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CMR INSTITUTE OF TECHNOLOGY, BENGALURU.

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Internal Assessment Test III –January 2019

Sub:	Calculus and Linear Algebra					Sub Code:	18MAT11	
Date:	03/01/2019	Duration:	90 mins	Max Marks:	50	Sem / Sec:	I / G , from I to M.	OBE

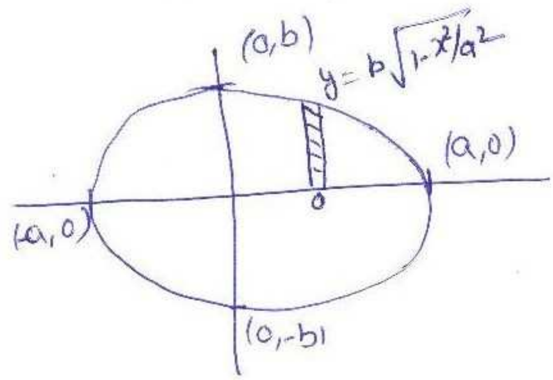
Question 1 is compulsory and answer any SIX questions from the rest.

		MARKS	CO	RBT
1.	(a) Find the area of a quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.	[04]	CO3	L3
	(b)Find the pedal equation for $r(1-cos\theta)$=2a.	[04]	CO4	L3
2.	The density at any point (x,y) of a lamina is $\frac{\sigma}{a}(x+y)$, where σ and a are constants. The lamina is bounded by the lines $x=0, y=0, x=a, y=b$. find the position of its centre of gravity.	[07]	CO3	L3
3.	State and prove the relation between Beta and Gamma functions. Hence find the value of $\Gamma\left(\frac{1}{2}\right)$.	[07]	CO3	L3
4.	Show that $\int_0^\infty \sqrt{y}\, e^{-y^2} dy \times \int_0^\infty \frac{e^{-y^2}}{\sqrt{y}} dy = \frac{\pi}{2\sqrt{2}}$	[07]	CO3	L3

5.	Derive an expression for the Angle between the radius vector and tangent.	[07]	CO1	L3
6.	Find the Angle between the given two polar curves $r = a log\theta$ and $r = \frac{a}{log\theta}$.	[07]	CO1	L3
7.	Find the Centre and Circle of curvature for the curve $xy = c^2$ at (c, c) .	[07]	CO1	L3
8.	Find the Evolute of the parabola $y^2 = 4ax$.	[07]	CO1	L3

III IAT Solutions

Q1. (a) Area $= \iint dx\,dy$



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\text{Area} = \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} dy\,dx$$

$$= \int_0^a [y]_0^{b\sqrt{1-x^2/a^2}} dx = \frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx = \frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= \frac{b}{a}\left[0 + \frac{a^2}{2}\sin^{-1}(1) - 0\right] = \frac{ab}{2}\cdot\frac{\pi}{2} = \frac{\pi ab}{4}$$

(b) Given curve $r(1-\cos\theta) = 2a$ — (1)

Pedal equation $p = r\sin\phi$ — (2)

from (1), $\log r + \log(1-\cos\theta) = \log 2a$

diff w.r.t. 'θ'

$$\frac{1}{r}\frac{dr}{d\theta} + \frac{\sin\theta}{1-\cos\theta} = 0 \Rightarrow \cot\phi = -\frac{2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2}$$

$$= -\cot\theta/2 = \cot(-\theta/2)$$

$\Rightarrow \phi = -\frac{\theta}{2}$ — (3)

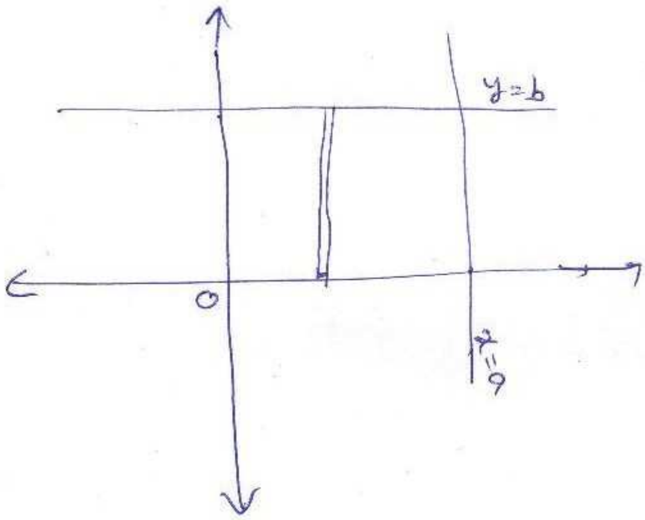
from (2) & (3), $p = r\sin(-\theta/2) = -r\sin\theta/2$. — (4)

from (1), $2r\sin^2\theta/2 = 2a \Rightarrow \sin^2\theta/2 = a/r$

from (4), $p^2 = r^2\sin^2\theta/2 \Rightarrow \boxed{p^2 = ar}$

Q2. Given $\rho = \frac{\sigma}{a}(x+y)$.

