Faculty :	Mr. Rahul Nyamangoudar	Class :	ECE 5th Sem - A & B
Subject :	Information Theory and Coding	Subject Code :	15EC54

Q 1.

a. Define source entropy and average source information rate

Answer:

Source Entropy:

The entropy of a source indicates the minimum amount of bits required to represent a symbol on an average. Entropy of a source emitting q possible symbol $s_1, s_2, \dots, s_q$ with probabilities $p_1, p_2, \dots, p_q$ in a statistical independent sequence is given by,

$$H(S) = \sum_{i=1}^{M} p_i \log_2\left(\frac{1}{p_i}\right) \quad \text{bits/symbol}$$

1M for Definition, 1M for Equation = 2M

Average source information rate:

The average source information rate R is defined as the product of the average information content per symbol and the symbol rate r_s. It represents the rate of symbols generated by the source in bits/sec or symbols/sec.

$$R = H(S) * r_s \quad \text{bits/sec}$$

1M for Definition, 1M for Equation = 2M

Total – 4 Marks

b. A discrete memoryless source emits one of five symbols once every millisecond. The symbol probabilities are $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}$ respectively. Find the source entropy and information rate.

Solution

Given: Discrete Memoryless Source

No. of Symbols = 5 (Let s_1, s_2, s_3, s_4, s_5 be the symbols)

Probabilities → $P(s_1) = \frac{1}{2}, P(s_2) = \frac{1}{4}, P(s_3) = \frac{1}{8}, P(s_4) = \frac{1}{16}, P(s_5) = \frac{1}{16}$

Symbol rate $(r_s) = 1 \text{ symbols}/1 \text{ milli second} = 10^3 \text{symbols/second}$

Source Entropy:

$$H(S) = \sum_{i=1}^{5} p_i \log_2\left(\frac{1}{p_i}\right) \quad \text{bits/symbol}$$

$$H(s) = \frac{1}{2}\log_2(2) + \frac{1}{4}\log_2(4) + \frac{1}{8}\log_2(8) + 2 * \frac{1}{16}\log_2(16)$$

$$H(s) = 1.875\ bits/symbol$$

1M for Entropy formula, 2M for Calculation = 3 Marks

Information rate:

$$R = H(s).r_s$$

$$R = 1.875 \frac{bits}{\cancel{symbol}} * \frac{10^3 \cancel{symbols}}{second}$$

$$R = 1875\ bits/sec$$

1M for information rate, 1M for symbol rate, 1M for Calculation = 3M

Total – 6 Marks

Q 2.

a. How do you measure information? Justify your answer.

Answer:

The amount of information in a message depends only on the uncertainty of the underlying event rather than its actual content and is measured as:

$$I(m_k) = \log_2\left(\frac{1}{p_k}\right)$$

Proof:

Considering an information source that emits one of the $'q'$ possible messages $m_1, m_2, \ldots, m_q$ with probabilities $p_1, p_2, \ldots, p_q$ respectively, such that $p_1 + p_2 + \ldots + p_q = 1$. Then,

The event that is most likely to happen (high probability), is having lesser information and an event that is less likely to occur (lower probability) has more information. Based on this, information of event can be related to the probability of event as,

$$I(m_k) \propto \frac{1}{p_k} \quad (2.1)$$

Information content I_k must approach $'0'$ as p_k approaches $'1'$, that is

$$I(m_k) \rightarrow 0 \text{ as } p_k \rightarrow 1 \quad (2.2)$$

Information content is non-negative since each message will have some information and in worst case, it can be equal to zero,

$$I(m_k) \geq 0 \quad \text{for} \quad 0 \leq p_k \leq 1 \quad (2.3)$$

Information content is zero for an event that will happen definitely and for an event which will not occur at all.

For two messages m_k and m_j,

$$I(m_k) < I(m_j) \quad \text{for} \quad p_k > p_j \tag{2.4}$$

For two independent messages m_k and m_j, the total information conveyed is sum of information conveyed by each message individually,

$$I(m_k \text{and } m_j) \triangleq I(m_k m_j) = I(m_k) + I(m_j) \tag{2.5}$$

A continuous function of p_k that satisfies the constraints specified in equations (2.1) to (2.6) is a logarithmic function and thus measure of information can be defined as,

$$I(m_k) = \log_2\left(\frac{1}{p_k}\right) \tag{2.6}$$

1M for How? & 5M for Justification = 6M

Total – 6 Marks

b. A black and white TV picture consists of 640 lines of picture information. Assume that each line consists of 480 picture elements (pixels) and that each pixel has 256 brightness levels. The picture is repeated at the rate of 30 frames/sec. Calculate the average rate of information conveyed by a TV set to a viewer.

