Improvement Test solution:

Subject: Automata Theory & Computability.

Sem: V B.

Dept: ISE.

① State and Prove Pumping Theorem for context-free languages.

Statement:- If L is a context-free language, then:

$\exists K \geq 1$ ($\forall$ strings $w \in L$ where $|w| \geq K$)

$\exists u, v, x, y, z$ ($w = uvxyz$,

$vy \neq \varepsilon$

$|vxy| \leq K$ and

$\forall q \geq 0$ ($uv^q xy^q z$) is in L)

Proof:- If L is context-free then it is generated by some context-free grammar $G = (V, \Sigma, R, S)$ with n nonterminal symbols and branching factor b. Let K be b^{n+1}. Any string that can be generated by G and whose parse tree contains no paths with repeated nonterminals must have length less than or equal to b^n.

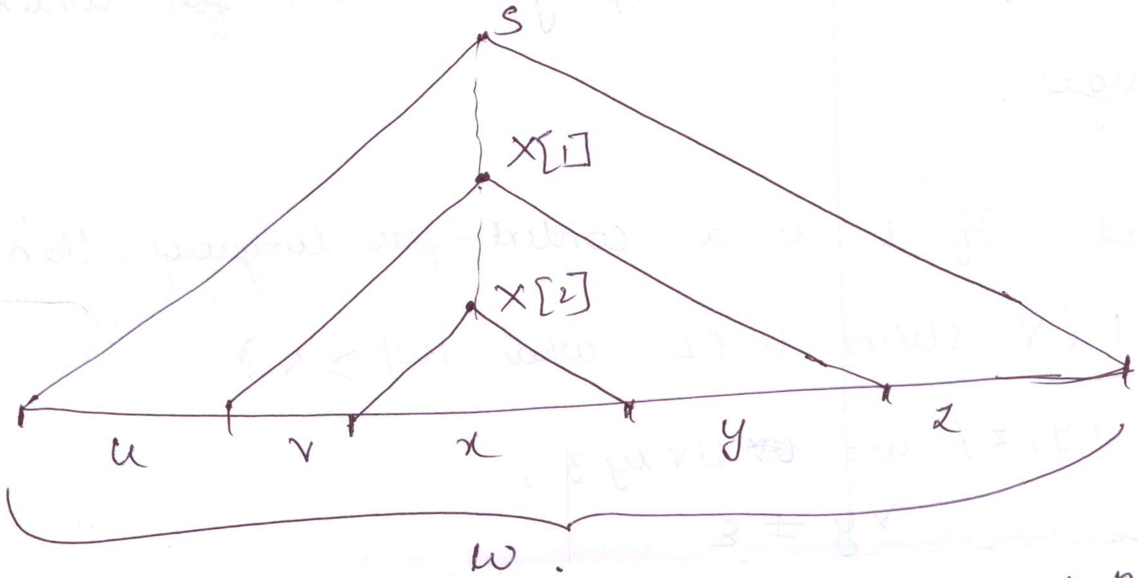
Let w be any string in $L(G)$ where $|w| \geq K$.

Let T be any smallest parse tree for w.

~~Let~~ T must have height at least $n+1$.

Choose some path T of length at least $n+1$.

Let X be the bottommost repeated non terminal along that path. Then w can be rewritten as $uvxyz$ as shown below.



The tree rooted at [1] has height at most $n+1$. Thus its yield vxy, has length less than or equal to b^{n+1} which is K. And $vy \neq \varepsilon$ $(S \overset{*}{\Rightarrow} uXz \overset{*}{\Rightarrow} uxz.)$ (rule 1)

Finally v and y can be pumped: uxz must be in L. And, for any $q \geq 1$, $uv^q x y^q z$ must be in L because,

$S \overset{*}{\Rightarrow} uXz \overset{*}{\Rightarrow} uvXyz \overset{*}{\Rightarrow} uvxyz$ (rule 2) could have been immediately used at X[1]. And for any $q \geq 1$ $uv^q x y^q z$ must be in L because rule$_1$ could have been used q times before finally using rule 2.