bounded by $x=0, y=0, x=a, y=b$



$$M_A = \iint \rho\, dx\, dy$$

$$\bar{x} = \frac{1}{M_A}\iint \rho x\, dx\, dy$$

$$\bar{y} = \frac{1}{M_A}\iint \rho y\, dx\, dy$$

$$M_A = \int_0^b\int_0^a \frac{\sigma}{a}(x+y)\, dx\, dy$$

$$= \frac{\sigma}{a}\int_0^b \left(\frac{x^2}{2}+xy\right)_0^a dy = \frac{\sigma}{a}\int_0^b\left(\frac{a^2}{2}+ay\right)dy = \frac{\sigma}{a}\left(\frac{a^2}{2}y+\frac{ay^2}{2}\right)_0^b$$

$$= \frac{\sigma}{a}\left(\frac{a^2 b}{2}+\frac{ab^2}{2}\right) = \frac{\sigma b(a+b)}{2}$$

$$\bar{x} = \frac{1}{M_A}\int_0^a\int_0^b \frac{\sigma}{a}(x+y).x\, dy\, dx = \frac{\sigma}{a}\cdot\frac{2}{\sigma b(a+b)}\int_0^a \left(x^2y+\frac{xy^2}{2}\right)_{y=0}^b dx$$

$$= \frac{2}{ab(a+b)}\int_0^a\left(x^2b + x\frac{b^2}{2}\right)dx = \frac{2}{ab(a+b)}\left(\frac{x^3b}{3}+\frac{x^2b^2}{4}\right)_0^a$$

$$= \frac{2}{ab(a+b)}\left(\frac{a^3b}{3}+\frac{a^2b^2}{4}\right) = \frac{2a}{a+b}\left(\frac{a}{3}+\frac{b}{4}\right) = \frac{a(4a+3b)}{6(a+b)}$$

$$\bar{y} = \frac{1}{M_A}\int_{y=0}^b\int_{x=0}^a \frac{\sigma}{a}(x+y).y\, dx\, dy = \frac{\sigma}{aM_A}\int_0^b\left(\frac{x^2y}{2}+xy^2\right)_0^a dy$$

$$\bar{y} = \frac{\sigma}{a} \cdot \frac{2}{\sigma b(a+b)} \int_0^b \left(\frac{a^2 y}{2} + ay^2 \right) dy$$

$$= \frac{2}{ab(a+b)} \left(\frac{a^2y^2}{4} + \frac{ay^3}{3} \right)_0^b = \frac{2}{ab(a+b)} \left(\frac{a^2b^2}{4} + \frac{ab^3}{3} \right)$$

$$= \frac{2b}{(a+b)} \left(\frac{a}{4} + \frac{b}{3} \right) = \frac{b(3a+4b)}{6(a+b)}$$

Q3. To prove, $\beta(m,n) = \dfrac{\Gamma m \, \Gamma n}{\Gamma(m+n)}$

Proof:- we have by definition of Beta & Gamma,

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \, \cos^{2n-1}\theta \, d\theta \quad \text{------ (1)}$$

$$\Gamma n = 2\int_0^\infty e^{-x^2} x^{2n-1} \, dx \quad \text{---- (2)}$$

$$\Gamma m = 2\int_0^\infty e^{-y^2} y^{2m-1} \, dy \quad \text{----- (3)}$$

$$\Gamma(m+n) = 2\int_0^\infty e^{-r^2} r^{2(m+n)-1} \, dr \quad \text{---- (4)}$$

Now, $\Gamma m \, \Gamma n = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} x^{2n-1} y^{2m-1} \, dx\, dy$

Changing RHS, to polar co-ordinates

$x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,d\theta\,dr$.