Solution:

Given: No. of lines - 640

No. of pixels per line – 480

No. of brightness levels per pixel – 256

No. of frames per second – 30

Average information per pixel considering that all levels to be equi-probable.is

$$H(S) = \log_2 256 = 8 \; bits \; per \; pixel$$

No. of pixels per frame or image is

$$N_p = 640 * 480 = 307.2 * 10^3 \; pixels \; per \; frame$$

Thus Average rate of information conveyed by a TV set to viewer is

$$R = H(S) * No. of \; pixel \; per \; frame \; (N_p) * r_s(in \; frames \; per \; second)$$

$$R = 8\frac{bits}{\cancel{pixel}} * 307.2 * 10^3 \frac{\cancel{pixels}}{\cancel{frame}} * 30\frac{\cancel{frames}}{second}$$

$$R = 73.728 \; Mbps$$

1M for H(S), 1M for N_p, 2M for calculation = 4M

Total – 4 Marks

Q 3.

For the Markov source shown in Figure Q3. Find G_1, G_2 and Verify $G_1 > G_2 > H(S)$.

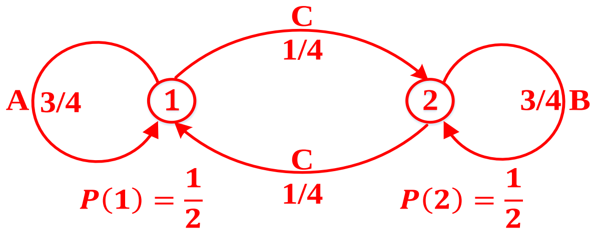

Figure Q3.

Solution:

Given $S = \{A, B, C\}$

$States = \{1, 2\}$

$P(1) = \frac{1}{2}$

$P(2) = \frac{1}{2}$

Firstly finding entropy of each state:

$$H_i = \sum_{j=1}^{2} p_{ij} \log_2 \left(\frac{1}{p_{ij}}\right)$$

$$H_1 = \frac{3}{4}\log_2\left(\frac{4}{3}\right) + \frac{1}{4}\log_2(4)$$

$$H_1 = 0.8113 \; bits/symbol$$

$$H_2 = \frac{3}{4}\log_2\left(\frac{4}{3}\right) + \frac{1}{4}\log_2(4)$$

$$H_2 = 0.8113 \; bits/symbol$$

Thus entropy of source is

$$H(S) = \sum_{i=1}^{2} P(i) * H_i$$

$$H(S) = \frac{1}{2} * 0.8113 + \frac{1}{2} * 0.8113$$

$$H(S) = 0.8113 \; bits/symbol$$

Then to find $G_1$and G_2, the tree structure is represented as in Figure 3.1. Correspondingly messages of length 1 & 2 with probabilities are listed in Table 3.1.

