CMR INSTITUTE OF TECHNOLOGY

USN | | | | | | | | | | |

CMRIT

Internal Assessment Test I

Sub:	ENGINEERING MATHEMATICS I						Code:	15MAT11
Date:	20/ 9 / 2017	Duration:	90 mins	Max Marks:	50	Sem: I	Sec:	A,B,C,D,F,G

Q1 is compulsory. Answer any 7 questions from the rest.

		Marks	OBE CO	OBE RBT
1.	Evaluate the following limits: a) $\lim_{x\to1}\left[\frac{x}{x-1}-\frac{1}{\log x}\right]$ b) $\lim_{x\to0}\left[\frac{a^x+b^x+c^x}{3}\right]^{1/x}$	[03+05]	CO1	L3
2.	Find the n[th] derivative of $\frac{x^2}{(x+1)^2(x+2)}$	[06]	CO1	L3
3.	If $x=\sin t$ *and* $y=\cos mt$, prove that $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}+(m^2-n^2)y_n=0$.	[06]	CO1	L3
4.	Find the angle between the curves $r=a(1+\cos\theta)$ and $r^2=a^2\cos 2\theta$.	[06]	CO2	L3

PTO

5.	Find the pedal equation of the curve $r^m=a^m(\cos m\theta+\sin m\theta)$.	[06]	CO2	L3
6.	Show that the radius of curvature at any point on the cycloid $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$ is $4a\cos(\theta/2)$.	[06]	CO2	L3
7.	Show that for the curve $r(1-cos\theta)=2a$, ρ^2varies as r^3, where ρ is the radius of curvature.	[06]	CO2	L3
8.	Show that the tangents to the cardioid $r=a(1+\cos\theta)$ at the points $\theta=\frac{\pi}{3}$ and $\theta=\frac{2\pi}{3}$ are respectively parallel and perpendicular to the initial line.	[06]	CO2	L3
9.	Obtain the Maclaurin's expansion of $e^{tan^{-1}x}$ upto term containing x^5.	[06]	CO1	L3

Solution

1 a) $\lim_{x \to 1} \left[\frac{x}{x-1} - \frac{1}{\log x}\right]$ $(\infty - \infty)$

$= \lim_{x \to 1} \frac{x \log x - (x-1)}{(x-1)\log x}$ $\left(\frac{0}{0}\right)$ — 1 mark

$= \lim_{x \to 1} \frac{1 + \log x - 1}{\frac{x-1}{x} + \log x}$ (L' Hospital's Rule)

$= \lim_{x \to 1} \frac{x \log x}{x - 1 + x \log x}$ $\left(\frac{0}{0}\right)$ — 1 mark

$= \lim_{x \to 1} \frac{1 + \log x}{1 + 1 + \log x}$ (L' Hospital's Rule)

$= \frac{1}{2}$ — 1 mark

b) $\lim_{x \to 0} \left(\frac{a^x + b^x + c^x}{3}\right)^{1/x}$ (1^∞)

Let $k = \lim_{x \to 0} \left(\frac{a^x + b^x + c^x}{3}\right)^{1/x}$

$\log k = \lim_{x \to 0} \frac{1}{x} \log\left(\frac{a^x + b^x + c^x}{3}\right)$ $(\infty \times 0)$ — 1 mark

$= \lim_{x \to 0} \frac{\log\left(\frac{a^x + b^x + c^x}{3}\right)}{x}$ $\left(\frac{0}{0}\right)$

$= \lim_{x \to 0} \frac{\frac{3}{a^x + b^x + c^x}\left(\frac{a^x \log a + b^x \log b + c^x \log c}{3}\right)}{1}$ (L' Hospital's Rule)

$= \frac{\log a + \log b + \log c}{a^0 + b^0 + c^0} = \frac{1}{3}\log(abc)$ — 2 marks