$$\Gamma m \Gamma n = 4 \int_0^\infty \int_0^{\pi/2} e^{-r^2} (r\cos\theta)^{2n-1} (r\sin\theta)^{2m-1}\, r\, dr\, d\theta$$

$$= 4 \int_0^\infty \int_0^{\pi/2} e^{-r^2} r^{2(m+n)-1} \cos^{2n-1}\theta \sin^{2m-1}\theta\, dr\, d\theta$$

$$= \left[2\int_0^\infty e^{-r^2} r^{2(m+n)-1}\, dr\right]\left[2\int_0^{\pi/2} \cos^{2n-1}\theta \sin^{2m-1}\theta\, d\theta\right]$$

$$= \Gamma(m+n)\ \beta(m+n) \Rightarrow \beta(m,n) = \frac{\Gamma m\, \Gamma n}{\Gamma(m+n)}$$

Now, taking $m = 1/2, n = 1/2 \Rightarrow \frac{(\Gamma 1/2)^2}{\Gamma 1} = \beta(1/2, 1/2)$

$$(\Gamma \tfrac{1}{2})^2 = 2\int_0^{\pi/2} d\theta = 2[\theta]_0^{\pi/2} = 2.\pi/2 \Rightarrow (\Gamma \tfrac{1}{2})^2 = \pi \Rightarrow \boxed{\Gamma \tfrac{1}{2} = \sqrt{\pi}}$$

Q4. Let $I_1 = \int_0^\infty \sqrt{y}\, e^{-y^2} dy$, $I_2 = \int_0^\infty \frac{e^{-y^2}}{\sqrt{y}} dy$

We, know that, $\Gamma n = 2\int_0^\infty e^{-x^2} x^{2n-1} dx$

$\therefore I_1 =$ for $I_1 =$ $2n-1 = 1/2$; for $I_2 =$ $2n-1 = -1/2$

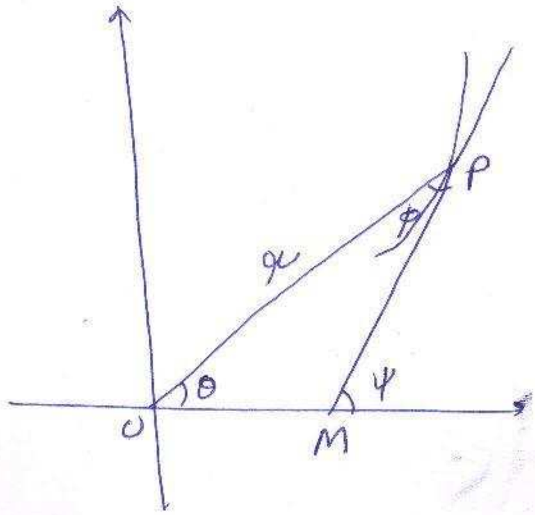
$n = 3/4$; $n = 1/4$

$\therefore I_1 = \frac{\Gamma 3/4}{2}$; $I_2 = \frac{\Gamma 1/4}{2}$

$\Rightarrow I_1 \times I_2 = \frac{1}{4} \Gamma 1/4\, \Gamma 3/4 = \frac{1}{4}. \pi\sqrt{2} = \pi/2\sqrt{2}$

Proved

Q5. Let $r = f(\theta)$ be the given curve.
$OP = r$. Let ψ be the angle between tangent at P and initial line.
Let ϕ be the angle between radius vector and tangent at P.



By triangle OPM,

~~$\psi = \phi + \theta \Rightarrow \tan\psi = \tan\theta + \tan$~~

$$\psi = \theta + \phi \Rightarrow \tan\psi = \tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} \quad \text{—— (1)}$$

$\therefore$ By Geometry $\tan\psi = \dfrac{dy}{dx}$,

we know that $x = r\cos\theta$, $y = r\sin\theta$

$$\tan\psi = \frac{dy/d\theta}{dx/d\theta} = \frac{r\cos\theta + r_1\sin\theta}{-r\sin\theta + r_1\cos\theta}, \quad \boxed{r_1 = \frac{dr}{d\theta}}$$

$$\tan\psi = \frac{\dfrac{r_1\sin\theta + r\cos\theta}{r_1\cos\theta}}{\dfrac{r_1\cos\theta - r\sin\theta}{r_1\cos\theta}} = \frac{\tan\theta + \dfrac{r}{r_1}}{1 - \dfrac{r}{r_1}\tan\theta} \quad \text{—— (2)}$$

