$AC - B^2$	$36 > 0$	$36 > 0$	$-36 < 0$	$-36 < 0$
Conclusion	Max. pt.	Min. pt.	Saddle pt.	Saddle pt.

Maximum value of $f(x, y)$ is $f(4, 0) = 64 - 240 + 288 = 112$

Minimum value of $f(x, y)$ is $f(6, 0) = 216 - 540 + 432 = 108$

Thus, $\boxed{\textbf{Maximum value is 112 and minimum value is 108.}}$

[51] *Examine the function* $xy(a - x - y)$ f...

Module-2

3(a) Given $\phi = x^2yz + 4xz^2$ at $(1,-2,-1)$

and direction $2i - j + 6K$

so $\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} K$

i.e, $\nabla\phi = (2xyz + 4z^2)\,i + (x^2z)\,j + (x^2y + 8xz)\,k$

$[\nabla\phi]_{(1,-2,-1)} = 8i - j - 10k$

The unit vector in the direction of $2i - j - 2k$ is

$\hat{n} = \frac{2i - j - 2K}{\sqrt{4+1+4}} = \frac{2i - j - 2K}{3}$; The required

directional derivative is

$\nabla\phi \cdot \hat{n} = \frac{(8i - j - 10k)\cdot(2i - j - 2k)}{3}$

Thus $\nabla\phi \cdot \hat{n} = \frac{(8)(2) + (-1)(-1) + (-10)(-2)}{3} = \frac{37}{3}$

$$\boxed{\nabla\phi \cdot \hat{n} = 37/3}$$

(2b)

Given $\vec{F} = (y+z)i + (z+x)j + (x+y)k$

Now
$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix}$$

$$= i\left(\frac{\partial}{\partial y}(x+y) - \frac{\partial}{\partial z}(z+x)\right) - j\left(\frac{\partial}{\partial x}(x+y) - \frac{\partial}{\partial z}(y+z)\right)$$
$$+ k\left(\frac{\partial}{\partial x}(z+x) - \frac{\partial}{\partial y}(y+z)\right)$$

$$= i(1-1) - j(1-1) + k(1-1) = \vec{0}$$

$\therefore \boxed{\nabla \times \vec{F} = \vec{0}} \rightarrow$ Given vector $\vec{F}$ is irrotational.

For scalar potential (ϕ) we have $\nabla\phi = \vec{F}$

$$\frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k = (y+z)i + (z+x)j + (x+y)k$$

$\Rightarrow \quad \frac{\partial \phi}{\partial x} = y+z \Rightarrow \phi = xy + xz + f(y,z)$ — (1)

$\frac{\partial \phi}{\partial y} = z+x \Rightarrow \phi = zy + xy + f(x,z)$ — (2)

$\frac{\partial \phi}{\partial z} = x+y \Rightarrow \phi = xz + yz + f(x,y)$ — (3)

By (1) & (2) & (3)

$$\boxed{\phi = xy + yz + zx}$$

(3c) we have for the cylindrical system

$$\vec{r} = \rho\cos\phi\, i + \rho\sin\phi\, j + zk$$

Let $\hat{e}_\rho, \hat{e}_\phi, \hat{e}_z$ be the basic unit vectors of this system.

Further we have, $h_1 = 1$, $h_2 = \rho$, $h_3 = 1$ for the cylindrical system.

Using the definition of the unit vectors,

$$\hat{e}_\rho = \frac{1}{h_1}\frac{\partial \vec{r}}{\partial \rho} = \cos\phi\, i + \sin\phi\, j + 0.k$$

$$\hat{e}_\phi = \frac{1}{h_2}\frac{\partial \vec{r}}{\partial \phi} = \frac{1}{\rho}(-\rho\sin\phi\, i + \rho\cos\phi\, j + 0.k)$$

$$= -\sin\phi\, i + \cos\phi\, j + 0k$$

$$\hat{e}_z = \frac{1}{h_3}\cdot\frac{\partial \vec{r}}{\partial z} = \frac{1}{1}(0i + 0j + 1k) = 0i + 0j + 1k$$

Now, $\hat{e}_\rho \cdot \hat{e}_\phi = -\cos\phi\sin\phi + \sin\phi\cos\phi = 0$, $\hat{e}_\phi \cdot \hat{e}_z = 0$

$\hat{e}_z \cdot \hat{e}_\rho = 0$, Thus the cylindrical system is orthogonal.