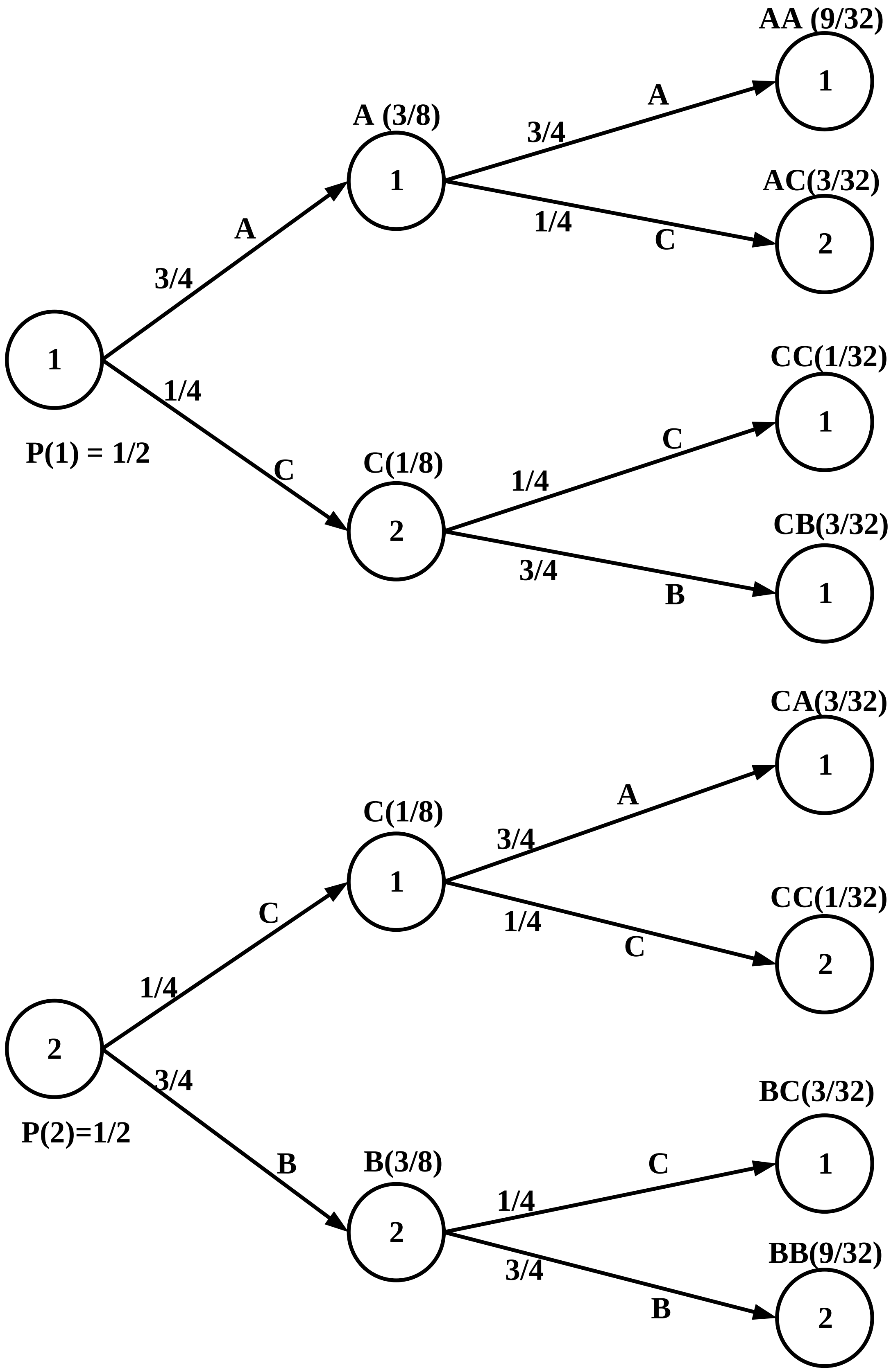

Figure 3.1: Tree diagram representation

Table 3.1 Messages of length 1 & 2 with their probabilities

Messages of length 1	**Messages of length 2**
$P(A) = \frac{3}{8}$	$P(AA) = \frac{9}{32}$
$P(B) = \frac{3}{8}$	$P(AC) = \frac{3}{32}$
$P(C) = \frac{1}{8} + \frac{1}{8} = \frac{2}{8}$	$P(CC) = \frac{1}{32} + \frac{1}{32} = \frac{2}{32}$
	$P(CB) = \frac{3}{32}$
	$P(CA) = \frac{3}{32}$
	$P(BC) = \frac{3}{32}$
	$P(BB) = \frac{9}{32}$

$$G_N = \frac{1}{N} \sum_{S^N} P(i) \log_2 \left(\frac{1}{P(i)} \right)$$

Thus, entropy per symbol of the sequence of symbols of length '1', i.e. G_1

$$G_1 = \sum_{i=A}^{C} P(i) \log_2 \left(\frac{1}{P(i)} \right)$$

$$G_1 = 2 * \frac{3}{8} \log_2 \left(\frac{8}{3} \right) + \frac{2}{8} \log_2 \left(\frac{8}{2} \right)$$

$$G_1 = 1.5613 \ bits \ per \ symbol$$

Thus, entropy per symbol of the sequence of symbols of length '2', i.e. G_2

$$G_2 = \frac{1}{2} \sum_{i=AA}^{CC} P(i) \log_2 \left(\frac{1}{P(i)} \right)$$

$$G_2 = \frac{1}{2} \left\{ 2 * \frac{9}{32} * \log_2 \left(\frac{32}{9} \right) + 4 * \frac{3}{32} * \log_2 \left(\frac{32}{3} \right) + \frac{2}{32} * \log_2 \left(\frac{32}{2} \right) \right\}$$

$$G_2 = 1.28 \ bits \ per \ symbol$$

Hence $G_1 > G_2 > H(S)$

1M each for H_1, H_2, H(S), G_1, G_2, 3M for tree, 2M for Table = 6M

Total – 6 Marks

Q 4.

a. State the properties of entropy.

Let $S \equiv \{s_1, s_2, s_3, \dots, s_q\}$ be the set of symbols emitted from a zero-memory source with probabilities $\{p_1, p_2, p_3, \dots, p_q\}$ respectively. Let the entropy of zero-memory source be $H(S)$, then

(I) $H(S)$ is a continuous function of $\{p_1, p_2, p_3, \dots, p_q\}$.

