CMR
INSTITUTE OF
TECHNOLOGY

USN

Internal Assesment Test - II

Sub:	Microwave and Antennas							Code:	15EC71
Date:	15/ 10 / 2018	Duration:	90 mins	Max Marks:	50	Sem:	7th	Branch:	ECE (A,B,C,D)
Answer Any FIVE FULL Questions									

	Marks	OBE CO	OBE RBT
1. Explain the following terms as related to antenna system:	[10]	CO4	L2

(i) HPBW

The angular beam width at the half power level or Half-Power-Beam Width (HPBW) (or 3dB beamwidth)

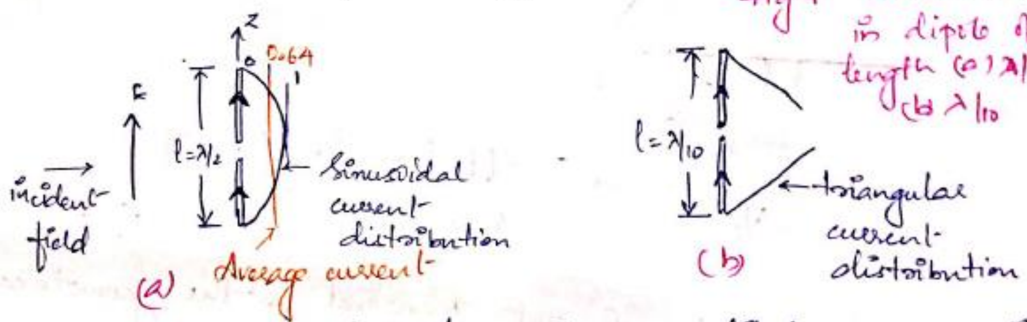
(ii) Effective length

Effective height may be defined as the ratio of the induced voltage to the incident field i.e.

$$h_e = \frac{V}{E} \quad (m) \qquad (1.39)$$

Consider a vertical dipole of length $l = \lambda/2$ immersed in an incident field E (fig 7)

Fig 7 Current dbn in dipole of length (a) $\lambda/2$ (b) $\lambda/10$



If the current distribution of the dipole were uniform its effective height would be l.

But the actual current distribution is nearly sinusoidal with an average value $\frac{2}{\pi} = 0.64$ of the maximum.

∴ effective ~~length~~ height $h = 0.64 l$.

(iii) Beam efficiency

1.6 Beam Efficiency

The (total) beam area Ω_A (or beam solid angle) consists of the main beam area (Ω_M) plus the minor-lobe area (or solid angle) Ω_m.

Thus, $\Omega_A = \Omega_M + \Omega_m$ ——— (1.21)

The ratio of main beam area to the total beam area is called (main) beam efficiency, ε_M.

Thus, Beam efficiency, $\varepsilon_M = \frac{\Omega_M}{\Omega_A}$ (dimensionless) ——— (1.22)

The ratio of minor-lobe area (Ω_m) to the total beam area is called stray factor.

Thus, stray factor, $\varepsilon_M = \frac{\Omega_m}{\Omega_A}$ ——— (1.23)

It follows that:

$$\varepsilon_M + \varepsilon_m = 1 \quad \text{——— (1.24)}$$

(iv) Gain

Gain of antenna is the ratio of maximum radiation intensity in given direction to the maximum radiation intensity from a reference antenna produced in the same direction with same power input.

i.e. $G = \frac{\text{Max}^m \text{ Rad}^n \text{ intensity from test antenna}}{(\text{Max}^m \text{ rad}^n \text{ intensity from reference antenna with same power input})}$

$= \frac{P_{max}(\text{Ant. under test})}{P_{max}(\text{ref. ant.}) \text{ (with same power i/p)}}$ ——— 1.30 (a)

$= \frac{P_{max}(\text{AUT})}{P_{max}(\text{ref. ant.})} \times G(\text{ref. ant})$ ——— 1.30 (b)

(v) Isotropic radiator

An isotropic radiator is a theoretical point source of electromagnetic or sound waves which radiates the same intensity of radiation in all directions. It has no preferred direction of radiation. It radiates uniformly in all directions over a sphere centred on the source.