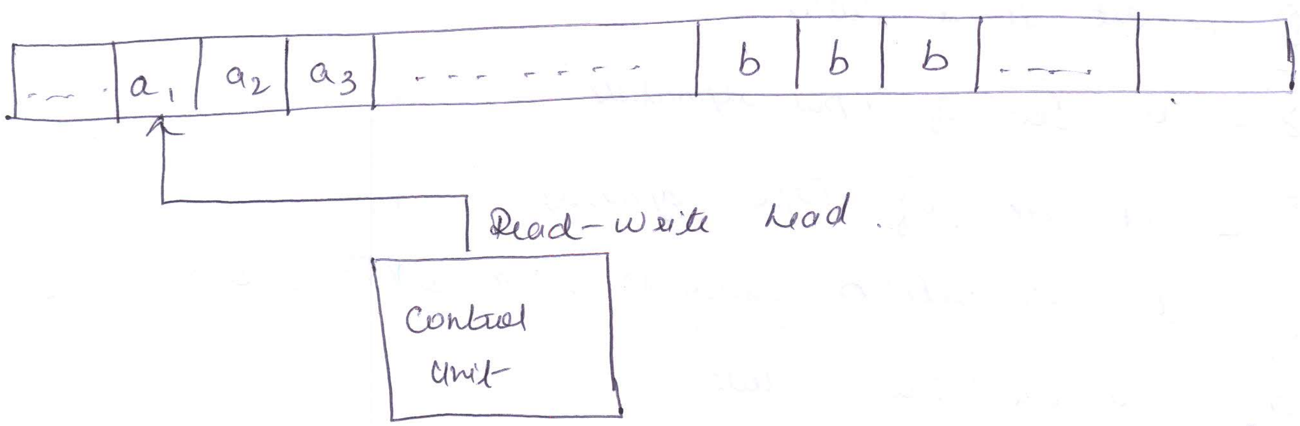
So, if L is a context-free language, every "long" string in L must be pumpable. ②

The pumped region can be pumped out once or pumped in any number of times, in all cases resulting in another string that is also in L.

(2) Explain with a diagram the operation of Turing machines? Give formal definition of Turing machine.

⇒ Turing machine is a finite automaton connected to read-write head with the following components.

- Tape
- Read-write head
- Control-unit



Tape: is used to store information and the tape is divided into cells. Each cell can store information of only one symbol. The string to be scanned will be stored from the leftmost position on the tape. The string to be scanned should end with blanks. The tape is assumed to be infinite both on left side and right side of the string.

* <u>Read-write head</u> :- The read-write head can read a symbol from where it is pointing to and it can write into the tape to where it points to.

* <u>Control unit</u>: The reading from the tape is determined by the control unit. The different moves performed by the machine depends on the current scanned symbol and the current state. The control unit consults action table i.e transition table and carry out tasks.

Formal definition of Turing Machine.

$M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$

Q - is set finite states

Σ - is set of input alphabets.

Γ - is set of Tape symbols

δ - is transition function from $Q \times \Gamma$ to $Q \times \Gamma \times \{L, R\}$

q_0 - is the start state

B - is a special symbol indicating blank character.

$F \subseteq Q$ is a set of final states.

(3) Prove that context free languages are closed under union, concatenation and Kleene closure (star). ③.

⇒ The Context-free languages are closed under union:

If L_1 and L_2 are context free languages, then there exists context free grammars $G_1 = (V_1, \Sigma_1, R_1, S_1)$ & $G_2 = (V_2, \Sigma_2, R_2, S_2)$ such that $L_1 = L(G_1)$ and $L_2 = L(G_2)$.

Build a new grammar G such that $L(G) = L(G_1) \cup L(G_2)$.

G will contain all the rules of both G_1 and G_2.

