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Internal Assessment Test – III

Sub:	**Engineering Mathematics - III**					Sub Code:	17MAT31	Branch:	CS,EEE, CV
Date:	19.11.2018	Duration:	90 min's	Max Marks:	50	Sem/ Sec:	III/ CS-A, EEE-A,CV-A		OBE

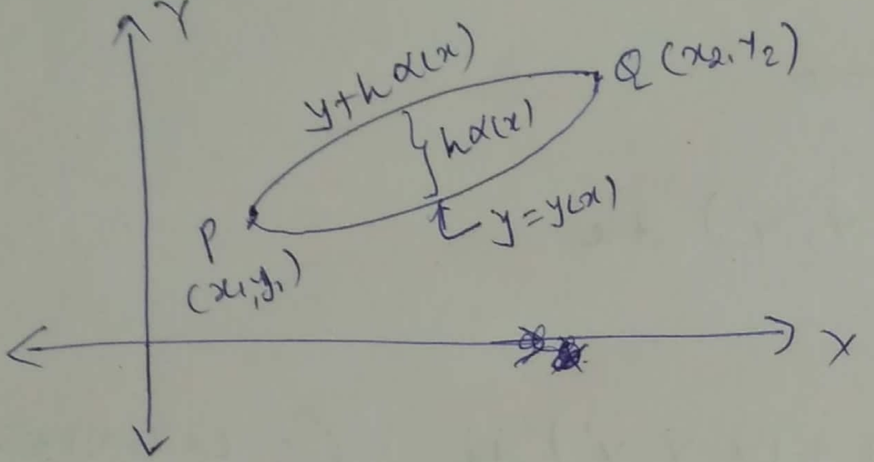
	Question 1 is compulsory. Answer any SIX questions from Question 2 to 8.	MARKS	CO	RBT
1.	State and prove Euler's theorem.	[08]	CO3	L3
2.	Find the infinite Fourier transform of $f(x) = \begin{cases} 1, & \lvert x\rvert \leq a \\ 0, & \lvert x\rvert > a \end{cases}$, Where **a** is a positive constant. Hence evaluate $\int_0^{\infty} \frac{\sin(ax)}{x}\, dx$.	[07]	CO2	L3
3.	Prove that geodesics on a plane are straight lines.	[07]	CO3	L3
4.	Find the Fourier sine Transform of $e^{-\lvert x\rvert}$ Hence show that $\int_0^{\infty} \frac{x\sin(mx)}{1+x^2}\, dx = \frac{\pi e^{-\lvert m\rvert}}{2}$, m > 0.	[07]	CO2	L3

5.	Solve the variational problem $\delta \int_0^{\pi} (y'^2 - y^2 + 4y\cos x)\, dx = 0$, $y(0)=y(\pi) = 0$	[07]
6.	Find inverse Fourier transform of $F(\alpha) = e^{-\alpha^2}$.	[07]
7.	Solve the integral equation $$\int_0^{\infty} f(x)\, \sin ax\, dx = \begin{cases} 1-\alpha, & 0 \le \alpha < 1 \\ 0, & \alpha > 1. \end{cases}$$	[07]
8.	Find the curve on which the functional $I = \int_0^1 (y'^2 + 12xy)\, dx$ with $y(0) = 3, y(1) = 6$ can be determined.	[07]

IAT- III

Maths - III   CS-A, EEE-A, CV-A.

Q.1 Let I be an Extremum along some curve $y = y(x)$ joining $P(x_1, y_1)$ and $Q(x_2, y_2)$.



— 1

→ Consider a function $\alpha(x)$ with $\alpha(x_1) = 0$ and $\alpha(x_2) = 0$. Then $Y(x) = y(x) + h\alpha(x)$ is a family of curves passing through P and Q.

$$\therefore \quad Y(x_1) = y(x_1) + h\alpha(x_1) = y(x_1) + 0 = y_1$$
$$Y(x_2) = y(x_2) + h\alpha(x_2) = y(x_2) = y_2.$$

— 1

where h is a parameter.

→ When h is small, $Y(x)$ is a neighbouring curve of $y(x)$.

When $h=0$, $\boxed{Y(x) = y(x)}$ the Extremal.