2. (a)	Define directivity and extend the result in terms of beam area.	[5]	CO4	L3

Directivity of an antenna = ratio of maximum power density $P(\theta, \phi)_{max}$ (watts/m²) to its average value over a sphere as observed in far field of an antenna.

Thus, $D = \dfrac{P(\theta,\phi)_{max}}{P(\theta,\phi)_{av}}$ (Directivity from pattern) —— (1.25)

The average power density over a sphere is given by:

$$P(\theta,\phi)_{av} = \frac{1}{4\pi}\int_{\phi=0}^{\phi=2\pi}\int_{\theta=0}^{\theta=\pi} P(\theta,\phi)\sin\theta\, d\theta\, d\phi$$

$$= \frac{1}{4\pi}\iint_{4\pi} P(\theta,\phi)\, d\Omega \quad (W/Sr) \qquad \text{—— (1.26)}$$

$$\therefore \text{Directivity}, D = \frac{P(\theta,\phi)_{max}}{\frac{1}{4\pi}\iint_{4\pi} P(\theta,\phi)\, d\Omega} = \frac{1}{\frac{1}{4\pi}\iint_{4\pi}\left[P(\theta,\phi)/P(\theta,\phi)_{max}\right] d\Omega} \qquad \text{—— (1.27)}$$

and

$$D = \frac{4\pi}{\iint_{4\pi} P_n(\theta,\phi)\, d\Omega} = \frac{4\pi}{\Omega_A} \quad \text{(Directivity from beam area } \Omega_A\text{)} \qquad \text{—— (1.28)}$$

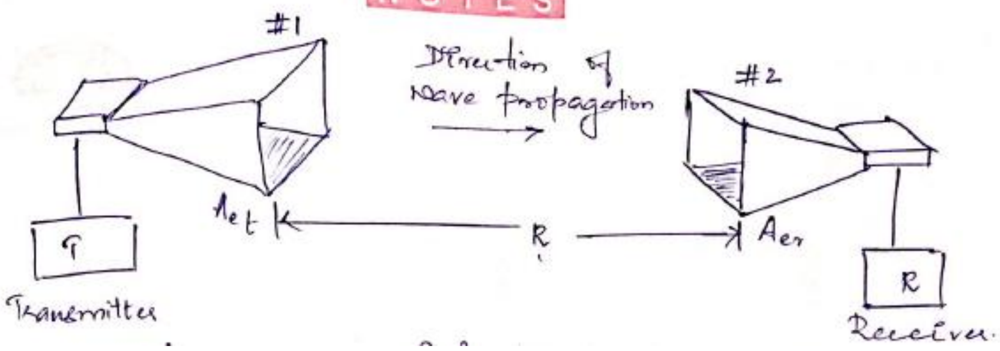
↳ Normalized power pattern

Thus, the directivity is the ratio of the area of a sphere (4π Sr) to the beam area Ω_A of the antenna.

Smaller the beam area, larger the directivity D.

(b) State and prove Friis transmission formula. [5] CO4 L2

The Friis transmission formula relates the power received to the power transmitted btwn two antennas separated by a distance $\boxed{R > 2L^2/\lambda}$, where L is the ~~distance~~ (largest) dimension of either antenna. The formula gives the power received over a radio communication link.



$$\frac{P_r}{P_t} = \left(\frac{\lambda}{4\pi R}\right)^2 G_{0t}\, G_{0r} \quad \text{————} (1.52)$$

Eqns (1.50), (1.51) & (1.52) are known as Friis transmission Eqn.