$= \log (abc)^{1/3}$ — 1 mark

$k = (abc)^{1/3}$ — 1 mark

2 $$\frac{x^2}{(x+1)^2(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+2)}$$

$$x^2 = A(x+1)(x+2) + B(x+2) + C(x+1)^2$$

Put $x = -1$

$$1 = B$$

Put $x = -2$

$$4 = C$$

Equating coeff of x^2,

$$1 = A + C \Rightarrow A = 1 - C = -3$$

$$\frac{x^2}{(x+1)^2(x+2)} = \frac{-3}{x+1} + \frac{1}{(x+1)^2} + \frac{4}{(x+2)}$$ — 3 marks

The n^{th} derivative $= -3 \cdot \frac{(-1)^n \cdot n!}{(x+1)^{n+1}} + \frac{(-1)^n (n+1)!}{(x+1)^{n+2}} + \frac{4(-1)^n n!}{(x+2)^{n+1}}$

$$= (-1)^n n! \left[\frac{-3}{(x+1)^{n+1}} + \frac{(n+1)}{(x+1)^{n+2}} + \frac{4}{(x+2)^{n+1}} \right]$$ — 3 marks

3. $x = \sin t$, $y = \cos mt$

$t = \sin^{-1} x$

$y = \cos(m \sin^{-1} x)$ — 1 mark

$$y_1 = -\sin(m \sin^{-1} x) \cdot \frac{m}{\sqrt{1-x^2}}$$

$\sqrt{1-x^2} \cdot y_1 = -m \sin(m \sin^{-1} x)$ — 1 mark

$$\sqrt{1-x^2}\, y_2 - \frac{2x}{2\sqrt{1-x^2}} y_1 = -m \cos(m \sin^{-1} x) \cdot \frac{m}{\sqrt{1-x^2}}$$

$(1-x^2) y_2 - x y_1 + m^2 y = 0$ — 1 mark

$$(1-x^2) y_{n+2} + n(-2x) y_{n+1} + \frac{n(n-1)}{\cancel{2}} (-\cancel{2}) y_n - x y_{n+1} - n y_n + m^2 y_n = 0$$

$(1-x^2) y_{n+2} - (2n+1) x y_{n+1} + (m^2 - n^2) y_n = 0$ — 3 marks

4. $r = a(1+\cos\theta)$ $\quad r^2 = a^2\cos 2\theta$

$\log r = \log a + \log(1+\cos\theta)$ $\quad 2\log r = 2\log a + \log(\cos 2\theta)$

$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta}$ $\quad \frac{\cancel{2}}{r}\frac{dr}{d\theta} = \frac{-2\sin 2\theta}{\cos 2\theta} = -\cancel{2}\tan 2\theta$

$= \frac{-\cancel{2}\sin\theta/2 \cos\theta/2}{\cancel{2}\cos^2\theta/2}$ $\quad \cot\varphi_2 = \cot\left(\frac{\pi}{2}+2\theta\right)$

$= -\tan\theta/2$ $\quad \varphi_2 = \frac{\pi}{2}+2\theta$ — 3 marks

$\cot\varphi_1 = \cot\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$

$\varphi_1 = \frac{\pi}{2} - \frac{\theta}{2}$

$|\varphi_1 - \varphi_2| = \frac{3\theta}{2}$ — 1 mark

To find θ:

$\cancel{a^2}(1+\cos\theta)^2 = \cancel{a^2}\cos 2\theta$.

$1+\cos^2\theta + 2\cos\theta = \cos 2\theta = 2\cos^2\theta - 1$

$\cos^2\theta - 2\cos\theta - 2 = 0$

$\cos\theta = \frac{2\pm\sqrt{4+8}}{2} = \frac{2\pm 2\sqrt{3}}{2} = 1\pm\sqrt{3}$

Since $|\cos\theta| \le 1$, $\quad \theta = \cos^{-1}(1-\sqrt{3})$

$\therefore |\varphi_1 - \varphi_2| = \frac{3}{2}\cos^{-1}(1-\sqrt{3})$ — 2 marks

5. $r^m = a^m(\cos m\theta + \sin m\theta)$

$m\log r = m\log a + \log(\cos m\theta + \sin m\theta)$

$\frac{\cancel{m}}{r}\frac{dr}{d\theta} = \frac{\cancel{m}(-\sin m\theta + \cos m\theta)}{\cos m\theta + \sin m\theta}$ — 2 marks

