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# Internal Assessment Test II – December 2018

| Sub: | Calculus and Linear Algebra | | | | | Sub Code: | 18MAT11 |
|---|---|---|---|---|---|---|---|
| Date: | 03/12/2018 | Duration: | 90 mins | Max Marks: | 50 | Sem / Sec: | I / ALL SECTIONS |

**Question 1 is compulsory and answer any SIX questions from the rest.**

| | | MARKS | OBE CO | OBE RBT |
|---|---|---|---|---|
| 1. | (a) If $u = x + 3y^2 - z^3$, $v = 4x^2yz$ and $w = 2z^2 - xy$ find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ at (1,-1,0). | [04] | CO2 | L3 |
| | (b) Evaluate: $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dx\,dy\,dz$. | [04] | CO3 | L3 |
| 2. | Obtain the Maclaurin's expansion of $\log(\sec x + \tan x)$ upto the first three non-vanishing terms. | [07] | CO2 | L3 |
| 3. | If $u = f(4x-5y, 5y-6z, 6z-4x)$ then prove that $\frac{1}{4}\frac{\partial u}{\partial x} + \frac{1}{5}\frac{\partial u}{\partial y} + \frac{1}{6}\frac{\partial u}{\partial z} = 0$. | [07] | CO2 | L3 |
| 4. | Find the extreme values of $f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$. | [07] | CO2 | L3 |


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5. A rectangular box, open at the top, is to have a volume of 32 cubic ft. Find the dimensions of the box requiring least material for its construction. [07] CO2 L3

6. Evaluate the double integral $\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dx\,dy$ by changing the order of integration. [07] CO3 L3

7. By changing to polar coordinates, evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} y\sqrt{x^2+y^2}\,dxdy$. [07] CO3 L3

8. Evaluate: $\lim_{x\to a}\left(2-\frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$ and $\lim_{x\to 0}\left(\frac{1^x+2^x+3^x}{3}\right)^{\left(\frac{1}{x}\right)}$ . [07] CO2 L3

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# $2^{nd}$ Internal Assessment Scheme and Solⁿ



Sub: Calculus and Linear Algebra

Sub code: 18MAT11

Date: 03/12/2018

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Sem/Sec: I/All sections

1) a) If $u = x + 3y^2 - z^3$, $v = 4x^2yz$, $w = 2z^2 - xy$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 6y & -3z^2 \\ 8xyz & 4x^2z & 4x^2y \\ -y & -x & 4z \end{vmatrix} \quad \text{—— (2)}$$

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} \text{ at } (1,-1,0) = \begin{vmatrix} 1 & -6 & 0 \\ 0 & 0 & -4 \\ 1 & -1 & 0 \end{vmatrix} \quad \text{—— (1)}$$

$$[J]_{(1,-1,0)} = 20 \quad \text{—— (1)}$$

b) Evaluation of $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dy\,dx\,dz$

$$I = \int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dy\,dx\,dz$$

$$I = \int_{-1}^{1}\int_{0}^{z}\left[xy + \frac{y^2}{2} + zy\right]_{x-z}^{x+z} dx\,dz \quad \text{—— (1)}$$

$$I = \int_{-1}^{1}\int_{0}^{z}(4xz + 2z^2)\,dx\,dz \quad \text{—— (1)}$$

$$I = \int_{-1}^{1}(2z^3 + 2z^3)\,dz \quad \text{—— (1)}$$

$$\boxed{I = 0} \quad \text{—— (1)}$$

2 a) Maclaurin's expansion of $\log(\sec x+\tan x)$ up to the three non vanishing terms.

W.K.T $y(x) = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \cdots$ (1)

Given $y(x) = \log(\sec x + \tan x)$

$y(0) = \log 1 = 0$

$y_1(x) = \frac{1}{\cancel{(\sec x + \tan x)}} \cancel{(\sec x + \tan x)} \sec x$

$y_1(x) = \sec x$

$y_1(0) = 1$

$y_2(x) = \sec x \tan x$

$y_2(0) = 0$

$y_3(x) = \sec x [1 + 2\tan^2 x]$

$y_3(0) = 1$

$y_4(0) = 0$

$y_5(0) = 5$ (5)