3. (a) The effective aperture of transmitting and receiving antennas in a communication system are 8 λ^2 and 12 λ^2 respectively with a separation of 1.5 km between them. The E.M wave is travelling with a frequency of 6MHz and the total input power is 25KW. Find the power received by the receiving antenna.	[5]	CO4	L3

$A_{et} = 8\lambda^2$

$A_{er} = 12\lambda^2$

$R = 1.5\,km$

$P_t = 25\,kW$; $f = 6\,MHz$

$$P_r = P_t \cdot \frac{A_{et} \cdot A_{er}}{r^2 \lambda^2} \qquad \lambda = \frac{3\times10^8}{6\times10^6}$$

$$\lambda = 50\,m$$

$$= \frac{25\times10^3 \times 8\lambda^2 \times 12\lambda^2}{(1.5\times10^3)^2 \cdot \lambda^2}$$

$$= 1.066\lambda^2$$

$$P_r = 1.066\,(50)^2 = 2.667\,kW$$

(b) Define the following with respect to antenna: (i) Radiation pattern (power and field pattern)	[5]	CO4	L1

* In other words, the graphical representation of radiation of an antenna as a function of direction is given the name radiation pattern of the antenna.

- If the radiation from antenna is expressed in terms of field strength E (V/m), the radiation pattern is called as 'field strength Pattern'

- If radiation in a given direction is expressed in terms of power per unit solid angle, then the resulting pattern is 'Power pattern'.

* Power pattern $\propto$ (field strength pattern)^2

(ii) Effective Aperture

1.8 Antenna apertures

Consider a receiving antenna to be a rectangular e.m horn immersed in the field of a uniform plane wave as in fig(6)

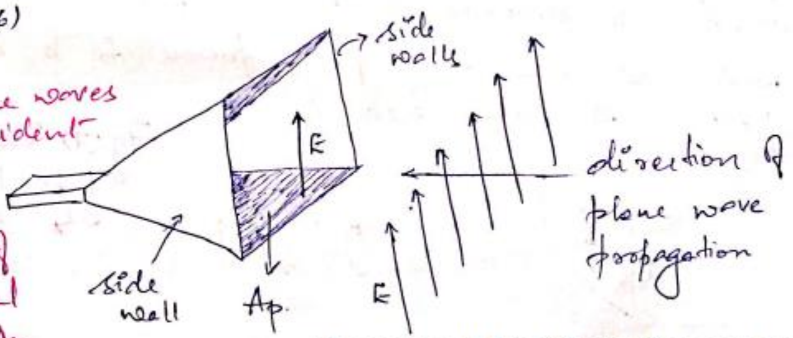


Fig (6). Plane waves incident on e.m horn of physical aperture, Ap

Scanned by CamScanner

Let the Poynting vector (or power density) of the plane wave be S watts per sq-meter and the area (physical aperture) of the horn be Ap m^2.

If horn extracts all the power from the wave over its entire physical aperture, then the total power (P) absorbed from the wave is.

$$P = \frac{E^2}{Z} A_p = S A_p. \qquad (1.32)$$

Thus the horn is regarded as having an aperture the total power it extracts from a passing wave being proportional to the aperture or area of its mouth.

The field response of the horn is not uniform across the aperture Ap since $\vec{E}$ at the sidewalls should be ideally zero ($\uparrow\uparrow\uparrow\uparrow$).

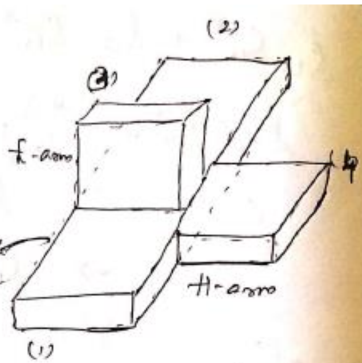
Thus effective aperture (Ae) of the horn is < physical aperture (Ap) &

$$\therefore \text{Aperture efficiency, } \varepsilon_{ap} = \frac{A_e}{A_p} \qquad (1.33)$$