Consider ②

$$I = \int_{x_1}^{x_2} f(x, y(x)+h\alpha(x), y'+h\alpha'(x))\, dx$$, is a

function of h. ——— 1

→ A nessary condition for I to be extremum is that $\frac{dI}{dh} = 0$. ——— 1

$$I = \int_{x_1}^{x_2} f(x, Y, Y')\, dx$$

$$\therefore \frac{dI}{dh} = \int_{x_1}^{x_2} \frac{\partial}{\partial x} f(x, Y, Y')\, dx$$ ($\because$ Leibnitz's rule for diff. under integral sign)

$$\therefore \frac{dI}{dh} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial h} + \frac{\partial f}{\partial Y}\frac{\partial Y}{\partial h} + \frac{\partial f}{\partial Y'}\frac{\partial Y'}{\partial h} \right) dx$$

$$= \int_{x_1}^{x_2} \left(0 + \frac{\partial f}{\partial Y}\alpha(x) + \frac{\partial f}{\partial y'}\alpha'(x) \right) dx$$ — 1 03

$$= \int_{x_1}^{x_2} \frac{\partial f}{\partial Y}\alpha(x)\, dx + \int_{x_1}^{x_2} \frac{\partial f}{\partial Y'}\alpha'(x)\, dx$$

$$= \int_{x_1}^{x_2} \frac{\partial f}{\partial Y}\alpha(x)\, dx + \left[\frac{\partial f}{\partial Y'}\alpha(x) \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial Y'}\right) \alpha(x)\, dx$$

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \frac{\partial f}{\partial y} \alpha(x)\, dx + 0 - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \alpha(x)\, dx$$

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \alpha(x) \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) dx$$

When I is Extremum, $\frac{dI}{dh} = 0$ & $Y(x) = y(x)$

$$\therefore\ 0 = \int_{x_1}^{x_2} \alpha(x) \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) dx = 0$$

— 1

Since, $\alpha(x)$ is arbitrary, we have,

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) = 0}$$

Q.2 Fourier Transform of $f(x)$ is

$$F(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx$$

— 1

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (1)\, e^{iux}\, dx$$

— 1

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{iua} - e^{-iua}}{iu} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \frac{2}{u} \left[\frac{e^{iua} - e^{-iua}}{2i} \right]$$ — 1

$$= \sqrt{\frac{2}{\pi}} \frac{1}{u} \sin(ua)$$

1

$$\boxed{F(u) = \sqrt{\frac{2}{\pi}} \frac{\sin(au)}{u}}$$ — 1

Applying inverse Fourier transform on 1

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(u) e^{-iux} du$$ — 1

$$= \frac{\cancel{\sqrt{2}}}{\sqrt{\pi}\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\sin(au)}{u} e^{-iux} du$$

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(au)}{u} e^{-iux} du$$

take $x = 0$, $f(0) = 1$ — 1

$$\therefore \quad 1 = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(au)}{u} du$$

$$1 = \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin(au)}{u} du$$

changing the variable u to x,

$$\boxed{\int_0^{\infty} \frac{\sin(ax)}{x} dx = \pi/2}$$ — 1

Q.3 Let S be the arc length of a curve joining 2 points $A(x_1, y_1)$ and $B(x_2, y_2)$ on a plane,

$$S = \int_{x_1}^{x_2} ds \quad \text{— } 1$$

$$S = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx \quad \text{— } 1$$

S is min if Euler's equation is satisfied,

here, $f(x, y, y') = \sqrt{1 + y'^2}$ — 1

Euler's equation : $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ — 1

$$\Rightarrow \quad 0 - \frac{d}{dx}\left(\frac{2y'}{2\sqrt{1+y'^2}}\right) = 0$$

$$\Rightarrow \quad \frac{\sqrt{1+y'^2}\, y'' - y' \dfrac{2y'\, y''}{2\sqrt{1+y'^2}}}{(1+y'^2)} = 0$$

2

$$\Rightarrow \quad y''(1+y'^2) - y''\, y'^2 = 0$$

$\Rightarrow \quad y'' = 0 \quad \Rightarrow \quad y' = c_1$ Integrating

$$\Rightarrow \boxed{y = x c_1 + c_2} \quad \text{— } 1$$

Q.4 Fourier sine transform of f(x) is

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-|x|} \sin(ux)\, dx \quad \text{— } 1$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^{\infty} e^{-x} \sin(ux)\, dx \right] \quad \left(\because |x| = x \text{ as } x > 0 \right)$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-x}}{1+u^2} \left\{ -\sin(ux) - u\cos ux \right\} \right]_0^{\infty}$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - \frac{1}{1+u^2} (0 - u) \right] \quad \text{2}$$

$$\boxed{F_s[f(x)] = \sqrt{2/\pi} \left(\frac{u}{1+u^2} \right)} \quad \text{— } 1$$

Applying inverse Fourier Transform,

$$f(x) = \sqrt{2/\pi} \int_0^{\infty} \sqrt{2/\pi} \left(\frac{u}{1+u^2} \right) \sin(ux)\, du$$

$$e^{-|x|} = \frac{2}{\pi} \int_0^{\infty} \frac{u \sin(ux)}{(1+u^2)}\, du \quad \text{2}$$

changing the variable u to x and x to m,

$$\boxed{\int_0^{\infty} \frac{x \sin(mx)}{(1+x^2)}\, dx = \frac{\pi}{2} e^{-|m|}} \quad ; m > 0$$

1

Q.5 here $f(x, y, y') = y'^2 - y^2 + 4y\cos x$ — 1

Euler eqⁿ: $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ — 1

$$(-2y + 4\cos x) - \frac{d}{dx}(2y') = 0$$

$$-2y + 4\cos x - 2y'' = 0$$

$$\therefore \boxed{y'' + y = 2\cos x}$$ — 1

A.E. $m^2 + 1 = 0 \Rightarrow m = \pm i$

$\therefore$ C.F. $= (c_1 \cos x + c_2 \sin x)$ — 1

P.I. $= \frac{1}{D^2+1} 2\cos x$

$a^2 = -1$

$$= \frac{x}{\cancel{2}D}\cancel{2}\cos x$$

$$= x \sin x$$ — 1

$\therefore$ $y =$ C.F. + P.I.

