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Internal Assessment Test III –January 2019

Sub:	Calculus and Linear Algebra					Sub Code:	18MAT11	
Date:	03/01/2019	Duration:	90 mins	Max Marks:	50	Sem / Sec:	I / A to F, N and O	OBE

Question 1 is compulsory and answer any SIX questions from the rest.

	Question	MARKS	CO	RBT
1.	(a) Find the area of a quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.	[04]	CO3	L3
	(b) A copper ball originally at 80^0C cools down to 60^0C in 20 minutes, if the temperature of the air being 40^0C. What will be the temperature of the ball after 40 minutes from the original?	[04]	CO4	L3
2.	The density at any point (x,y) of a lamina is $\frac{\sigma}{a}(x+y)$, where σ and a are constants. The lamina is bounded by the lines $x=0, y=0, x=a, y=b$. Find the position of its centre of gravity.	[07]	CO3	L3
3.	State and prove the relation between Beta and Gamma functions. Hence find the value of $\Gamma\left(\frac{1}{2}\right)$.	[07]	CO3	L3
4.	Show that $\int_0^\infty \sqrt{y}\, e^{-y^2} dy \times \int_0^\infty \frac{e^{-y^2}}{\sqrt{y}} dy = \frac{\pi}{2\sqrt{2}}$.	[07]	CO3	L3

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5.	Solve $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$.	[07]	CO4	L3
6.	Solve $(r sin\theta - r^2)d\theta - (cos\theta)dr = 0$.	[07]	CO4	L3
7.	Find the orthogonal trajectories of the family of curves $\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$, where λ is the parameter.	[07]	CO4	L3
8.	Obtain the general solution and the singular solution of the following equation as Clairaut's equation : $xp^3 - yp^2 + 1 = 0$.	[07]	CO5	L3

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IAT - III

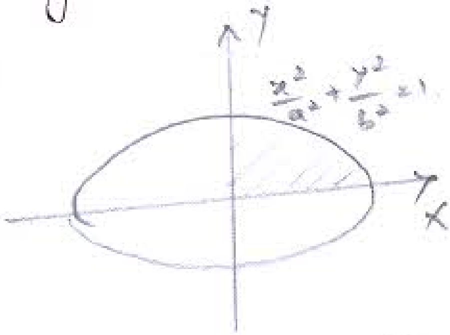
Engineering Mathematics - I

Calculus And Linear Algebra - 18MAT11.

1

a) $x^2/a^2 + y^2/b^2 = 1$

$$\text{Area of I quad} = \iint_R dx\,dy = \int_{x=0}^{a}\int_{y=0}^{b/a\sqrt{a^2-x^2}} dy\,dx \quad \text{— (1)}$$



$$= \int_{x=0}^{a} (y)_0^{b/a\sqrt{a^2-x^2}}\, dx$$

$$= \int_{x=0}^{a} \frac{b}{a}\sqrt{a^2-x^2}\, dx = \frac{b}{a}\left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1} x/a\right]_{x=0}^{a} \quad \text{— (2)}$$

$$= \frac{b}{a}\cdot\frac{a^2}{2}\cdot\pi/2 = \frac{\pi ab}{4}$$

$\therefore$ Area of I Quadrant $= \frac{\pi ab}{4}$ Sq. units — (1)

b) Temp of air $= 40^\circ C = t_2$.

Initial temperature, $t_1 = 80^\circ C$.

Given $T = 60^\circ C$ when $t = 20$ mins.

To find :- $T = ?$ when $t = 40$ mins.

By Newton's law of Cooling,

$$T = t_2 + (t_1 - t_2)e^{-kt}.$$

②

Since $(T)_{t=20\,mins} = 60^\circ$,

$$60 = 40 + (80-40)\, e^{-k(20)}.$$

$$\Rightarrow e^{-20k} = \frac{1}{2}$$

$$\Rightarrow k = \frac{-1}{20} \ln(0.5) = 0.03466. \text{ — (2)}$$

To find :- T when $t = 40$ mins.