(dimensionless)

5.	With a neat sketch, explain the operation of a magic tee. Derive its S matrix representation. Also mention its application.	[10]	CO2	L3

Hybrid.
Magic Tee



from previous chapter,
we have:
~~the S-matrix for~~

$$[S]_{EH} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{12} & S_{22} & S_{23} & S_{24} \\ S_{13} & S_{23} & S_{33} & S_{34} \\ S_{14} & S_{24} & S_{34} & S_{44} \end{bmatrix} \Rightarrow \text{with symmetry ppty.}$$

Due to H-arm $\rightarrow S_{14} = S_{24}$ ——— (1.a)

Due to E-arm $\rightarrow S_{23} = -S_{13}$ ——— (1.b)

Due to geometry of junction, signal fed to port 4 cannot induce dominant mode in port 3 & vice-versa.

$\therefore$ port (3) & (4) $\rightarrow$ isolated ports.

i.e. $S_{34} = S_{43} = 0$. ——— (1.c)

Assume port 3 & 4 are matched to the junction. $\Rightarrow \therefore S_{33} = 0$ & $S_{44} = 0$. ——— (1.d)

Now, $$[S]_{EH} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{12} & S_{22} & -S_{13} & S_{14} \\ S_{13} & -S_{13} & 0 & 0 \\ S_{14} & S_{14} & 0 & 0 \end{bmatrix}$$

Now from unitary ppty:

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{12} & S_{22} & -S_{13} & S_{14} \\ S_{13} & -S_{13} & 0 & 0 \\ S_{14} & S_{14} & 0 & 0 \end{bmatrix} \begin{bmatrix} S_{11}^* & S_{12}^* & S_{13}^* & S_{14}^* \\ S_{12}^* & S_{22}^* & -S_{13}^* & S_{14}^* \\ S_{13}^* & -S_{13}^* & 0 & 0 \\ S_{14}^* & S_{14}^* & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ for $i=1$ $\left\{ \sum_{k=1}^{4} S_{ki} S_{kj} = 1 \ \forall i=j \right\}$.

$$|S_{11}|^2 + |S_{12}|^2 + |S_{13}|^2 + |S_{14}|^2 = 1. \quad \text{——— } (2a)$$

for $i=2$

$$|S_{12}|^2 + |S_{22}|^2 + |S_{13}'|^2 + |S_{14}|^2 = 1 \quad \text{——— } (2b)$$

for $i=3$

$$|S_{13}|^2 + |S_{13}|^2 + 0 + 0 = 1. \quad \text{——— } (2\text{-}c)$$

for $i=4$

$$|S_{14}|^2 + |S_{14}|^2 + 0 + 0 = 1 \quad \text{——— } (2.d)$$

from $(2\cdot c) \Rightarrow \therefore S_{13} = \frac{1}{\sqrt{2}}$. ——— $(3.a)$

from $(2.d) \Rightarrow S_{14} = \frac{1}{\sqrt{2}}$. ——— $(3.b)$

Comparing eqns $(2a)$ & $(2.b)$ we get:-

$$S_{11} = S_{22} \quad \text{——— } (3.c)$$

Substituting $(3.a)$ and $(3.b)$ in $(2(a))$; we get:

$$|S_{11}|^2 + |S_{12}|^2 = 0 \rightarrow \text{This eqn is valid if}$$

$$S_{11} = S_{12} = 0. \quad \text{——— } (4.a)$$

$\hookrightarrow$ only one soln:

$\therefore$ from eqn. $(3.c)$ $S_{22} = S_{11} = 0.$ ——— $(4.b)$

$$\therefore [S] = \begin{bmatrix} 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \end{bmatrix} \quad \text{——— } (5)$$

6.	State the properties of S-parameters. Starting from the impedance matrix equation, prove the symmetry property of a reciprocal network. Also prove the unitary property of S parameter.	[10]	CO2	L3