$\therefore y(x) = 0 + x(1) + \frac{x^2}{2} \times 0 + \frac{x^3}{6} \times 1 + \frac{x^4}{24} \times 0 + \frac{x^5}{120} \times 5 + \cdots$ (1)

$\log(\sec x + \tan x) = x + \frac{x^3}{6} + \frac{x^5}{24} + \cdots$

3) a). $u = f(4x-5y,\ 5y-6z,\ 6z-4x)$

To prove: $\frac{1}{4} u_x + \frac{1}{5} u_y + \frac{1}{6} u_z = 0$

$u \to (p, q, r) \to (x, y, z)$

By chain rule

$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial x} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x}$ (1)

$u_x = u_p(4) + u_q \cdot (0) + u_r \cdot (-4)$ (1)

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} \quad \text{—(1)}$$

$$\frac{\partial u}{\partial y} = u_p(-5) + u_q(5) + u_r(0) \quad \text{—(1)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} \quad \text{—(1)}$$

$$u_z = u_p(0) + u_q(-6) + u_r(6) \quad \text{—(1)}$$

$$\therefore \frac{1}{4}\frac{\partial u}{\partial x} + \frac{1}{5}\frac{\partial u}{\partial y} + \frac{1}{6}\frac{\partial u}{\partial z}$$

$$= \cancel{u_p} + \cancel{u_r} - \cancel{u_p} + \cancel{u_q} - \cancel{u_q} + \cancel{u_r}$$

$= 0$ Proved. —(1)

(4) To find the Extreme values of $f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$

$f_x = P = 3x^2 + 3y^2 - 6x$ and $f_y = 6xy - 6y$ —(1)

we shall find the points $f_x = 0$, $f_y = 0$

$\therefore x^2 + y^2 - 2x = 0$ and $xy - y = 0$ —(1)

the Stationary points are $(0,0)$, $(1,1)$, $(1,-1)$, $(2,0)$ —(2)

| | $(0,0)$ | $(1,1)$ | $(1,-1)$ | $(2,0)$ |
|---|---|---|---|---|
| $r = f_{xx} = 6x-6$ | $-6 < 0$ | $0$ | $0$ | $6 > 0$ |
| $t = f_{yy} = 6x-6$ | $-6$ | $0$ | $0$ | $6$ |
| $s = f_{xy} = 6y$ | $0$ | $6$ | $-6$ | $0$ |
| $rt - s^2$ | $36 > 0$ | $-36 < 0$ | $-36 < 0$ | $36 > 0$ |
| Conclusion | Max. pt | Saddle pt | Saddle pt | Min. pt |

$\therefore f_{max}$ at $(0,0) = 4$

$f_{min}$ at $(2,0) = 0$ —(3)

5 Given Volume = 32 cubic. ft (4)

Let $x, y, z$ be the length, breadth, and height of the given rectangular box, V ~~Th~~ be its volume and S be its surface area.

Then $V = xyz$

and $S = 2(xy + yz + zx) - xy = 2yz + 2zx + xy$

$\therefore$ ~~$V = 32$~~ $xyz = 32$

Since we are using least material for construction, S is minimum

We need to find $x, y, z$ such that S is minimum, subject to the condition $V = 32$.

~~Let $F = 2(xy) + 2yz + xy$~~

Let $F = xy + 2yz + 2zx + \lambda(xyz)$ — (2)

$F_x = y + 2z + \lambda(yz)$

$F_y = x + 2z + \lambda xz$

$F_z = 2y + 2x + \lambda xy$

Form the eqns $F_x = 0,\ F_y = 0,\ F_z = 0$

ie $y + 2z + \lambda yz = 0 \implies \lambda = \dfrac{-(y+2z)}{yz} \rightarrow (1)$

$x + 2z + \lambda xz = 0 \implies \lambda = -(x+2z)/xz \rightarrow (2)$

$2y + 2x + \lambda xy = 0$

$\implies \lambda = \dfrac{-(2y+2x)}{xy} \rightarrow (3)$

From (1), (2) & (3)

$$\frac{-(y+2z)}{yz} = \frac{-(x+2z)}{xz} = \frac{-(2y+2x)}{xy} \quad — (2)$$

Consider $\frac{-(y+2z)}{yz} = \frac{-(x+2z)}{xz}$

$$\Rightarrow \frac{y+2z}{y} = \frac{x+2z}{x}$$

$$\Rightarrow \cancel{xy} + 2zx = \cancel{xy} + 2zy$$

$$\Rightarrow \cancel{2z}x = \cancel{2z}y \Rightarrow \boxed{x=y} \text{ --- } (1)$$

Consider $\frac{-(x+2z)}{\cancel{x}z} = \frac{-(2x+2y)}{xy}$

$$\Rightarrow yx + \cancel{2zx} = \cancel{2xz} + 2yz$$

$$\Rightarrow yx = 2yz$$

$$\Rightarrow z = \frac{x}{2}$$

Now $y=x$ & $z=\frac{x}{2}$.