(4a) Given $\vec{F} = \nabla(xy^3z^2)$. To Find $div(\vec{F})$ and $curl(\vec{F})$ at $(1,-1,1)$

we have

$$\vec{F} = \frac{\partial}{\partial x}(xy^3z^2)\,i + \frac{\partial}{\partial y}(xy^3z^2)\,j + \frac{\partial}{\partial z}(xy^3z^2)\,k$$

$$\vec{F} = y^3z^2\,i + 3xy^2z^2\,j + 2xy^3z\,k$$

Now

$$div(\vec{F}) = \nabla\cdot\vec{F} = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)\left(y^3z^2 i + 3xy^2z^2 j + 2xy^3z k\right)$$

$$\nabla\cdot\vec{F} = 0 + 6xyz^2 + 2xy^3$$

at $(1,-1,1)$

$$[\nabla\cdot\vec{F}]_{(1,-1,1)} = 6\times1\times(-1)\times1 + 2\times1\times(-1)^3\times1$$

$$= -6 - 2 = -8$$

and

$$curl(\vec{F}) = \nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3z^2 & 3xy^2z^2 & 2xy^3z \end{vmatrix}$$

$$= i\left[6xy^2z - 6xy^2z\right] - j\left[2y^3z - 2y^3z\right] + k\left[3y^2z^2 - 3y^2z^2\right]$$

$$= 0i + 0j + 0k = \vec{0}$$

(4 b) Given $\vec{F} = \dfrac{xi + yj}{x^2+y^2} = \dfrac{x}{x^2+y^2} i + \dfrac{y}{x^2+y^2} j$

Now

$$div \vec{F} = \nabla . \vec{F} = \frac{\partial}{\partial x} \frac{x}{(x^2+y^2)} + \frac{\partial}{\partial y} \frac{y}{x^2+y^2}$$

$$= \frac{(x^2+y^2).1 - x(2x)}{(x^2+y^2)^2} + \frac{(x^2+y^2).1 - y(2y^2)}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2x^2+x^2+y^2-2y^2}{(x^2+y^2)^2} = 0$$

i.e $di \vec{F} = \nabla . \vec{F} = 0 \rightarrow$ Vector is solenoidal.

Now $curl \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} & 0 \end{vmatrix}$

$$= i\left[0-0\right] - j\left[0-0\right] + k\left[\frac{y(2x)}{(x^2+y^2)^2} - \frac{x(2y)}{(x^2+y^2)}\right]$$

$$= 0i + 0j + 0k = \vec{0}$$

i.e $curl \vec{F} = \nabla \times \vec{F} = \vec{0} \rightarrow$

so Given vector $\vec{F}$ is solenoidal and irrotational both.

(4c) Given

$$\vec{F} = 2y\hat{i} - z\hat{j} + 3x\hat{k}$$

$\gg$ $$\vec{F} = F_r \hat{e}_r + F_\theta \hat{e}_\theta + F_\phi \hat{e}_\phi \quad \text{——— (1)}$$

where F_r, F_θ, F_ϕ are to be determined.

$$\therefore \vec{F}\cdot\hat{e}_r = F_r, \quad \vec{F}\cdot\hat{e}_\theta = F_\theta, \quad \vec{F}\cdot\hat{e}_\phi = F_\phi \quad \text{——— (2)}$$

In terms of spherical polar coordinates, the given function.

$$\vec{F} = 2(r\sin\theta\sin\phi)\hat{i} + (r\cos\theta)\hat{j} + 3(r\sin\theta\cos\phi)\hat{k} \quad \text{——— (3)}$$

Also we have, $\hat{e}_r = \frac{1}{h_1}\frac{\partial\vec{r}}{\partial r}$; $\hat{e}_\theta = \frac{1}{h_2}\frac{\partial\vec{r}}{\partial\theta}$;

$$\hat{e}_\phi = \frac{1}{h_3}\frac{\partial\vec{r}}{\partial\phi}$$

where $\vec{r} = (r\sin\theta\cos\phi)\hat{i} + (r\sin\theta\sin\phi)\hat{j} + r\cos\theta\,\hat{k}$

and $h_1 = 1,\ h_2 = r,\ h_3 = r\sin\theta$

$\therefore\ \hat{e}_r = \sin\theta\cos\phi\, i + \sin\theta\sin\phi\, j + \cos\theta\, k$

$\hat{e}_\theta = \cos\theta\cos\phi\, i + \cos\theta\sin\phi\, j - \sin\theta\, k$

$\hat{e}_\phi = -\sin\phi\, i + \cos\phi\, j$

Thus we have from (2), the required F_r, F_θ, F_ϕ is as follows.