Ppty 2: Symmetry of [S] for a reciprocal n/w

A reciprocal device has the same transmission characteristics in either direction of a pair of ports & is thus characterized by a symmetric scattering matrix:

$$S_{ij} = S_{ji} \quad (i \neq j)$$

i.e. $[S]_t = [S]$

Proof:

for a reciprocal n/w, the impedance matrix eqn is : $[V] = [Z][I]$ ——— (1)

Now, steady state voltage & current at i^{th} port :

$$V_i = V_i^+ + V_i^- \quad \text{———} \quad (2.a)$$

$$\text{and } I_i = \frac{V_i^+}{Z_{oi}} - \frac{V_i^-}{Z_{oi}} \quad \text{———} \quad (2.b)$$

from (2.a) & (2.b) :

$$V_i^+ = \frac{1}{2}(V_i + Z_{oi} I_i) \quad \text{———} \quad (3.a)$$

$$\text{and } V_i^- = \frac{1}{2}(V_i - Z_{oi} I_i) \quad \text{———} \quad (3.b)$$

Avg. incident Power at i^{th} port :

$$P_{avg_i} = \frac{1}{2} V_i I_i^* = \frac{|V_i^+|^2}{2 Z_{oi}^*} \quad \text{———} \quad (4)$$

(complex power)

In this normalization process, characteristic impedance is normalized to unity.

$$\text{Now, } a_i = \frac{V_i^+}{\sqrt{Z_{oi}}} = \frac{1}{2}\left(\frac{V_i}{\sqrt{Z_{oi}}} + \sqrt{Z_{oi}}\, I_i\right) \Rightarrow \text{from (3.a)} \quad \text{———} \quad (9)$$

$$\& \; b_i = \frac{V_i^-}{\sqrt{Z_{oi}}} = \frac{1}{2}\left(\frac{V_i}{\sqrt{Z_{oi}}} - \sqrt{Z_{oi}}\, I_i\right) \Rightarrow \text{from}$$

If $\sqrt{Z_{oi}} = 1$ (normalized to 1), then

$$a_i = \frac{1}{2}(V_i + I_i) \quad \text{———} \quad (10.a)$$

$$\& \; b_i = \frac{1}{2}(V_i - I_i) \quad \text{———} \quad (10.b)$$

$$\text{or } \; V_i = a_i + b_i \quad \text{———} \quad (11.a)$$

$$\text{and } I_i = a_i - b_i \quad \text{———} \quad (11.b)$$

$\therefore$ eqn (1) $\Rightarrow$ $[V] = [Z]([a]-[b])$

$$\Rightarrow [a]+[b] = [Z]([a]-[b])$$

$$\Rightarrow ([Z]+[U])[b] = ([Z]-[U])[a] \quad \text{—— (12)}$$

$\{[U] = \text{unit matrix}\}$

S-matrix eqn is:

$$[b] = [S][a]$$

comparing (12) with this:

$$[S] = \frac{[b]}{[a]} = \frac{[Z]-[U]}{[Z]+[U]} = \frac{[R]}{[Q]} \text{ (say)} \quad \text{—— (13)}$$

Now since Z-matrix is symmetric,

we have: $([Z]-[U])_t = [Z]-[U]$

$\&$ $([Z]+[U])_t = [Z]+[U]$.