(II) Extremal Property:

Lower Bound on Entropy:

Entropy has a minimum value when one of the probability p_i for $1 \leq i \leq q$ is equal to '1' and hence rest all other probabilities is zero. Thus

$$H(S) \geq 0$$

Upper Bound on Entropy:

Entropy has a maximum value when all the individual probabilities are equal, that is $p_1 = p_2 = \ . \ . \ . = p_q$. Where each probability p_i is equal to $1/q$ (q is number of symbols).

$$H(S) \leq \log_2 q$$

Thus

$$0 \leq H(S) \leq \log_2 q$$

(III) Additive Property

For a set of symbols $\bar{S} \equiv \{s_{11}, s_{12}, \dots, s_{1N}, s_2, s_3, \dots s_q\}$ emitted from the same source with probabilities $\{r_{11}, r_{12}, \dots, r_{1N}, p_2, p_3, \dots, p_q\}$ and Entropy $H(\bar{S})$. Where symbol s_1 is split into multiple symbols $\{s_{11}, s_{12}, \dots, s_{1N}\}$ having probabilities $\{r_{11}, r_{12}, \dots, r_{1N}\}$ respectively. Such that, $r_{11} + r_{12} + r_{13} + \ . \ . \ . + r_{1N} = p_1$,

$$H(\bar{S}) \geq H(S)$$

(IV) Entropy function is a symmetrical function of all variables, that is

$$H(p_1, p_2, p_3, \dots, p_q) = H(p_{\sigma(1)}, p_{\sigma(2)}, p_{\sigma(3)}, \dots, p_{\sigma(q)})$$

Where, σ denotes a permutation of $(1, \dots, q)$.

1M for I, III, IV, 2M for II = 5M

Total – 5 Marks

b. The international Morse code uses a sequence of dots and dashes to transmit letters of the English alphabet. The dash is represented by a current pulse that has a duration of 3 units and the dot has a duration of 1 unit. The probability of occurrence of a dash is 1/3 of the probability of occurrence of a dot.

i. Calculate the information content of a dot and a dash.

ii. Calculate the average information in the dot-dash code.

iii. Assume that the dot lasts 2 msec, which is the same time interval as the pause between symbols. Find the average rate of information transmission.

Solution:

Given $\mathrm{S} = \{\mathrm{dot}, \mathrm{dash}\}$

Duration of dash = 3 units and Duration of dot = 1 unit

$$\mathrm{P}(dash) = \frac{1}{3}\mathrm{P}(dot) \quad (4.1)$$

w.k.t,

$$P(dot) + P(dash) = 1 \quad (4.2)$$

Using equation (4.1) in equation (4.2), we have

$$P(dot) + \frac{1}{3}P(dot) = 1$$

$$\therefore P(dot) = \frac{3}{4}$$

$$P(dash) = \frac{1}{4}$$

(i) Information of dot and dash is

$$Information(dot) = \log_2\left(\frac{1}{P(dot)}\right) = \log_2\left(\frac{4}{3}\right) = 0.415\ bits$$

$$Information(dash) = \log_2\left(\frac{1}{P(dash)}\right) = \log_2(4) = 2\ bits$$

(ii) Average information of dot-dash code is

$$H(S) = P(dot)\log_2\left(\frac{1}{P(dot)}\right) + P(dash)\log_2\left(\frac{1}{P(dash)}\right)$$

$$H(S) = \frac{3}{4} * \log_2\left(\frac{4}{3}\right) + \frac{1}{4} * \log_2(4)$$

$$H(S) = 0.8113\ bits/symbol$$

(iii) $duration(dot) = 1\ unit = 2msec$

$duration(dash) = 3\ units = 6msec$

$duration(pause) = 2\ msec$

Considering a message of length N,

$$\text{Number of dots} = \frac{3}{4} * N = 0.75 * N\ dots$$

$$\text{Number of dashes} = \frac{1}{4} * N = 0.25 * N\ dots$$

$$\text{Number of pauses} = N - 1 \cong N\ pauses\ (for\ large\ N)$$

$$\text{Average duration/symbol} = \frac{0.75 * N * 2 * 10^{-3} + 0.25 * N * 6 * 10^{-3} + N * 2 * 10^{-3}}{N}$$

$$= \frac{\not{N}(1.5 + 1.5 + 2) * 10^{-3}}{\not{N}} = 5\ msecs/symbol$$

$$\therefore \text{symbol rate } (r_s) = 1\ symbol/5\ msec = 200\ symbols/sec$$

Hence average information rate is

$$R = H(S) * r_s$$

$$R = 0.8113 \frac{bits}{\cancel{symbols}} * 200 \frac{\cancel{symbols}}{sec} = 162.26\ bits/sec$$

1M for (i), 1M for (ii), 3M for (iii) = 5M

Total - 5 Marks

Q 5.

a. Show that entropy of nth extension of a zero-memory source 'S' is $H(S^n) = nH(S)$, where H(S) is entropy of zero-memory source.