$= c_1 \cos x + c_2 \sin x + x \sin x$ — 1

$y = c_1 \cos x + (c_2 + x)\sin x$

$y(0) = 0 \Rightarrow 0 = c_1 + 0 \Rightarrow \boxed{c_1 = 0}$

$\therefore \boxed{y = (c_2 + x)\sin x}$ — 1

Q.6 $\quad F(\alpha) = e^{-\alpha^2}$ 8

consider $\alpha = u \quad \therefore F(u) = e^{-u^2}$

Applying inverse F.T.,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(u)\, e^{-iux}\, du \quad \text{— 1}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2}\, e^{-iux}\, du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(u^2+iux)}\, du \quad \text{— 1}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(u^2+iux+\frac{i^2x^2}{4}\right) + i^2x^2/4}\, du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(u+ix/2\right)^2}\, e^{-x^2/4}\, du$$

take $\quad u + ix/2 = t \Rightarrow du = dt$ 3

$$= \frac{e^{-x^2/4}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2}\, dt$$

$$= \frac{e^{-x^2/4}}{\sqrt{2}\,\cancel{\sqrt{\pi}}} \left(\cancel{\sqrt{\pi}}\right) \quad \text{— 1}$$

$$\boxed{f(x) = \frac{e^{-x^2/4}}{\sqrt{2}}} \quad \text{— 1}$$

Q.7

$$\int_0^\infty f(x)\sin(\alpha x)\,dx = \begin{cases} 1-\alpha, & 0 \le \alpha < 1 \\ 0, & \alpha > 1 \end{cases}$$

Multiplying $\sqrt{2/\pi}$ on both the side,

$$\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(\alpha x)\,dx = \sqrt{\frac{2}{\pi}}\begin{cases} 1-\alpha, & 0 \le \alpha < 1 \\ 0, & \alpha > 1 \end{cases} \quad \text{--- (1)}$$

Consider $\alpha = u$

$$\sqrt{2/\pi}\int_0^\infty f(x)\sin(ux)\,dx = \sqrt{2/\pi}\begin{cases} 1-u, & 0 \le u < 1 \\ 0, & u > 1 \end{cases}$$

Fourier sine transform of $f(x)$ is,

$$F_S(u) = \sqrt{2/\pi}\int_0^\infty f(x)\sin(ux)\,dx \quad \text{--- (1)}$$

$$\therefore F_S(u) = \sqrt{2/\pi}\begin{cases} 1-u, & 0 \le u < 1 \\ 0, & u > 1 \end{cases} \quad \text{--- (1)}$$

Applying inverse Fourier sine transform on $F_S(u)$,

$$f(x) = \sqrt{2/\pi}\int_0^\infty F_S(u)\sin(ux)\,du \quad \text{--- (1)}$$

$$= \sqrt{2/\pi}\int_0^1 \sqrt{2/\pi}\,(1-u)\sin(ux)\,du$$

$$= \frac{2}{\pi}\left[(1-u)\left\{-\frac{\cos(ux)}{x}\right\} - (-1)\left\{-\frac{\sin(ux)}{x^2}\right\}\right]_0^1$$

$$= \frac{2}{\pi}\left[0 - \frac{\sin(x)}{x^2} - \left(\frac{-1}{x}\right)\right] \quad -\ 2$$

$$\boxed{f(x) = 2/\pi \left(\frac{1}{x} - \frac{\sin x}{x^2}\right)} \quad -\ 1$$

Q.8 here $f(x, y, y') = y'^2 + 12xy$ — 1

Euler eqn : $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ — 1

$$12x - \frac{d}{dx}(2y') = 0$$

$$12x - 2y'' = 0$$

$$\therefore\ y'' = 6x \quad -\ 1$$

$$\Rightarrow\ y' = 3x^2 + c_1 \quad -\ 1$$

$$\Rightarrow\ \boxed{y = x^3 + c_1 x + c_2} \quad -\ 1$$

$y(0) = 3 \Rightarrow \boxed{3 = c_2}$

$y(1) = 6 \Rightarrow 6 = 1 + c_1 + c_2$

$\therefore\ 5 = c_1 + c_2$

$\boxed{c_1 = 2}$ — 1

$$\therefore\ \boxed{y = x^3 + 3x + 2} \quad -\ 1$$