$\cot\varphi = \frac{-\tan m\theta + 1}{1+\tan m\theta}$ (dividing num and den by $\cos m\theta$)

$= \frac{\tan\frac{\pi}{4} - \tan m\theta}{1+\tan\frac{\pi}{4}\tan m\theta} = \tan\left(\frac{\pi}{4} - m\theta\right)$

— 1 mark

$$\cot \varphi = \cot\left(\frac{\pi}{2} - \left(\frac{\pi}{4} - m\theta\right)\right)$$

$$= \cot\left(\frac{\pi}{4} + m\theta\right)$$

$$\phi = \frac{\pi}{4} + m\theta .$$ — 1 mark

$$p = r\sin\varphi = r\sin\left(\frac{\pi}{4} + m\theta\right)$$

$$= r\left[\sin\frac{\pi}{4}\cos m\theta + \cos\frac{\pi}{4}\sin m\theta\right]$$

$$= \frac{r}{\sqrt{2}}\left[\cos m\theta + \sin m\theta\right]$$

$$= \frac{r}{\sqrt{2}} \frac{r^m}{a^m} = \frac{r^{m+1}}{\sqrt{2}\, a^m}$$

$$r^{m+1} = \sqrt{2}\, a^m . p .$$ — 2 marks

6. $x = a(\theta + \sin\theta)$ $\quad y = a(1 - \cos\theta)$

$$\frac{dx}{d\theta} = a(1 + \cos\theta) \qquad \frac{dy}{d\theta} = a\sin\theta .$$

$$\frac{dy}{dx} = \frac{\cancel{a}\sin\theta}{\cancel{a}(1+\cos\theta)} = \frac{2\sin\theta/2\cos\theta/2}{2\cos^2\theta/2} = \tan\theta/2$$ — 1 mark

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx} = \frac{1}{2}\sec^2\theta/2 \cdot \frac{1}{a(1+\cos\theta)}$$

$$= \frac{1}{2}\frac{\sec^2\theta/2}{a \cdot 2\cos^2\theta/2} = \frac{1}{4a\cos^4\theta/2}$$ — 2 marks

$$\rho = \frac{[1 + y'^2]^{3/2}}{y''} = \frac{\left[1 + \tan^2\theta/2\right]^{3/2}}{\frac{1}{4a\cos^4\theta/2}} = \sec^3\theta/2 \cdot 4a\cos^4\theta/2$$

$$= 4a\cos\theta/2$$ — 3 marks

7. $r(1-\cos\theta) = 2a$ —(1)

$\log r + \log(1-\cos\theta) = \log 2a.$

Diff, $\frac{1}{r}\frac{dr}{d\theta} = -\frac{\sin\theta}{1-\cos\theta} = -\frac{2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2} = -\cot\theta/2$ — 1mk —(2)

$r_1 = \frac{dr}{d\theta} = -r\cot\theta/2$ —— 1 mk

$r_2 = \frac{d^2r}{d\theta^2} = +\frac{r}{2}\operatorname{cosec}^2\theta/2 - r_1\cot\theta/2$

$= \frac{r}{2}\operatorname{cosec}^2\frac{\theta}{2} + r\cot^2\theta/2$ —— 1mk

$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2} = \frac{(r^2+r^2\cot^2\theta/2)^{3/2}}{r^2+2r^2\cot^2\theta/2 - \frac{r^2}{2}\operatorname{cosec}^2\frac{\theta}{2} - r^2\cot^2\theta/2}$

$= \frac{r^3\operatorname{cosec}^3\theta/2}{r^2+r^2\cot^2\theta/2 - \frac{r^2}{2}\operatorname{cosec}^2\theta/2}$

$= \frac{r\operatorname{cosec}^3\theta/2}{\operatorname{cosec}^2\theta/2 - 1/2\operatorname{cosec}^2\theta/2} = 2r\operatorname{cosec}\theta/2$