Sub in $xyz = 32$

$$\Rightarrow x(x)\left(\frac{x}{2}\right) = 32$$

$$\Rightarrow x^3 = 64$$

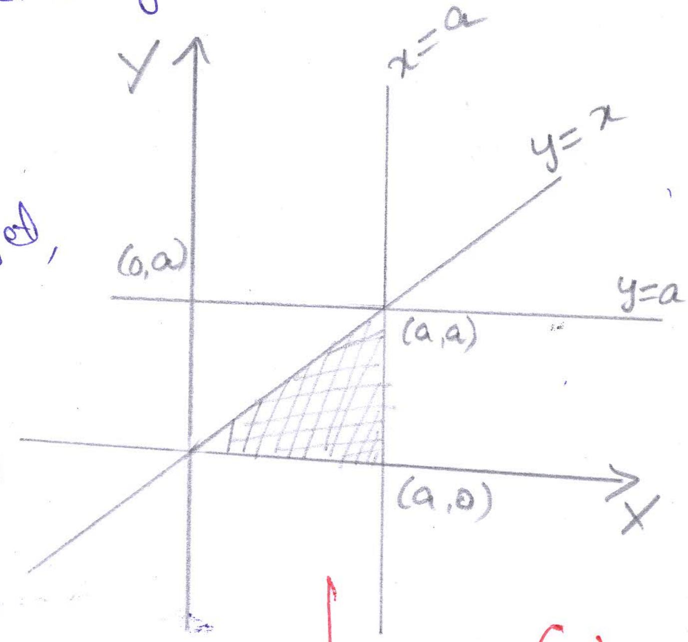
$$\Rightarrow x = 4$$

$\therefore y=4$, $z=2$. $\Big\}$ — (1)

Thus the demensions of the given rectangular box are 4, 4, 2. — (1)

6 $\int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy$ (5)

Let $I = \int_{y=0}^{a} \int_{x=y}^{a} \frac{x}{x^2+y^2}\, dx\, dy$




(2)

By changing the order we get,

$I = \int_{x=0}^{a} \int_{y=0}^{x} \frac{1}{x^2+y^2}\, x\, dy\, dx$ — (2)

$= \int_{x=0}^{a} x \frac{1}{x} \left[\tan^{-1}\left(\frac{y}{x}\right)\right]_{y=0}^{x} dx$

$= \int_{x=0}^{a} \left[\tan^{-1} 1 - \tan^{-1} 0\right] dx$

$= \int_{x=0}^{a} \left(\frac{\pi}{4} - 0\right) dx$

$= \frac{\pi}{4} \left[x\right]_{x=0}^{a}$

$= \frac{\pi a}{4} = \frac{a\pi}{4}$ — (3)

7 $\int_0^a \int_0^{\sqrt{a^2-y^2}} y\sqrt{x^2+y^2}\, dx\, dy$

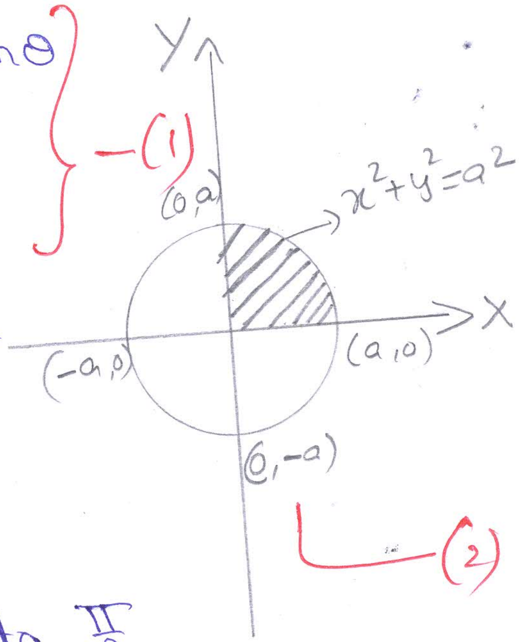
Let $I = \int_{y=0}^{a} \int_{x=0}^{\sqrt{a^2-y^2}} y\sqrt{x^2+y^2}\, dx\, dy$