$\therefore$ eqn (15) $\Rightarrow$ $[S]_t = ([Z]-[U])([Z]+[U])^{-1}$

$$= [R][Q]^{-1} = [S] \quad \text{—— (16)}$$

$\{$from (13)$\}$

Thus $[S]_t = [S]$ for a symmetrical junction

Ppty 3. Unitary ppty for a lossless junction

statement: for any lossless network, the sum of the products of each term of any one row or of any column of the S-matrix multiplied by its complex conjugate is unity.

for a lossless n-port device, total power leaving N-ports must be equal to $\rightarrow$ the total power input to these ports, so that

$$\sum_{n=1}^{N} |b_n|^2 = \sum_{n=1}^{N} |a_n|^2 \quad \text{—— (17-a)}$$

or $\sum_{n=1}^{N} \left| \sum_{i=1}^{n} S_{ni}\, a_i \right|^2 = \sum_{n=1}^{N} |a_n|^2$ —— (17-b)

If only ith port is excited & all other ports are matched (i.e. $a_n = 0$ except a_i) so that

$$\sum_{n=1}^{N} |S_{ni}\, a_i|^2 = |a_i|^2 \qquad (18)$$

$$\sum_{n=1}^{N} |S_{ni}|^2 = \sum_{n=1}^{N} S_{ni}\, S_{ni}^* = 1 \qquad (19)$$

∴ for a lossless junction:

$$\sum_{n=1}^{N} S_{ni} \cdot S_{ni}^* = 1 \Rightarrow \text{unity property.} \qquad (20a)$$

If all $a_n = 0$, except a_i & a_k

then $$\sum_{n=1}^{N} S_{nk} \cdot S_{ni}^* = 0 \;;\; i \neq k. \qquad (20.b)$$

7. (a) The S-Parameters of a 2 port network are given by $S_{11}=0.2 \angle 0°$, $S_{22}=0.1 \angle 0°$, $S_{12}=0.6 \angle 90°$ and $S_{21}= 0.6 \angle 90°$. Is the network reciprocal and lossless? [4] CO2 L3

$$[S] = \begin{bmatrix} 0.2\angle 0° & 0.6\angle 90° \\ 0.6\angle 90° & 0.1\angle 0° \end{bmatrix}$$

Since $[S]$ is symmetric, the network is reciprocal. For lossless network, unitary property must be satisfied:

$$S_{1i}\cdot S_{1i}^* + S_{2i}\cdot S_{2i}^* = 1 = |S_{11}|^2 - |S_{21}|^2$$

$$S_{1k}\cdot S_{1i}^* + S_{2k}\cdot S_{2i}^* = 0$$

$$[S^*]\,[S]_T = [U]$$

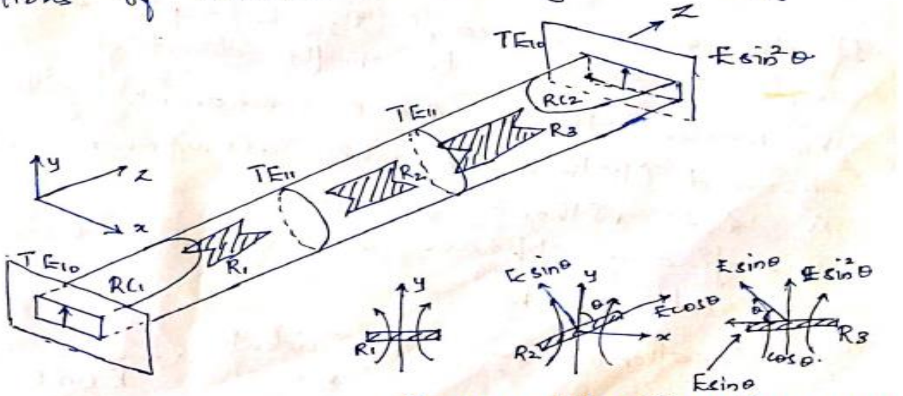
Here,

$$S_{21} = S_{12} \text{ leads to } |S_{11}|^2 + |S_{12}|^2 = (0.2)^2 + (0.6)^2 = 0.04 + .36 = 0.40 < 1$$

Therefore, the network is not lossless.