Proof:

Let, the q^n symbols of n^{th} extension of source S, S^n, be $\{\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_{q^n}\}$. Where each σ_i corresponds to some sequence of length $'n'$ of the s_i. Let $P(\sigma_i)$ represents the probability of σ_i, where σ_i corresponds to sequence of $s_i's$ represented as

$$\sigma_i = s_{i_1}, s_{i_2}, \ldots, s_{i_n} \qquad (5.1)$$

Since occurrence of each individual symbol in σ_i is independent of the other,

$$p(\sigma_i) = p_{i_1} p_{i_2} \ldots p_{i_n} \qquad (5.2)$$

With $p_{i_1} = P(S_{i_1})$.

The Entropy of n^{th} extension of source can be written as,

$$H(S^n) = \sum_{S^n} p(\sigma_i) \log_2 \left(\frac{1}{p(\sigma_i)}\right) \qquad (5.3)$$

Where

$$\sum_{S^n} p(\sigma_i) = \sum_{S^n} p_{i_1} p_{i_2} \ldots p_{i_n} = \sum_{i_1=1}^{q} \sum_{i_2=1}^{q} \cdots \sum_{i_n=1}^{q} p_{i_1} p_{i_2} \ldots p_{i_n}$$

$$= \sum_{i_1=1}^{q} p_{i_1} \sum_{i_2=1}^{q} p_{i_2} \;.\;.\;. \sum_{i_n=1}^{q} p_{i_n} = 1$$

Equation (5.3) can be written as,

$$H(S^n) = \sum_{S^n} p(\sigma_i) \log_2 \left(\frac{1}{p_{i_1} p_{i_2} \dots p_{i_n}} \right)$$

$$= \sum_{S^n} p(\sigma_i) \log_2 \left(\frac{1}{p_{i_1}} \right) + \sum_{S^n} p(\sigma_i) \log_2 \left(\frac{1}{p_{i_2}} \right) + \;.\;.\;. + \sum_{S^n} p(\sigma_i) \log_2 \left(\frac{1}{p_{i_n}} \right) \quad (5.4)$$

If we take just the 1st term of the sum of summations given in above equation, we have

$$\sum_{S^n} p(\sigma_i) \log_2 \left(\frac{1}{p_{i_1}} \right) = \sum_{i_n=1}^{q} p_{i_1} p_{i_2} \dots p_{i_n} \log_2 \left(\frac{1}{p_{i_1}} \right)$$

$$= \sum_{i_1=1}^{q} p_{i_1} \log_2 \left(\frac{1}{p_{i_1}} \right) \sum_{i_2=1}^{q} p_{i_2} \;.\;.\;. \sum_{i_n=1}^{q} p_{i_n}$$

$$= \sum_{i_1=1}^{q} p_{i_1} \log_2 \left(\frac{1}{p_{i_1}} \right) = H(S)$$

Thus,

$$\sum_{S^n} p(\sigma_i) \log_2 \left(\frac{1}{p_{i_1}} \right) = H(S) \quad (5.5)$$

Using equation (5.5) to evaluate other terms in equation (5.4), we obtain

$$H(S^n) = nH(S)$$

2M till Equation (5.3), 3M for Equation (5) & (6) = 5M

Total – 5 Marks

b. For a zero-memory source S with source alphabet $\{A, C, D, I, M, R, T\}$ having probabilities $\left\{\frac{1}{27}, \frac{1}{3}, \frac{1}{9}, \frac{1}{9}, \frac{1}{27}, \frac{1}{27}, \frac{1}{3}\right\}$. Construct ternary–compact code using Huffman encoding procedure. Using the bit-representations obtained encode the message "CMRITADDA".