$F_r = 2r\sin^2\theta\sin\phi\cos\phi + r\cos\theta\sin\theta\sin\phi$
$\quad + 3r\sin\theta\cos\theta\cos\phi$

$F_\theta = 2r\sin\theta\cos\theta\sin\phi\cos\phi - r\cos^2\theta\sin\phi$
$\quad - 3r\sin^2\theta\cos\phi.$

$F_\phi = -2r\sin\theta\sin^2\phi - r\cos\theta\cos\phi$

Q5 a) Find the rank of the matrix $A = \begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & 2 & 4 & 3 \\ 3 & 6 & 12 & 9 \\ 3 & 3 & 3 & 6 \end{bmatrix}$.

Sol^n:-

$$A = \begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & 2 & 4 & 3 \\ 3 & 6 & 12 & 9 \\ 3 & 3 & 3 & 6 \end{bmatrix}$$

$R_3 \to \frac{1}{3} \times R_3$, $R_4 \to \frac{1}{4} \times R_4$, $R_1 \leftrightarrow R_2$

$$\cancel{\simeq \begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & 2 & 4 & 3 \\ & & & \\ & & & \end{bmatrix}}$$

$$\simeq \begin{bmatrix} 1 & 2 & 4 & 3 \\ 2 & 1 & -1 & 3 \\ 1 & 2 & 4 & 3 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 - R_1$

$$\simeq \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & -3 & -9 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -3 & -1 \end{bmatrix}$$

$R_4 \to 3R_4 - R_2$

$$\simeq \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & -3 & -9 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to \frac{-1}{3} \times R_2$

$$A \simeq \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, rank $(A) = 2$.

b) Using Gauss Jordan method, solve the system of equations $x+y+z=9$, $x-2y+3z=8$; $2x+y-z=3$.

Sol^n:

$$[A:b] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 1 & -2 & 3 & 8 \\ 2 & 1 & -1 & 3 \end{array}\right]$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$

$$\simeq \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 0 & -3 & 2 & -1 \\ 0 & -1 & -3 & -15 \end{array}\right]$$

$R_1 \to 3R_1 + R_2$, $R_3 \to 3R_3 - R_2$

$$\simeq \left[\begin{array}{ccc:c} 3 & 0 & 5 & 26 \\ 0 & -3 & 2 & -1 \\ 0 & 0 & -11 & -44 \end{array}\right]$$

$R_3 \to \frac{-1}{11} \times R_4$

$$\simeq \left[\begin{array}{ccc:c} 3 & 0 & 5 & 26 \\ 0 & -3 & 2 & -1 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$R_1 \to R_1 - 5R_3, \quad R_2 \to R_2 - 2R_3$

$$\simeq \left[\begin{array}{ccc:c} 3 & 0 & 0 & 6 \\ 0 & -3 & 0 & -9 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$\Rightarrow 3x = 6, \quad -3y = -9, \quad z = 4.$

Hence the sol^n of the given system of eq^n is

$x = 2, \; y = 3, \; z = 4.$

c) Diagonalize the matrix $\begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}$.

Sol^n Let $A = \begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}$

The characteristic eq^n is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -1-\lambda & 3 \\ -2 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-1-\lambda)(4-\lambda) + 6 = 0$$

$$\Rightarrow -4 + \lambda - 4\lambda + \lambda^2 + 6 = 0$$

$\Rightarrow \lambda^2 - 3\lambda + 2 = 0$

$\Rightarrow \lambda^2 - 2\lambda - \lambda + 2 = 0$

$\Rightarrow \lambda(\lambda-2) - 1(\lambda-2) = 0$

$\Rightarrow (\lambda-2)(\lambda-1) = 0$

$\Rightarrow \lambda = 1, 2.$

The eigen values of A are 1, 2

If $\lambda = 1$,

$(A - \lambda I)\,X = O$

$\Rightarrow \left(\begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$\Rightarrow \left(\begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$\Rightarrow \begin{pmatrix} -2 & 3 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$R_2 \rightarrow R_2 - R_1$