(b) Explain with a neat sketch a precision type variable attenuator. Also derive the S-matrix. [6] CO2 L2

Precision type variable attenuator uses a circular wave guide section (c) containing a very thin tapered resistive card (R_2). Both sides of this are connected to axisymmetric sections of circular to rectangular waveguide



tapered transitions (RC1 and RC2)

- Centre circular section can be rotated by 360° w.r.t fixed sections of circular to rectangular W-G transitions.
- Induced current on the resistive card R_2 due to incident signal is dissipated as heat producing attenuation of txtd s/g.
- Incident TE_{10} dominant wave in rect WG is converted into dominant TE_{11} mode in circular WG
- R1 and R3 are placed ⊥r to elec fld of TE_{10} mode so that it doesn't absorb the energy. But any component parallel to this plane will be readily absorbed.

∴ pure TE_{11} mode is excited in circular WG section.

- If the resistive card R_2 is kept at an angle θ relative to E-fld direction of TE_{11} mode, then $E\cos\theta$ component gets absorbed (‖r to card) while component $E\sin\theta$ is txd without attenuation.

This finally appears as component $E\sin^2\theta$ in rect. O/p WG.

∴ attenuation of incident wave: $\alpha = \frac{E}{E\sin^2\theta} = \frac{1}{\sin^2\theta}$

∴ S matrix of an ideal precision rotary attenuator is :-

$$[S] = \begin{bmatrix} 0 & \sin^2\theta \\ \sin^2\theta & 0 \end{bmatrix}.$$

8.	Explain various losses in Microstrip lines.	[10]	CO3	L2

The characteristic impedance and wave-propagation velocity of a microstrip line was analyzed in Section 11-1-1. The other characteristic of the microstrip line is its attenuation. The attenuation constant of the dominant microstrip mode depends on geometric factors, electrical properties of the substrate and conductors, and on the frequency. For a nonmagnetic dielectric substrate, two types of losses occur in the dominant microstrip mode: (1) dielectric loss in the substrate and (2) ohmic skin loss in the strip conductor and the ground plane. The sum of these two losses may be expressed as losses per unit length in terms of an attenuation factor α. From ordinary transmission-line theory, the power carried by a wave traveling in the positive z direction is given by

$$P = \tfrac{1}{2} VI^* = \tfrac{1}{2}(V_+ e^{-\alpha z} I_+ e^{-\alpha z}) = \tfrac{1}{2}\frac{|V_+|^2}{Z_0} e^{-2\alpha z} = P_0 e^{-2\alpha z} \qquad (11\text{-}1\text{-}10)$$

where $P_0 = |V_+|^2/(2Z_0)$ is the power at $z = 0$.

The attenuation constant α can be expressed as

$$\alpha = -\frac{dP/dz}{2P(z)} = \alpha_d + \alpha_c \qquad (11\text{-}1\text{-}11)$$

where α_d is the dielectric attenuation constant and α_c is the ohmic attenuation constant.

The gradient of power in the z direction in Eq. (11-1-11) can be further expressed in terms of the power loss per unit length dissipated by the resistance and the power loss per unit length in the dielectric. That is,

$$\begin{aligned} -\frac{dP(z)}{dz} &= -\frac{d}{dz}(\tfrac{1}{2}VI^*) \\ &= \tfrac{1}{2}\left(-\frac{dV}{dz}\right)I^* + \tfrac{1}{2}\left(-\frac{dI^*}{dz}\right)V \\ &= \tfrac{1}{2}(RI)I^* + \tfrac{1}{2}\sigma V^* V \\ &= \tfrac{1}{2}|I|^2 R + \tfrac{1}{2}|V|^2\sigma = P_c + P_d \end{aligned} \qquad (11\text{-}1\text{-}12)$$

where σ is the conductivity of the dielectric substrate board.