Solution:

Given No. of Symbols (q) – 7

No. of representation symbols (r) – 3 (ternary)

∴ No. of Stages is

$$\alpha = \frac{q - r}{r - 1}$$

$$\alpha = \frac{7 - 3}{3 - 1} = 2$$

(i) Considering composite symbol being placed as low as possible

Symbol	Code	Probabilities	Stage 1	Stage 2
C	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}(0)$
T	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}(1)$
D	20	$\frac{1}{9}$	$\frac{1}{9}(0)$	$\boldsymbol{\frac{1}{3}(2)}$
I	21	$\frac{1}{9}$	$\frac{1}{9}(1)$	
A	220	$\frac{1}{27}(0)$	$\boldsymbol{\frac{1}{9}(2)}$	
M	221	$\frac{1}{27}(1)$		
R	222	$\frac{1}{27}(2)$		

For this case, "CMRITADDA" can be encoded as

0,221,222,21,1,220,20,20,220

(i) Considering composite symbol being placed as high as possible

Symbol	Code	Probabilities	Stage 1	Stage 2
C	1	$\frac{1}{3}$	$\frac{1}{3}$	$\boldsymbol{\frac{1}{3}(0)}$
T	2	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}(1)$
D	01	$\frac{1}{9}$	$\boldsymbol{\frac{1}{9}(0)}$	$\frac{1}{3}(2)$
I	02	$\frac{1}{9}$	$\frac{1}{9}(1)$	
A	000	$\frac{1}{27}(0)$	$\frac{1}{9}(2)$	
M	001	$\frac{1}{27}(1)$		
R	002	$\frac{1}{27}(2)$		

For this case, "CMRITADDA" can be encoded as

1,001,002,02,2,000,01,01,000

4M for obtaining bit representations, 1M for encoding "CMRITADDA"= 5M

Total – 5 Marks

Q 6.

State and prove Kraft – McMillan Inequality for instantaneous code.

Answer:

Consider an instantaneous code with source alphabet $S = \{s_1, s_2, \ldots, s_q\}$ and code alphabet $X = \{x_1, x_2, \ldots, x_r\}$. Thus source symbols $\{s_1, s_2, \ldots, s_q\}$ are represented by code words $\{X_1, X_2, \ldots, X_q\}$ with lengths $\{l_1, l_2, \ldots, l_q\}$ respectively. Where each X_i is formed as sequences of symbols of code alphabet X.

Then a necessary and sufficient condition for the existence of an instantaneous code with word lengths $l_1, l_2, \ldots, l_q$ is that

$$\sum_{i=1}^{q} r^{-l_i} \leq 1 \qquad (6.1)$$

where $'r'$ is the number of different symbols in the code alphabet X.

Proof:

Consider the quantity,

$$\left(\sum_{i=1}^{q} r^{-l_i}\right)^n = \left(r^{-l_0} + r^{-l_1} + r^{-l_2} + \ .\ .\ .\ .\ + r^{-l_q}\right)^n \qquad (6.2)$$

where $'n'$ is a positive integer. Equation (6.2) represents the LHS of equation (6.1) for the n^{th} extension of source code words.

Expanding equation (6.2) will result in q^n terms, with each having a form

$$r^{-l_{i_1} - l_{i_2} - l_{i_3} - \ .\ .\ .\ .\ - l_{i_n}} = r^{-k} \qquad (6.3)$$

where,

$$l_{i_1} + l_{i_2} + l_{i_3} + \ .\ .\ .\ .\ + l_{i_n} = k \qquad (6.4)$$

For a code,

Let the smallest possible length of a code-word be unity i.e. $l_i = 1$, Then minimum value of $'k'$ would be

$$k|_{min} = n \qquad (6.5)$$

and let the largest value of code-word be $'l'$ i.e. $l_i = l$, then the maximum value of $'k'$ would be

$$k|_{max} = nl \qquad (6.6)$$

If N_k denote the number of terms of the form r^k and also the number of code-words of length $'k'$, then we can rewrite equation

$$\left(\sum_{i=1}^{q} r^{-l_i}\right)^n = \sum_{k=n}^{nl} N_k r^{-k} \tag{6.7}$$

If a code is uniquely decodable then,

$$N_k \leq r^k \tag{6.8}$$

Using inequality relation of equation (6.8) in equation (6.7), we have

$$\left(\sum_{i=1}^{q} r^{-l_i}\right)^n \leq \sum_{k=n}^{nl} r^k r^{-k}$$

Simplifying above equation,

$$\left(\sum_{i=1}^{q} r^{-l_i}\right)^n \leq \sum_{k=n}^{nl} 1$$

$$\left(\sum_{i=1}^{q} r^{-l_i}\right)^n \leq \sum_{k=n-\boldsymbol{n+1}}^{nl-\boldsymbol{n+1}} 1$$

$$\left(\sum_{i=1}^{q} r^{-l_i}\right)^n \leq \sum_{k=1}^{nl-n+1} 1$$

$$\left(\sum_{i=1}^{q} r^{-l_i}\right)^n \leq nl - n + 1$$

where, $nl - n + 1 = n(l-1) + 1 \cong nl + 1 \cong nl$, thus

$$\left(\sum_{i=1}^{q} r^{-l_i}\right)^n \leq nl \tag{9}$$

Taking n^{th} roots on both sides of above inequality, we get

$$\sum_{i=1}^{q} r^{-l_i} \leq (nl)^{1/n} \qquad \text{for all n}$$

For large $'n'$, as $n \to \infty$, we have

$$\lim_{n\to\infty} (nl)^{1/n} = 1 \tag{10}$$

Thus

$$\sum_{i=1}^{q} r^{-l_i} \leq 1$$

Which is nothing but the inequality given in equation (6.1).