We add to G a new symbol, S, and two new rules

$S \rightarrow S_1$ and $S \rightarrow S_2$.

The two new rules allow G to generate a string iff at least one of G_1 or G_2 generate it.

So $G = (V_1 \cup V_2 \cup \{S\}, \Sigma_1 \cup \Sigma_2, R_1 \cup R_2 \cup \{S \rightarrow S_1, S \rightarrow S_2\}, S)$.

⇒ The Context free languages are closed under concatenation

If L_1 and L_2 are CFLs then there exists CFGs

$G_1 = (V_1, \Sigma_1, R_1, S_1)$ and $G_2 = (V_2, \Sigma_2, R_2, S_2)$ such that $L_1 = L(G_1)$ and $L_2 = L(G_2)$.

Build a new grammar G such that $L(G) = L(G_1) L(G_1)$.

G will contain rules of both G_1 and G_2.

Add to G a new start symbol S and new rule $S \rightarrow S_1 S_2$.

So $G = (V_1 \cup V_2 \cup \{S\}, \Sigma_1 \cup \Sigma_2, R_1 \cup R_2 \cup \{S \rightarrow S_1 S_2\}, S)$.

⇒ The Content-free languages are Closed under Kleene state.

If L_1 is a Content-free language, then there exists a content-free grammar $G_1 = (V_1, \Sigma_1, R_1, S_1)$ such that $L_1 = L(G_1)$.

Build a new grammar G such that $L(G) = L(G_1)^*$.

G will contain all the rules of G_1

Add to G a new start state S and two new rules

$S \to \varepsilon$ and $S \to SS_1$.

so $G = (V_1 \cup \{S\}, \Sigma_1, R_1 \cup \{\to \varepsilon, S \to SS_1\}, S)$.

(4)
(a) Show that $L = \{a^n b^n c^n \mid n >= 0\}$ is not content free.

Soln Let L be a content free, Let $w = a^n b^n c^n \in L$.

~~Note~~, $|w| \geq n$ and we split w into uvxyz such that

$|vxy| \leq n$ and $|vy| \geq 1$ or $vy \neq \epsilon$

and so $uv^q wx^q y \in L$ for $i = 0, 1, 2$ must satisfy according to pumping lemma.

Consider case: $u = a^j$, $v = a^k$ when $|vy| = j + k \geq 1$ and $|vxy| \leq n$. which is

$$\underbrace{\overbrace{a\cdots a}^{n}}_{uvxy}\underbrace{\overbrace{b\cdots b}^{n}\overbrace{c\cdots c}^{n}}_{z}$$

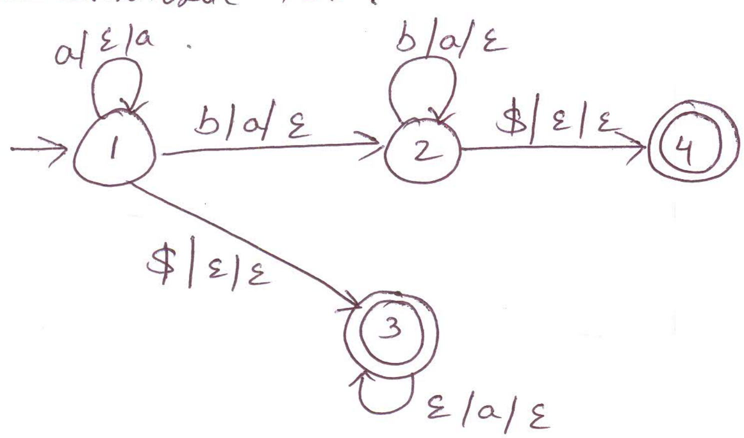
Pump one: $uv^2 wx^2 y \in L$ for $q = 2$ the language is $a^{n+j+k} b^n c^n \notin L$. Hence by contradiction the given language is not content free.

(b) Explain Deterministic CFL with example.