By Newton's Law of Cooling,

$$T = t_2 + (t_1 - t_2)\, e^{-kt}$$

$$= 40 + (80-40)\, e^{-(0.03466)(40)}$$

$$= 49.99$$

$$\simeq 50^\circ C. \text{ — (2)}$$

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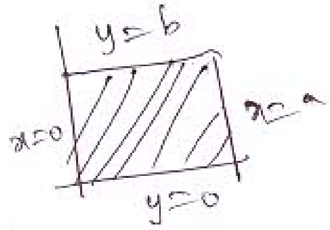
2) Given density is $\frac{\sigma}{a}(x+y)$

Shape of Lamina is shaded region



$$\bar{x} = \frac{\iint_A x \rho \, dx\, dy}{M_A} \qquad \bar{y} = \frac{\iint_A y \rho \, dx\, dy}{M_A}$$

$$M_A = \iint \rho\, dx\,dy = \int_{x=0}^{a}\int_{y=0}^{b} \frac{\sigma}{a}(x+y)\,dy\,dx$$

$$M_A = \frac{\sigma}{a}\int_{x=0}^{a}\left(xy+y^2/2\right)_0^b dx = \frac{\sigma}{a}\int_{x=0}^{a}\left(bx+b^2/2\right)dx$$

$$M_A = \frac{\sigma}{2a}\left(ba^2+b^2a\right) = \frac{\sigma ab}{2a}(a+b)$$

$$\boxed{M_A = \frac{\sigma ab}{2a}(a+b)} \quad \text{— (2)}$$

Nr of $\bar{x}$ $= \iint_A x\rho\,dx\,dy = \int_{x=0}^{a}\int_{y=0}^{b} x\frac{\sigma}{2a}(x+y)\,dy\,dx$

$$= \frac{\sigma}{a}\int_{x=0}^{a}\left(x^2y+xy^2/2\right)_0^b dx = \frac{\sigma}{a}\left(\frac{bx^2}{2}+\frac{b^2x}{2}\right)_{x=0}^{a}$$

$$= \frac{\sigma}{a}\left(\frac{bx^3}{3}+\frac{b^2x^2}{4}\right)_0^a = \frac{\sigma a^2 b}{a}\left(\frac{a}{3}+\frac{b}{4}\right)$$

Nr. of $\bar{x} = \frac{\sigma ab}{12}(4a+3b)$

$$\therefore \bar{x} = \frac{\frac{\sigma ab}{12}(4a+3b)}{\frac{\sigma b}{2}(a+b)} = \frac{a(4a+3b)}{6(a+b)} \quad \text{— (2)}$$

Similarly $\bar{y} = \frac{b(3a+4b)}{6(a+b)}$ $\quad\therefore (\bar{x},\bar{y}) = \left(\frac{a(4a+3b)}{6(a+b)}, \frac{b(3a+4b)}{6(a+b)}\right)$

(2) (1)

3) $\beta(m,n) = \dfrac{\Gamma m \, \Gamma n}{\Gamma m+n}$

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta \quad \text{— (1)}$$

$$\Gamma n = 2\int_0^{\infty} e^{-x^2} x^{2n-1} dx \;\&\; \Gamma m = 2\int_0^{\infty} e^{-y^2} y^{2m-1} dy \quad \text{— (2)}$$

$$\Gamma m+n = 2\int_0^{\infty} e^{-r^2} r^{2(m+n)-1} dr. \quad \text{— (3)}$$

— (2)

$$\Gamma m . \Gamma n = 4\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} x^{2n-1} y^{2m-1} dx dy.$$

Substitute $x = r\cos\theta$ $y = r\sin\theta \Rightarrow x^2+y^2 = r^2$

r varies from 0 to ∞ & θ from 0 to $\pi/2$

— (1)

$$\therefore \Gamma m . \Gamma n = 4\int_0^{\infty}\int_0^{\pi/2} e^{-r^2} (r\cos\theta)^{2n-1} (r\sin\theta)^{2m-1} r \, dr d\theta$$

$$= 2\int_0^{\infty} e^{-r^2} r^{2(m+n)-1} dr . \; 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta.$$

— (1)

$\therefore \Gamma m \, \Gamma n = \Gamma m+n \;\; \beta(m,n)$ from (1) & (3) — (1)