$= \frac{2r}{\sin\theta/2} = \frac{2r}{\sqrt{\frac{1-\cos\theta}{2}}} = \frac{2r}{\sqrt{\frac{2a}{2r}}} = \frac{2r^{3/2}}{\sqrt{a}}$

$\rho^2 = \frac{4}{a}r^3 \Rightarrow \rho^2 \propto r^3$ —— 3 marks

OR

From (2), $\cot\varphi = -\cot\theta/2 = \cot(-\theta/2)$ —— 2 marks

$\Rightarrow \varphi = -\theta/2.$

$\therefore$ the pedal eqn is $p = r\sin\varphi = -r\sin\frac{\theta}{2}$

$= -r\sqrt{\frac{a}{r}} = -\sqrt{ar}$

or, $p^2 = ar$ —— 2 marks

Diff w.r.t p, $2p = a\frac{dr}{dp}.$

$\rho = r\frac{dr}{dp} = \frac{r.2p}{a} \Rightarrow \rho^2 = \frac{4r^2}{a^2}.p^2 = \frac{4r^2.ar}{a^2} = \frac{4}{a}r^3$

$\Rightarrow \rho^2 \propto r^3$ —— 2 marks

8. $r = a(1+\cos\theta)$

$\log r = \log a + \log(1+\cos\theta)$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta} = \frac{-2\sin\theta/2\,\cos\theta/2}{2\cos^2\theta/2}$$

$$\cot\phi = -\tan\theta/2 = \tan(-\theta/2)$$

$$= \cot(\pi/2 + \theta/2)$$

$$\phi = \frac{\pi}{2} + \frac{\theta}{2} \qquad \text{— 2 marks}$$

$$\tan\psi = \tan(\theta+\phi)$$

$$= \tan\left(\theta + \frac{\pi}{2} + \frac{\theta}{2}\right)$$

$$= \tan\left(\frac{\pi}{2} + \frac{3\theta}{2}\right) \qquad \text{— 2 marks}$$

At $\theta = \frac{\pi}{3}$, $\tan\psi = \tan\pi = 0 \Rightarrow$ tangent is parallel to initial line. — 1 mk

At $\theta = \frac{2\pi}{3}$, $\tan\psi = \tan\frac{3\pi}{2} = \infty \Rightarrow$ tangent is perpendicular to initial line. — 1 mk

9. $y = e^{\tan^{-1}x}$.

$y(x) = y(0) + x\,y_1(0) + \frac{x^2}{2!}y_2(0) + \cdots$ — 1 mk

$y = e^{\tan^{-1}x} \qquad y(0) = 1$

$$y_1 = \frac{e^{\tan^{-1}x}}{1+x^2} \Rightarrow y_1 = \frac{y}{1+x^2}\,; \quad y_1(0) = 1$$

$y_1(1+x^2) = y$.

$(1+x^2)\,y_{n+1} + 2nx\,y_n + n(n-1)\,y_{n-1} = y_n$

$(1+x^2)\,y_{n+1} + (2nx-1)\,y_n + n(n-1)\,y_{n-1} = 0$

At $x=0$, $y_{n+1} +$ [illegible] $-1)\,y_n + n(n-1)\,y_{n-1} = 0$ — 2 mks

$$y_{n+1}(0) = y_n(0) - n(n-1)\,y_{n-1}(0)$$

$$n=1 \Rightarrow y_2(0) = y_1(0) = 1$$

$$n=2 \Rightarrow y_3(0) = y_2(0) - 2y_1(0)$$
$$= 1 - 2 = -1$$

$$n=3 \Rightarrow y_4(0) = y_3(0) - 6\,y_2(0)$$
$$= -1 - 6 = -7$$

$$n=4 \Rightarrow y_5(0) = y_4(0) - 12\,y_3(0)$$
$$= -7 + 12 = 5 \quad \text{— 2 mks}$$

$$\therefore\ y(x) = 1 + x + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(-7) + \frac{x^5}{5!}(5)$$

(upto 5^{th} deg term)

$$= 1 + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{7x^4}{24} + \frac{x^5}{24} \quad \text{— 1 mk}$$