A language L is deterministic context free iff L\$ can be accepted by some deterministic PDA when \$ marks end-of-string.

Eg Let $L = a^* \cup \{a^n b^n : n > 0\}$.

Consider any PDA M that accepts L.
When PDA begins reading a's, M must push them onto stack in case there are going to be b's.
But if runs out of input without seeing b's, it needs a way to pop those a's from the stack before it can accept. Without an end-of-string marker, there is no way to allow the popping of a's.
With an end-of-string marker, we can build the following deterministic PDA.



(5)

(a) obtain a Turing machine to accept the language $L(M) = \{0^n 1^n 2^n \mid n >= 1\}$.

Solution

$M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$

where $Q = \{q_0, q_1, q_2, q_3, q_4, q_5, q_6\}$

$\Sigma = \{0, 1, 2\}$.

$\Gamma = \{0, 1, 2, X, Y, Z, B\}$

q_0 is the start state.

B is blank character.

$F = \{q_6\}$ is the final state

δ is

$\delta(q_0, 0) = (q_1, X, R)$ $\quad$ $\delta(q_0, Y) = (q_4, Y, R)$

$\delta(q_1, 0) = (q_1, 0, R)$, $\delta(q_1, 1) = (q_2, Y, R)$, $\delta(q_1, Y) = (q_1, Y, R)$

$\delta(q_2, 1) = (q_2, 1, R)$, $\delta(q_2, 2) = (q_3, Z, L)$ $\quad$ $\delta(q_2, Z) = (q_2, Z, R)$

$\delta(q_3, 0) = (q_3, 0, L)$, $\delta(q_3, Z) = (q_3, Z, L)$, $\delta(q_3, Y) = (q_3, Y, L)$

$\delta(q_3, 1) = (q_3, 1, L)$.

$\delta(q_4, Z) = (q_5, Z, R)$ $\quad$ $\delta(q_4, Y) = (q_4, Y, R)$

$\delta(q_5, Z) = (q_5, Z, R)$

$\delta(q_6, B) = (q_6, B, R)$.

(b) List out the undecidable questions.

* Given a context-free language language L is $L = \Sigma^*$?
* Given a context-free language L, is the complement of L context-free?
* Given a context-free language L is L regular?
* Given two context-free languages L_1 & L_2 is $L_1 = L_2$?
* Given two context-free language L_1 & L_2, is $L_1 \subseteq L_2$?
* Given two context-free languages L_1 and L_2 is $L_1 \cap L_2 = \phi$
* Given a context-free language L, is L inherently ambiguous?
* Given a context free grammar G, is G ambiguous.

(6). Prove that deterministic CFLs are not closed under union and Intersection.

(1) The deterministic CFL are not closed under union.

Let $L_1 = \{a^i b^j c^k : i, j, k \geq 0 \text{ and } i \neq j\}$.

Let $L_2 = \{a^i b^j c^k : i, j, k \geq 0 \text{ \& } j \neq k\}$.

Let $L' = L_1 \cup L_2$

$= \{a^i b^j c^k : i, j, k \geq 0 \text{ and } ((i \neq j) \text{ or } (j \neq k))\}$.

Let $L'' = \neg L'$

$= \{a^i b^j c^k : i, j, k \geq 0$ and $i = j = k\} \cup$

$\{w \in \{a, b, c\}^* :$ the letter are out of order$\}$.

Let $L''' = L'' \cap a^* b^* c^*$

$= \{a^n b^n c^n : n \geq 0\}$. is not context free as

per pumping theorem.

⇒ Nonclosure under Intersection.

Let $L_1 = \{a^i b^j c^k : i, j, k \geq 0$ and $i = j\}$.

Let $L_2 = \{a^i b^j c^k : i, j, k \geq 0$ and $j = k\}$.

Let $L' = L_1 \cap L_2$

$= \{a^n b^n c^n : n \geq 0\}$. is not context free.