Hence $\beta(m,n) = \dfrac{\Gamma m \, \Gamma n}{\Gamma m+n.}$

To P.T $\Gamma\frac{1}{2} = \sqrt{\pi}$

$$\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n} \quad \therefore \beta(\tfrac{1}{2},\tfrac{1}{2}) = \frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma \frac{1}{2}+\frac{1}{2}} = \frac{(\Gamma\frac{1}{2})^2}{1}$$

$$\beta(\tfrac{1}{2},\tfrac{1}{2}) = (\Gamma\tfrac{1}{2})^2$$

Also by using formula

$$\beta(\tfrac{1}{2},\tfrac{1}{2}) = 2\int_0^{\pi/2} \sin^{2\times\frac{1}{2}-1}\theta \cos^{2\times\frac{1}{2}-1}\theta\, d\theta$$

$$(\Gamma\tfrac{1}{2})^2 = 2\int_0^{\pi/2} d\theta = 2\times\frac{\pi}{2} = \pi$$

(2)

$$\therefore (\Gamma\tfrac{1}{2})^2 = \pi \Rightarrow \boxed{\Gamma\tfrac{1}{2} = \sqrt{\pi}}$$

4. $\int_0^\infty \sqrt{y}\, e^{-y^2} dy \quad \int_0^\infty \frac{e^{-y^2}}{\sqrt{y}} dy = \frac{\pi}{2\sqrt{2}}$

Let $I_1 = \int_0^\infty \sqrt{y}\, e^{-y^2} dy = \int_0^\infty e^{-y^2} y^{+1/2} dy$.

W.K.T $\boxed{\Gamma n = 2\int_0^\infty e^{-x^2} x^{2n-1} dx}$; here $2n-1 = +1/2$

$\Rightarrow n = 3/4$

$\therefore \cancel{I_1} \cancel{\int} \Rightarrow I_1 = \frac{1}{2}\Gamma\frac{3}{4}$ — (3)

Let $I_2 = \int_0^\infty \frac{e^{-y^2}}{\sqrt{y}} dy = \int_0^\infty e^{-y^2} \cdot y^{-1/2} dy$.

Comparing with Γn formula $2n-1 = -1/2$

$\Rightarrow n = 1/4$

$\therefore I_2 = \frac{1}{2}\Gamma\frac{1}{4}$ — (3)

$\therefore I_1 \cdot I_2 = \frac{1}{4}\Gamma\frac{3}{4}\Gamma\frac{1}{4}$

$= \frac{1}{4} \cdot \pi\sqrt{2}$

$\therefore I_1 I_2 = \frac{\pi}{2\sqrt{2}}$ Hence proved.

— (1)

5. $(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0.$ — (1)

$M(x,y) = 3x^2y^4 + 2xy = xy(3xy^3+2)$

$N(x,y) = 2x^3y^3 - x^2 = x^2(2xy^3-1)$

$$\frac{\partial M}{\partial y} = 12x^2y^3 + 2x$$

$$\frac{\partial N}{\partial x} = 6x^2y^3 - 2x$$

Eqn. (1) is not exact; as $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ — (2)

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 6x^2y^3 + 4x$$

$= 2x(3xy^3+2)$ .... close to M — (1)

$$\therefore \frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{xy(3xy^3+2)} \times 2x(3xy^3+2)$$

$$= \frac{2}{y} = g(y).$$

$$I.F = e^{-\int g(y)\,dy} = e^{-2\int \frac{1}{y}\,dy} = e^{-2\log y}$$

$$= e^{\log y^{-2}} = \frac{1}{y^2}$$ — (1)

Multiplying eqn (1) with $\frac{1}{y^2}$,

$\left(3x^2y^2 + \frac{2x}{y}\right)dx + \left(2x^3y - \frac{x^2}{y^2}\right)dy = 0.$ — (2)

— (1)

Now, $M(x,y) = 3x^2y^2 + \frac{2x}{y}$

$N(x,y) = 2x^3y - \frac{x^2}{y^2}$

$$\frac{\partial M}{\partial y} = 6x^2y - \frac{2x}{y^2}$$

$$\frac{\partial N}{\partial x} = 6x^2y - \frac{2x}{y^2}.$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

$\Rightarrow$ Eqn (2) is Exact.