$\Rightarrow \begin{pmatrix} -2 & 3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$\Rightarrow -2x_1 + 3x_2 = 0$

$\Rightarrow 2x_1 = 3x_2$

$$\Rightarrow x_1 = \frac{3}{2} x_2$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 3/2 \\ 1 \end{pmatrix}.$$

For $\lambda = 1$, eigen vector is $\begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$;

For $\lambda = 2$

$$(A - \lambda I) X = 0$$

$$\Rightarrow \left(\begin{pmatrix} -1 & 3 \\ -2 & 4 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} -3 & 3 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$R_2 \to 3R_2 - 2R_1$

$$\Rightarrow \begin{pmatrix} -3 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$-3x_1 + 3x_2 = 0$$

$$\Rightarrow x_1 = x_2$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

CMR

For $\lambda = 2$, eigen vector $= \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

The modal matrix is. $P = \begin{pmatrix} 3/2 & 1 \\ 1 & 1 \end{pmatrix}$.

$$|P| = \begin{vmatrix} 3/2 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= \left(\frac{3}{2} \times 1\right) - (1 \times 1)$$

$$= 3/2 - 1 = 1/2.$$

$$P^{-1} = \frac{\begin{pmatrix} 1 & -1 \\ -1 & 3/2 \end{pmatrix}}{1/2}$$

$$P^{-1} = 2\begin{pmatrix} 1 & -1 \\ -1 & 3/2 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 3 \end{pmatrix}.$$

$D = P^{-1} A P$

$$= \begin{pmatrix} 2 & -2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 3/2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -2 \\ -4 & 6 \end{pmatrix}\begin{pmatrix} 3/2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

The diagonal matrix is $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

Q.6

a) Investigate the values of μ & λ such that the equations $x+y+z=6$, $x+2y+3z=10$, $x+2y+\lambda z=\mu$, may have (i) unique sol^n (ii) infinite sol^n (iii) no sol^n.

Sol^n:

$$[A:b] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\simeq \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\simeq \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$$

(i) unique sol^n:- $\lambda \neq 3$.

(ii) infinite sol^n:- $\lambda = 3$, $\mu = 10$.

(iii) no sol^n:- $\lambda = 3$, $\mu \neq 10$.

b) Using Gauss elimination method, solve the system of equations : $x+2y+z=3$, $2x+3y+3z=10$, $3x-y+2z=13$.

Sol^n:- $[A:b] = \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 3 & -1 & 2 & 13 \end{array}\right]$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$\simeq \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & -7 & -1 & 4 \end{array}\right]$

$R_3 \to \cancel{2R_3 + 7R_1}$ $R_3 - 7R_2$

$\simeq \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right]$

By backward subtitution,

$-8z = -24 \Rightarrow z = 3$

$-y + z = 4$

$\Rightarrow -y = 4 - 3 = 1$

$\Rightarrow y = -1$

$x + 2y + z = 3$

$x = 3 - 2(-1) - 3$

$\Rightarrow x = 2$

Scanned with OKEN Scanner

Hence, the sol^n is $x=2,\ y=-1,\ z=3$.

c) Find the eigen value & eigen vectors of the matrix
$A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$.

Sol^n:- The characteristic eq^n is,

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left[(5-\lambda)(1-\lambda)-1\right] - 1\left[1(1-\lambda)-3\right] + 3\left[1-3(5-\lambda)\right] = 0$$

$$\Rightarrow (1-\lambda)(5-5\lambda-\lambda+\lambda^2-1) - (1-\lambda-3) + 3(1-15+3\lambda) = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2-6\lambda+4) - (-\lambda-2) + 3(3\lambda-14) = 0$$

$$\Rightarrow \lambda^2 - 6\lambda + 4 - \lambda^3 + 6\lambda^2 - 4\lambda + \lambda + 2 + 9\lambda - 42 = 0$$

$$\Rightarrow -\lambda^3 + 7\lambda^2 - 36 = 0.$$

$$\Rightarrow (\lambda-3)(\lambda^2-4\lambda-12) = 0$$

Eigen values are $\lambda: 3, 6, -2$.