$x = \sqrt{a^2-y^2} \Rightarrow x^2+y^2 = a^2$

We have $x = r\cos\theta$, $y = r\sin\theta$

$\therefore x^2 + y^2 = r^2$ — (1)

and $dx\,dy = r\,dr\,d\theta$




(2)

Now $x^2 + y^2 = a^2$ & $x^2 + y^2 = r^2$

$\therefore r^2 = a^2 \Rightarrow \boxed{r = a}$

Since the region in the first quadrant, $\theta$ varies from $0$ to $\frac{\pi}{2}$.

$$\therefore I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} (r\sin\theta)\, r \cdot r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r^3 \sin\theta\, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \sin\theta \left(\frac{r^4}{4}\right)_{r=0}^{a} d\theta \quad \text{— (2)}$$

$$= \frac{a^4}{4} \int_{\theta=0}^{\pi/2} \sin\theta\, d\theta$$

$$= \frac{a^4}{4} \left(-\cos\theta\right)_{\theta=0}^{\pi/2}$$

$$= -\frac{a^4}{4}\left[\cos\frac{\pi}{2} - \cos 0\right]$$

$$= -\frac{a^4}{4}[0 - 1]$$

$$I = \frac{a^4}{4} \quad \text{— (2)}$$

8 $\lim_{x \to a} \left(2 - \frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)} = 1^{\infty}$ —— $\left(1/2\right)$ (6)

Let $K = \lim_{x \to a} \left(2 - \frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$

Taking log

$$\log K = \lim_{x \to a} \log \left(2 - \frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$$

$$= \lim_{x \to a} \tan\left(\frac{\pi x}{2a}\right) \log\left(2 - \frac{x}{a}\right) \quad \ldots \infty \times 0$$

$$= \lim_{x \to a} \frac{\log\left(2 - \frac{x}{a}\right)}{\cot\left(\frac{\pi x}{2a}\right)} \quad \ldots \frac{0}{0} \quad \text{—— (1)}$$

Applying L-Hospital's Rule

$$\log K = \lim_{x \to a} \frac{\left(\frac{1}{2 - x/a}\right)\left(-\frac{1}{a}\right)}{-\operatorname{cosec}^2\left(\frac{\pi x}{2a}\right)\left(\frac{\pi}{2a}\right)} \quad \text{—— (1)}$$

$$= \frac{-\frac{1}{a}}{-\pi/2a}$$

$$\log K = \frac{2}{\pi}$$

$$K = e^{2/\pi} \quad \text{—— (1)}$$

$$\lim_{x\to 0}\left(\frac{1^x+2^x+3^x}{3}\right)^{\frac{1}{x}} \quad --- \quad 1^\infty \quad — (1/2)$$

Let $K = \lim_{x\to 0}\left(\frac{1^x+2^x+3^x}{3}\right)^{1/x}$

Taking log

$$\log K = \lim_{x\to 0}\frac{\log\left(\frac{1^x+2^x+3^x}{3}\right)}{x} = \frac{0}{0} \quad — (1)$$

Applying L-Hospital's Rule

$$\log K = \lim_{x\to 0}\frac{\frac{1}{\left(\frac{1^x+2^x+3^x}{3}\right)}\left(\frac{1}{3}\right)\left[1^x\log 1+2^x\log 2+3^x\log 3\right]}{1} \quad — (1)$$

$$= \lim_{x\to 0}\left[\frac{\cancel{3}}{1^x+2^x+3^x}\right]\left(\frac{1}{\cancel{3}}\right)\left[1^x\log 1+2^x\log 2+3^x\log 3\right]$$

$$= \frac{1}{3}\left[0+\log 2+\log 3\right]$$

$$\log K = \frac{\log(6)}{3} = \log(6^{1/3})$$

$$K = 6^{1/3} \quad — (1)$$