The solution is,

$$\int M\,dx + \int N(y)\,dy = c$$

$$\Rightarrow \int \left(3x^2y^2 + \frac{2x}{y}\right) dx + \int 0.\,dy = c$$

$$\Rightarrow x^3y^2 + \frac{x^2}{y} = c. \quad \text{— (2)}$$

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6. $(r\sin\theta - r^2)\,d\theta - (\cos\theta)\,dr = 0.$

$$\Rightarrow (r\sin\theta - r^2)\,d\theta = \cos\theta\,dr$$

$$\Rightarrow \frac{dr}{d\theta} = \frac{r\sin\theta - r^2}{\cos\theta} = r\tan\theta - r^2\sec\theta.$$

$$\Rightarrow \frac{dr}{d\theta} - r\tan\theta = -r^2\sec\theta \quad \text{— (1).}$$

Eqn. (1) is Bernoulli equation. — (1)

Eqn. (1) $\div r^2$,

$$\frac{1}{r^2}\frac{dr}{d\theta} - \frac{1}{r}\tan\theta = -\sec\theta. \quad \text{— (2)}$$

— (1)

$\therefore$ Let $\frac{1}{r} = v \Rightarrow -\frac{1}{r^2}\frac{dr}{d\theta} = \frac{dv}{d\theta}$ — (1)

$$\text{Eqn (2)} \Rightarrow -\frac{dv}{d\theta} - v\tan\theta = -\sec\theta$$

$$\Rightarrow \frac{dv}{d\theta} + v\tan\theta = \sec\theta, \text{ linear in } v. \text{ — (1)}$$

$$I.F = e^{\int \tan\theta \, d\theta} = e^{\log(\sec\theta)} = \sec\theta \text{ — (1)}$$

The solution is,

$$v\sec\theta = \int \sec^2\theta \, d\theta + c$$

$$\Rightarrow \frac{\sec\theta}{r} = \tan\theta + c \text{ — (2).}$$

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7. We have, $\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$ — (1)

Differentiating (1) w.r.t x,

$$\frac{2x}{a^2} + \frac{2yy_1}{b^2+\lambda} = 0.$$

$$\Rightarrow \frac{yy_1}{b^2+\lambda} = -\frac{x}{a^2}$$

$$\Rightarrow b^2+\lambda = \frac{-a^2yy_1}{x} \text{ — (2) — (1)}$$

From (1), $b^2+\lambda = \frac{-a^2y^2}{x^2-a^2}$ — (3) — (1)

Equating (2) and (3),

$$\frac{\cancel{+}\cancel{a^2}\cancel{y}y_1}{x} = \frac{\cancel{+}\cancel{a^2}y\cancel{^2}}{x^2-a^2}$$

$$\Rightarrow \frac{y_1}{x} = \frac{y}{x^2-a^2}$$

$$\Rightarrow y_1(x^2-a^2) - xy = 0 \quad \text{— (1)}$$

Replacing y_1 by $-\frac{1}{y_1}$,

$$-\frac{1}{y_1}(x^2-a^2) - xy = 0 \quad \text{— (1)}$$

$$-\frac{dx}{dy}(x^2-a^2) = xy$$

$$\Rightarrow \frac{(a^2-x^2)}{x}dx = y\,dy$$

$$\Rightarrow \left(\frac{a^2}{x} - x\right)dx = y\,dy \quad \text{— (1)}$$

Integrating both sides,

$$a^2\log x - \frac{x^2}{2} = \frac{y^2}{2} + c$$

$$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + c - a^2\log x = 0$$

$\Rightarrow x^2+y^2-2a^2\log x + k = 0$, where $k = 2c$;

is the required Orthogonal trajectory.

— (2)

8. $xp^3 - yp^2 + 1 = 0$.

$\Rightarrow yp^2 = p^3x + 1$

$\Rightarrow y = px + \dfrac{1}{p^2}$, Clairaut's equation. (1)

$\therefore$ The general solution is,

$$y = cx + \frac{1}{c^2} \quad \text{—— (1)} \quad \text{—— (1)}$$

Partially differentiating (1) wrt the parameter 'c', we get $0 = x - \dfrac{2}{c^3}$ —— (1)

$$\Rightarrow \frac{2}{c^3} = x \Rightarrow \frac{c^3}{2} = \frac{1}{x}$$

$$\Rightarrow c^3 = \frac{2}{x}$$

$$\Rightarrow c = \left(\frac{2}{x}\right)^{1/3} \text{—— (1)}$$

Eqn. (1) $\Rightarrow y = \left(\dfrac{2}{x}\right)^{1/3} x + \dfrac{1}{(2/x)^{2/3}}$

$$= 2^{1/3} \cdot x^{2/3} + x^{2/3} \times 2^{-2/3}$$

$$2^{2/3} y = 2x^{2/3} + x^{2/3}$$

$$\Rightarrow 2^{2/3} y = 3x^{2/3}$$

$\Rightarrow 4y^3 = 27x^2$ is the required singular solution.

(3).