For $\lambda = 3$,

$$(A - \lambda I)(x) = 0$$

$$\Rightarrow \begin{pmatrix} 1-3 & 1 & 3 \\ 1 & 5-3 & 1 \\ 3 & 1 & 1-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$R_2 \to 2R_2 + R_1$, $R_3 \to 2R_3 + 3R_1$

$$\Rightarrow \begin{pmatrix} -2 & 1 & 3 \\ 0 & 5 & 5 \\ 0 & 5 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} -2 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$-2x_1 + x_2 + 3x_3 = 0$$

$$x_2 + x_3 = 0$$

$$\Rightarrow \boxed{x_2 = -x_3}$$

$-2x_1 + x_3 + 3x_3 = 0$

$\Rightarrow -2x_1 + 2x_3 = 0 \quad \Rightarrow \boxed{x_1 = x_3}$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ -x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

For $\lambda = 3$, eigen vector is $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

For $\lambda = 6$:-

$$\Rightarrow \begin{pmatrix} 1-6 & 1 & 3 \\ 1 & 5-6 & 1 \\ 3 & 1 & 1-6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$R_2 \to 5R_2 + R_1$; $R_3 \to 5R_3 + 3R_1$

$$\Rightarrow \begin{pmatrix} -5 & 1 & 3 \\ 0 & -4 & 8 \\ 0 & 8 & -16 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} -5 & 1 & 3 \\ 0 & -4 & 8 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$-5x_1 + x_2 + 3x_3 = 0$$

$$-4x_2 + 8x_3 = 0$$

$$\boxed{x_2 = 2x_3}$$

$$-5x_1 + 2x_3 + 3x_3 = 0$$

$$\Rightarrow \boxed{x_1 = x_3}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ 2x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

For $\lambda = 6$, eigen vector $= \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

For $\lambda = -2$,

$$\begin{pmatrix} 1+2 & 1 & 3 \\ 1 & 5+2 & 1 \\ 3 & 1 & 1+2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$R_2 \to 3R_2 - R_1$

$R_3 \to R_3 - R_1$

$$\Rightarrow \begin{pmatrix} 3 & 1 & 3 \\ 0 & 20 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$3x_1 + x_2 + 3x_3 = 0$$

$$20x_2 = 0$$

$$\Rightarrow \boxed{x_2 = 0}$$

$$3x_1 + 0 + 3x_3 = 0$$

$$\Rightarrow \boxed{x_1 = -x_3}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_3 \\ 0 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Hence, the eigen values are 3, 6, −2 & corresponding eigen vectors are

$$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Module - 4

Q-7

a) Determine whether the vectors $(8,0,5)$ is a linear combination of the vectors $(1,2,3)$, $(0,1,4)$, $(2,-1,1)$.

Sol^n: Let α, β, γ be any scalars such that

$$(8,0,5) = \alpha(1,2,3) + \beta(0,1,4) + \gamma(2,-1,1)$$

$$\alpha + 0\cdot\beta + 2\gamma = 8$$

$$2\alpha + \beta - \gamma = 0$$

$$3\alpha + 4\beta + \gamma = 5$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 8 \\ 2 & 1 & -1 & 0 \\ 3 & 4 & 1 & 5 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$$\simeq \left[\begin{array}{ccc|c} 1 & 0 & 2 & 8 \\ 0 & 1 & -5 & -16 \\ 0 & 4 & -5 & -19 \end{array}\right]$$

$R_3 \to R_3 - 4R_2$

$$\simeq \left[\begin{array}{ccc|c} 1 & 0 & 2 & 8 \\ 0 & 1 & -5 & -16 \\ 0 & 0 & 15 & 45 \end{array}\right]$$

By backward substitution,

$$15\gamma = 45$$

$$\Rightarrow \boxed{\gamma = 3}$$

$$\beta - 5\gamma = -16$$

$$\beta - 5(3) = -16$$

$$\boxed{\beta = -1}$$

$$\alpha + 0\cdot\beta + 2\gamma = 8$$

$$\Rightarrow \alpha = 8 - 2(3)$$

$$\Rightarrow \boxed{\alpha = 2}$$

Hence, the given vectors $(1,2,3)$, $(0,1,4)$, $(2,-1,1)$ is a linear combination of $(8,0,5)$.

$$(8,0,5) = 2(1,2,3) - 1(0,1,4) + 3(2,-1,1).$$

b) Find the basis & dimension of the subspace spanned by the vectors $(2,4,2)$, $(1,-1,0)$, $(1,2,1)$, $(0,3,1)$ in $V_3(R)$.