Comparing (1) & (2)

$$\tan\phi = \frac{r}{r_1} \Rightarrow \boxed{\tan\phi = r\frac{d\theta}{dr}}$$

Q6. Given

$$r = a\log\theta \quad ; \quad r = \frac{a}{\log\theta}$$

$$\Rightarrow \log r = \log a + \log(\log\theta) \quad ; \quad \log r = \log a - \log(\log\theta)$$

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\log\theta}\cdot\frac{1}{\theta} \quad ; \quad \frac{1}{r}\frac{dr}{d\theta} = 0 - \frac{1}{\log\theta}\cdot\frac{1}{\theta}$$

$$\cot\phi_1 = \frac{1}{\theta\log\theta} \quad ; \quad \cot\phi_2 = -\frac{1}{\theta\log\theta}.$$

to find intersection, solving given equations

$$a\log\theta = \frac{a}{\log\theta} \Rightarrow (\log\theta)^2 = 1 \Rightarrow \log\theta = 1 \Rightarrow \theta = e$$

$$\therefore \tan\phi_1 = e\log e = e \qquad \tan\phi_2 = -e\log e = -e$$

$$\therefore \tan(\phi_1 - \phi_2) = \frac{\tan\phi_1 - \tan\phi_2}{1+\tan\phi_1\tan\phi_2} = \frac{e+e}{1-e^2} = \frac{2e}{1-e^2}$$

$$|\phi_1 - \phi_2| = \tan^{-1}\left(\frac{2e}{1-e^2}\right)$$

Q7. Given $xy = c^2$

Centre of curvature $(\bar{x}, \bar{y})$ is given by

$$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2) \quad , \quad \bar{y} = y + \frac{(1+y_1^2)}{y_2}$$

diff given curve.

$$y + xy_1 = 0 \Rightarrow y_1 = -y/x \quad \text{at } (c,c)$$

$$= -c/c = -1$$

$$y_2 = -\left(\frac{xy_1 - y}{x^2}\right) = -\left(\frac{-c-c}{c^2}\right) = \frac{2}{c}$$

$$\bar{x} = c - \frac{(-1)}{(2/c)}(1+1) = c + \frac{c}{2}(2) = 2c$$

$$\bar{y} = c + \frac{(1+1)}{(2/c)} = c + \frac{c}{2}\cdot 2 = 2c$$

Circle of curvature is given by

$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$$

$$\rho = \frac{[1+y_1^2]^{3/2}}{y_2} = \frac{[1+1]^{3/2}}{(2/c)} = 2\sqrt{2}\cdot\frac{c}{2} = c\sqrt{2}$$

$\therefore$ Circle is given by

$$(x-2c)^2 + (y-2c)^2 = 2c^2$$

Q8. Given $y^2 = 4ax$

Parametric equation for parabola is $y = 2at$, $x = at^2$

$$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2) \qquad \bar{y} = y + \frac{(1+y_1^2)}{y_2}$$

$$y_1 = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t}$$

$$y_2 = -\frac{1}{t^2}\cdot\frac{dt}{dx} = \frac{-1}{t^2}\cdot\frac{1}{2at} = -\frac{1}{2at^3}$$

$$\bar{x} = at^2 - \frac{(1/t)}{(-1/2at^3)}\left(1+\frac{1}{t^2}\right) = at^2 + 2at^2\left(\frac{t^2+1}{t^2}\right)$$

$$\bar{x} = at^2 + 2at^2 + 2a = 3at^2 + 2a$$

$$\Rightarrow \quad t = \left[\frac{\bar{x}-2a}{3a}\right]^{1/2}$$

$$\bar{y} = 2at + \frac{1}{(-\frac{1}{2at^3})}\left(1+\frac{1}{t^2}\right)$$

$$= 2at - 2at^3 \frac{(t^2+1)}{\cancel{t^2}}$$

$$= 2at - 2at(t^2+1)$$

$$\bar{y} = -2at^3$$

$$\bar{y} = -2a\left[\frac{\bar{x}-2a}{3a}\right]^{3/2}$$

$$(\bar{y})^2 = 4a^2 \frac{[\bar{x}-2a)^3}{(3a)^3} \Rightarrow 27a^{\cancel{3}}(\bar{y})^2 = 4a^{\cancel{2}}(\bar{x}-2a)^3$$

Evolute, i.e, locus of $(\bar{x}, \bar{y})$ is given by

$\cancel{27y^3}$

$$\boxed{27ay^2 = 4(x-2a)^3}$$