2M for Statement, 8M for Derivation = 10M

Total – 10 Marks

Q 7.

State and prove Shannon's noiseless coding theorem. What do you infer from it?

Answer:

Let a block code with source symbols $s_1, s_2, \dots, s_q$ be represented by code words $X_1, X_2, \dots, X_q$. Let the probabilities of the source symbols be $p_1, p_2, \dots, p_q$ and the lengths of the code words be $p_1, p_2, \dots, p_q$. Let $H_r(S)$ represent entropy of $r-ary$ representation of the source. Then the average length can be made as close to entropy $H_r(S)$ by coding the n^{th} extension of source S i.e. S^n, rather than S.

Proof:

Length of the symbol can be related to probability as

$$l_i \approx \log_r\left(\frac{1}{p_i}\right) \qquad (7.1)$$

Thus,

- If $\log_r(1/p_i)$ is an integer, we should choose the word length l_i equal to this integer.
- If $\log_r(1/p_i)$ is not an integer, it might seem reasonable that a code could be found by selecting l_i as the first integer greater than this value.

Then, we may select l_i which has an integer value between

$$\log_r\left(\frac{1}{p_i}\right) \le l_i \le \log_r\left(\frac{1}{p_i}\right) + 1 \qquad (7.2)$$

Checking if the above lengths satisfy Kraft – McMillan inequality, since

$$\log_r\left(\frac{1}{p_i}\right) \le l_i \qquad (7.3)$$

Then

$$\frac{1}{p_i} \le r^{l_i}$$

Or

$$p_i \ge r^{-l_i}$$

Summing for all symbols, we have

$$\sum_{i=1}^{q} p_i \ge \sum_{i=1}^{q} r^{-l_i}$$

Thus

$$1 \ge \sum_{i=1}^{q} r^{-l_i} \qquad or \qquad \sum_{i=1}^{q} r^{-l_i} \le 1$$

Hence it satisfies Kraft – McMillan inequality.

Now considering equation (7.3) and multiplying it by p_i, we have

$$p_i \log_r\left(\frac{1}{p_i}\right) \le p_i l_i \le p_i \log_r\left(\frac{1}{p_i}\right) + p_i \quad (7.4)$$

Equation (7.4) which holds good for all the symbols, summing up for the symbols we have

$$\sum_{i=1}^{q} p_i \log_r\left(\frac{1}{p_i}\right) \le \sum_{i=1}^{q} p_i l_i \le \sum_{i=1}^{q} p_i \log_r\left(\frac{1}{p_i}\right) + \sum_{i=1}^{q} p_i \quad (7.5)$$

Where,

$$\sum_{i=1}^{q} p_i \log_r\left(\frac{1}{p_i}\right) = H_r(S)$$

and

$$\sum_{i=1}^{q} p_i l_i = L$$

Using above relations in equation (7.5), we have

$$H_r(S) \le L \le H_r(S) + 1 \quad (7.6)$$

Since equation (7.6) is valid for zero-memory source, we can apply it to n^{th} extension as well, thus

$$H_r(S^n) \le L_n \le H_r(S^n) + 1 \quad (7.7)$$

where, L_n indicates the average length on n^{th} extension of zero-memory source, we also know that

$$H_r(S^n) = nH_r(S)$$

Thus equation (7.7) can be rewritten as

$$nH_r(S) \le L_n \le nH_r(S) + 1 \quad (7.8)$$

Dividing equation (7.8) by $'n'$, we have

$$\boldsymbol{H_r(S) \le \frac{L_n}{n} \le H_r(S) + \frac{1}{n}} \quad (7.9)$$

Where, L_n/n indicates the average length of one zero-memory source symbol in an n^{th}extension symbol. So it is possible to make L_n/n as close to $H_r(S)$ as we wish by coding the n^{th} extension of S rather than S. Thus

$$\lim_{n\to\infty} \boldsymbol{\frac{L_n}{n} = H_r(S)} \quad (7.10)$$

Equation (7.9) is known as Shannon’s First Theorem or Shannon’s Noiseless Coding Theorem.