Sol:

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & -1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & 1 \end{bmatrix}$$

$R_2 \to 2R_2 - R_1, \quad R_3 \to 3R_3 - R_1$

$$\simeq \begin{bmatrix} 2 & 4 & 2 \\ 0 & -6 & -2 \\ 0 & 0 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

$R_4 \to 2R_4 + R_2$

$$\simeq \begin{bmatrix} 2 & 4 & 2 \\ 0 & -6 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore the basis for the subspace of $V_3(\mathbb{R})$ is

$\{(2,4,2), (0,-6,-2)\}$ (or) $\{(2,4,2), (1,-1,0)\}$.

c) Find the basis & dimension of the row space, column space & null space of the matrix $\begin{bmatrix} 1 & -1 & 1 & 3 & 2 \\ 2 & -1 & 1 & 5 & 1 \\ 0 & 1 & -1 & -1 & -3 \end{bmatrix}$

Sol^n Let $A = \begin{bmatrix} 1 & -1 & 1 & 3 & 2 \\ 2 & -1 & 1 & 5 & 1 \\ 0 & 1 & -1 & -1 & -3 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$

$\simeq \begin{bmatrix} 1 & -1 & 1 & 3 & 2 \\ 0 & 1 & -1 & -1 & -3 \\ 0 & 1 & -1 & -1 & -3 \end{bmatrix}$

$R_3 \to R_3 - R_2$

$\simeq \begin{bmatrix} 1 & -1 & 1 & 3 & 2 \\ 0 & 1 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Basis of the row space is $\{(1,-1,1,3,2), (0,1,-1,-1,3)\}$

(or)

$\{(1,-1,1,3,2), (2,-1,1,5,1)\}$

dim of the row space is 2.

Basis of the column space is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}$.

dim of the column space is 2.

For null space :-

$$\begin{bmatrix} 1 & -1 & 1 & 3 & 2 \\ 0 & 1 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = 0$$

$$\Rightarrow \quad x_1 - x_2 + x_3 + 3x_4 + 2x_5 = 0$$

$$x_2 - x_3 - x_4 - 3x_5 = 0$$

$$\boxed{x_2 = x_3 + x_4 + 3x_5}$$

$$x_1 = x_2 - x_3 - 3x_4 - 2x_5$$

$$= \cancel{x_3} + x_4 + 3x_5 - \cancel{x_3} - 3x_4 - 2x_5$$

$$\boxed{x_1 = -2x_4 + x_5}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2x_4 + x_5 \\ x_3 + x_4 + 3x_5 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ x_3 \\ x_3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -2x_4 \\ x_4 \\ 0 \\ x_4 \\ 0 \end{pmatrix} + \begin{pmatrix} x_5 \\ 3x_5 \\ 0 \\ 0 \\ x_5 \end{pmatrix}$$

$$= x_3 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 1 \\ 3 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Hence the basis of null space is

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Dim of the null space = 3.

Q. 8.

a) Find the coordinates of the vector $v = (0,1,3)$ w.r.t the basis $B = \{(1,1,0), (0,1,1), (1,0,1)\}$.

Sol: Let (α, β, γ) be the coordinates of the vector v w.r.t B.

$(0,1,3) = \alpha(1,1,0) + \beta(0,1,1) + \gamma(1,0,1)$

$\alpha + \gamma = 0 \Rightarrow \alpha = -\gamma$

$\alpha + \beta = 1 \Rightarrow -\gamma + \beta = 1$

$\beta + \gamma = 3 \qquad \gamma + \beta = 3$

$2\beta = 4$

$\Rightarrow \boxed{\beta = 2}$

$\gamma = 3 - \beta$

$\gamma = 3 - 2$

$\boxed{\gamma = 1} \Rightarrow \boxed{\alpha = -1}$

Hence, the coordinates are $(-1, 2, 1)'$.

b) Define inner product space. Given $u = (1,2,4)$, $v = (2,-3,5)$, $w = (4,2,-3)$ in $\mathbb{R}^3$. Find
(i) $\langle u, v\rangle$, (ii) $\langle v, w\rangle$, (iii) $\langle u, w\rangle$, (iv) $\|u\|$, (v) $\|v\|$

Sol^n Def^n: An inner product space is a vector space V over $\mathbb{R}$, $\langle\,,\rangle : V \times V \rightarrow \mathbb{R}$, satisfying the following properties:

(i) $\langle x, x\rangle \geq 0$ & $\langle x, x\rangle = 0 \Leftrightarrow x = 0$.