Substitution of Eq. (11-1-12) into Eq. (11-1-11) results in

$$\alpha_d \simeq \frac{P_d}{2P(z)} \quad \text{Np/cm} \qquad (11\text{-}1\text{-}13)$$

and

$$\alpha_c \simeq \frac{P_c}{2P(z)} \quad \text{Np/cm} \qquad (11\text{-}1\text{-}14)$$

Ohmic losses. In a microstrip line over a low-loss dielectric substrate, the predominant sources of losses at microwave frequencies are the nonperfect conductors. The current density in the conductors of a microstrip line is concentrated in a sheet that is approximately a skin depth thick inside the conductor surface and exposed to the electric field. Both the strip conductor thickness and the ground plane thickness are assumed to be at least three or four skin depths thick. The current density in the strip conductor and the ground conductor is not uniform in the transverse plane. The microstrip conductor contributes the major part of the ohmic loss. A diagram of the current density J for a microstrip line is shown in Fig. 11-1-7.

Because of mathematical complexity, exact expressions for the current density of a microstrip line with nonzero thickness have never been derived [10]. Several researchers [8] have assumed, for simplicity, that the current distribution is uniform and equal to I/w in both conductors and confined to the region $|x| < w/2$. With this assumption, the conducting attenuation constant of a wide microstrip line is given by

$$\alpha_c \simeq \frac{8.686 R_s}{Z_0 w} \quad \text{dB/cm} \quad \text{for } \frac{w}{h} > 1 \qquad (11\text{-}1\text{-}21)$$

Radiation losses. In addition to the conductor and dielectric losses, microstrip line also has radiation losses. The radiation loss depends on the substrate's thickness and dielectric constant, as well as its geometry. Lewin [12] has calculated the radiation loss for several discontinuities using the following approximations:

1. TEM transmission
2. Uniform dielectric in the neighborhood of the strip, equal in magnitude to an effective value
3. Neglect of radiation from the transverse electric (TE) field component parallel to the strip
4. Substrate thickness much less than the free-space wavelength

4.	Calculate the exact directivity and approximate directivity for the following sources having the following power patterns: (i) $U = U_m \sin^2\theta \sin^2\varphi$ (ii) $U = U_m \sin\theta \sin^3\varphi$ (iii) $U = U_m \sin^2\theta \sin^3\varphi$ Where U has a value only for $0 \le \theta \le \pi$ and $0 \le \varphi \le \pi$ and zero elsewhere.	[10]	CO4	L3

(ii) $U = U_m \cdot \sin\theta \sin^3\phi \quad , \quad 0 \le \theta \le \pi, \; 0 \le \phi \le \pi$

$$P_{rad} = \int_0^{\pi}\int_0^{\pi} (U_m \cdot \sin\theta \cdot \sin^3\phi) \cdot \sin\theta \cdot d\theta \cdot d\phi$$

$$= U_m \int_0^{\pi} \sin^2\theta \cdot d\theta \int_0^{\pi} \sin^3\phi \cdot d\phi$$

$$= U_m \cdot 2 \int_0^{\pi/2} \sin^2\theta \, d\theta \cdot 2 \int_0^{\pi/2} \sin^3\phi \cdot d\phi$$

$$= U_m \cdot 2 \cdot \frac{1}{2} \cdot \frac{\pi}{2} \cdot 2 \times \frac{2}{3}$$

$$P_{rad} = U_m \frac{2\pi}{3}$$

$$D = \frac{4\pi \cdot U_m}{P_{rad}} = \frac{4\pi \cdot U_m \cdot 3}{U_m \cdot \pi \cdot 2} = 2 \times 3 = 6.$$

(Exact directivity)

Approximate directivity, $D \simeq \dfrac{41{,}253^\circ}{90^\circ \times 90^\circ} = 5.092$

(iii) $$D = \frac{4\pi \cdot U_m}{P_{rad}} = \frac{4\pi \cdot U_m}{U_m \cdot 16/9} = \frac{9\pi}{4}$$

(i) $$D = \frac{4\pi \cdot U_m}{P_{rad}} = \frac{4\pi \cdot U_m}{U_m \cdot 2\pi/3} = 2 \times 3 = 6$$