(ii) $\langle \alpha x + y, z \rangle = \alpha \langle x, z \rangle + \langle y, z \rangle$.

(iii) $\langle x, y \rangle = \langle y, x \rangle$.

~~(i) $\langle u, u \rangle = \langle (2, -3, 5), \alpha$~~

(i) $\langle u, v \rangle = 2 \times 1 + 2 \times (-3) + 4 \times 5 = 2 - 6 + 20 = 16$.

(ii) $\langle v, w \rangle = 2 \times 4 + (-3)(2) + 5 \times (-3)$

$= 8 - 6 - 15 = -13$

(iii) $\langle u, w \rangle = 4 + 4 - 12 = -4$

(iv) $\|u\|^2 = \langle u, u \rangle = 1 + 4 + 16 = 21$

$\|u\| = \sqrt{21}$

(v) $\|v\|^2 = \langle v, v \rangle = 4 + 9 + 25 = 38$

$\|v\| = \sqrt{38}$.

c) What is a subspace? Prove that the subset $W = \{(x,y,z) \mid x-3y+4z=0\}$ of the vector space $\mathbb{R}^3$ is a subspace.

Sol: Def^n:- Let $V(F)$ be a vector space over the field F. A subset W of $V(F)$ is said to be a subspace of $V(F)$ if it is closed under vector addition & scalar multiplication,

i.e.,

(i) $0 \in W$

(ii) $x+y \in W$, if $x, y \in W$

(iii) $\alpha x \in W$, if $x \in W$, $\alpha \in F$.

$$W = \{(x,y,z) \mid x-3y+4z=0\}$$

(i) Zero element:-

$$0-3(0)+4(0) = 0$$

$$\Rightarrow (0,0,0) \in W.$$

(ii) Let $(x_1,y_1,z_1), (x_2,y_2,z_2) \in W$ & $\alpha, \beta \in F$

$\cancel{\alpha(x_1,y_1,z_1)+}$

$(x_1, y_1, z_1) \in W \Rightarrow x_1 - 3y_1 + 4z_1 = 0$

$\Rightarrow \alpha(x_1 - 3y_1 + 4z_1) = \alpha \cdot 0$

$\Rightarrow \alpha x_1 - 3\alpha y_1 + 4\alpha z_1 = 0 \quad -(1)$

$(x_2, y_2, z_2) \in W \Rightarrow x_2 - 3y_2 + 4z_2 = 0$

$\Rightarrow \beta(x_2 - 3y_2 + 4z_2) = 0$

$\Rightarrow \beta x_2 - 3\beta y_2 + 4z_2 = 0 \quad -(2)$

From (1) & (2), we have

$(\alpha x_1 - 3\alpha y_1 + 4\alpha z_1) + (\beta x_2 - 3\beta y_2 + 4\beta z_2) = 0$

$\Rightarrow (\alpha x_1 + \beta x_2) - 3(\alpha y_1 + \beta y_2) + 4(\alpha z_1 + \beta z_2) = 0$

$\Rightarrow (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2) \in W$

$\Rightarrow \alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2) \in W.$

$\Rightarrow$ W is a subspace of $\mathbb{R}^3$.

Module-5

9 a) S.T $T(x,y) = (x+y, x-y, y)$ is a L.T.

Let $a = (x_1, y_1)$ & $b = (x_2, y_2) \in V_2(R), R^2$

T.P (i) $T(a+b) = T(a) + T(b)$

(ii) $T(k.a) = k.T(a)$

(i) $a+b = (x_1+x_2, y_1+y_2)$

$T(a+b) = (x_1+x_2+y_1+y_2, x_1+x_2-y_1-y_2, y_1+y_2)$ — (1)

$T(a)+T(b) = (x_1+y_1, x_1-y_1, y_1) + (x_2+y_2, x_2-y_2, y_2)$

$T(a)+T(b) = (x_1+y_1+x_2+y_2, x_1+x_2-y_1-y_2, y_1+y_2)$ — (2)

from (1) & (2) $T(a+b) = T(a) + T(b)$

(ii) $T(ka) = T(kx_1, ky_1) = (kx_1+ky_1, kx_1-ky_1, ky_1)$

$= k(x_1+y_1, x_1-y_1, y_1)$

$= k\,T(a)$

Hence it is a L.T.

9(b) $T: R^2 \to R^2$ $\quad T(x, y) = (2x+4y,\ x+2y)$

$e_1 = (1,0),\quad e_2 = (0,1)$

$T(e_1) = T(1,0) = (2, 1)$

$T(e_2) = T(0,1) = (4, 2)$

$\therefore$ Matrix T is $\quad A = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$

$\det(A) = \begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} = 4-4 = 0$

$\Rightarrow$ T is singular.

To find kernal (T)

(ie) set of vector (x, y) such that $T(x, y) = (0,0)$

$$\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\left.\begin{array}{l} 2x+4y=0 \\ x+2y=0 \end{array}\right\} \Rightarrow x = -2y$

Take y as arbitrary k

$\Rightarrow x = -2k$

$\therefore \ker T = \{(-2k, k) / k \in R\}$

span $(-2,1)$.

9c) Verify Rank nullity $T: R^3 \to R^3$

$T(x, y, z) = (x+2y,\ y-z,\ x+2z)$

Soln

$T(e_1) = T(1,0,0) = (1, 0, 1)$

$T(e_2) = T(0,1,0) = (2, 1, 0)$

$T(e_3) = T(0,0,1) = (0, -1, 2)$

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}$$

To find Rank :

$R_3 \to R_3 - R_1$ $\quad A \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & -2 & 2 \end{pmatrix}$

$R_3 \to R_3 + 2R_2 \Rightarrow A \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$ Rank is 2.

To find nullity $\quad x + 2y = 0$

$y - z = 0$

Take $z = k$ arbitrary $\Rightarrow y = k$

k

$\& \ x = -2k.$

Kernal is spaned by $(-2,1,1)$

So dimension of kernal, nullity is 1.

$\text{rank}(T) + \text{nullity}(T) = 2+1 = 3$

& $\dim(\mathbb{R}^3) = 3$

Hence Rank nullity thm

Rank (T) + nullity of $(T) = \dim(V)$ is verified

— x —

10) a) $T: \mathbb{R}^3 \to \mathbb{R}^4$ $T(x,y,z) = (x+y+z,\ 2x+z,\ 2y-z,\ 6y)$

$T(e_1) = T(1,0,0) = (1, 2, 0, 0)$
$T(e_2) = T(0,1,0) = (1, 0, 2, 6)$
$T(e_3) = T(0,0,1) = (1, 1, -1, 0)$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 6 & 0 \end{bmatrix}$$ is the required matrix

10) b) $T: R^3 \to R^3$

$T(x, y, z) = (x+z, x-z, y)$

S.T T is invertible & find T^{-1}

$T(e_1) = T(1,0,0) = (1, 1, 0)$

$T(e_2) = T(0,1,0) = (0, 0, 1)$

$T(e_3) = T(0,0,1) = (1, -1, 0)$

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

T.P Invertible. $\det(A) = 1(1) + 0 + 1(1) = 2 \neq 0$

$\therefore T$ is invertible

To find T^{-1}, $T(x, y, z) = (a, b, c)$

$x+z = a \qquad x-z = b \qquad y = c.$

$x+z = a$

$x-z = b$

$2x = a+b \Rightarrow x = \frac{a+b}{2}$

$\therefore z = x - b$

$z = \frac{a+b}{2} - b$

$z = \frac{a-b}{2}$

$T^{-1}(a,b,c) = \left(\frac{a+b}{2}, c, \frac{a-b}{2}\right)$ or $T^{-1}(x, y, z) = \left(\frac{x+y}{2}, z, \frac{x-y}{2}\right)$

10) c) Verify Rank nullity

$T: R^3 \to R^2 \quad T(x,y,z) = (x+y, z)$

$T(e_1) = T(1,0,0) = (1,0)$

$T(e_2) = T(0,1,0) = (1,0)$

$T(e_3) = T(0,0,1) = (0,1)$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Rank$(A) = 2$.

To find nullity. ~~$x+y=0$~~, $x+y=0 \Rightarrow x=-y$

$z=0$

let $y=k$ be arbitrary.

$\Rightarrow (x,y,z) = (-k, k, 0) = k(-1,1,0)$

one basis vector $\Rightarrow$ nullity $= 1$

rank + nullity $= 2+1 = \dim(R^3) = 3$

Hence Rank nullity thm is verified.