Inference:

- Equation (7.9), tells us that we can make average length of r−ary code symbols per source symbol as small as, but no smaller than the entropy of source measured in r-ary units.
- Equation (7.10), tell us that we can make average length as close to entropy $H_r(S)$ by coding the n^{th}extension of source S rather than S.
- Equation (7.10) also tell us that the price that we pay for decreasing L_n/n is the increased coding complexity caused by the large number (q^n) of source symbols.

2M for Statement, 6M for Derivation, 2M for Inference = 10M

Total - 10 Marks

Q 8.

For a discrete memoryless source with source alphabet $S = \{A, B, C\}$ and with probabilities $P = \{0.1, 0.5, 0.4\}$:

i. Construct binary–Huffman code. Find its efficiency.
ii. Construct binary–Huffman code for second extension of discrete memoryless source. Find its efficiency.
iii. Verify Shannon's noiseless coding theorem using result of 8(i) and 8(ii)

Solution:

Given Source alphabet $S = \{A, B, C\}$

Probability of symbols $P = \{0.1,0.5,0.4\}$

Thus, Entropy of Source is,

$$H(S) = \sum_{i=1}^{3} p_i \log_2\left(\frac{1}{p_i}\right)$$

$$H(S) = 0.1 * \log_2\left(\frac{1}{0.1}\right) + 0.5 * \log_2\left(\frac{1}{0.5}\right) + 0.4 * \log_2\left(\frac{1}{0.4}\right)$$

$$\therefore H(S) = 0.33219 + 0.5 + 0.52877 = 1.361\ bits/symbol$$

i. Binary – Huffman code representation of source S are obtained in Table 8.1

Table 8.1: Binary representation for symbols of source S

Symbol (s_i)	Code	Probabilities (p_i)	Stage 1	Length (l_i)
B	0	0.5	0.5 (0)	1
C	10	0.4 (0)	**0.5 (1)**	2
A	11	0.1 (1)		2

Hence source $S = \{A, B, C\}$ is 90.73 % efficient.

Average length of above code is

$$L = \sum_{i=1}^{3} p_i l_i = 0.5 * 1 + 0.4 * 2 + 0.1 * 2 = 1.5\ bits/symbol$$

Hence efficiency of coding is

$$\eta_c = \frac{H(S)}{L} = \frac{1.361}{1.5} = 0.9073$$

0.5M each for H(S) & L; 1M each for Table 8.1 & η_c = 3M

ii. The second extension source symbols are obtained in Table 8.2

Table 8.2: Probability of each symbol of second extension source

Second Extension Symbol	Probability
AA	0.01
AB	0.05
AC	0.04
BA	0.05
BB	0.25
BC	0.2
CA	0.04
CB	0.2
CC	0.16

Entropy of second extension source is

$$H(S^2) = 2.H(S) = 2.722\ bits/symbol$$

Thus Huffman representation of second extension of source S is:

$$\alpha = \frac{9-2}{2-1} = 7\ stages$$

Binary representation of symbols of second extension of source S of Table 8.2 are obtained using Huffman – encoding procedure as in Table 8.3.

Average length of second extension of source is

$$L_2 = \sum_{i=1}^{9} \sigma_i l_i$$

$$L_2 = 0.25 * 2 + 2 * 0.2 * 2 + 0.16 * 3 + 2 * 0.05 * 5 + 0.04 * 5 + 0.04 * 6 + 0.01 * 6$$

$$L_2 = 2.78\ bits/symbol$$

Table 8.2: Binary representation for symbols of second extension of source S

σ_i	Code	$P(\sigma_i)$	Stage 1	Stage 2	Stage 3	Stage 4	Stage 5	Stage 6	Stage 7	l_i
BB	01	0.25	0.25	0.25	0.25	0.25	**0.35**	**0.4**	**0.6 (0)**	2
BC	10	0.2	0.2	0.2	0.2	0.2	0.25	0.35 (0)	0.4 (1)	2
CB	11	0.2	0.2	0.2	0.2	0.2	0.2 (0)	0.25 (1)		2
CC	001	0.16	0.16	0.16	0.16	**0.19 (0)**	0.2 (1)			3
AB	00000	0.05	0.05	**0.09**	**0.1 (0)**	0.16 (1)				5
BA	00001	0.05	0.05	0.05 (0)	0.09 (1)					5
AC	00011	0.04	**0.05 (0)**	0.05 (1)						5
CA	000100	0.04 (0)	0.04 (1)							6
AA	000101	0.01 (1)								6

Hence efficiency of second extension of source is

$$\eta_c^2 = \frac{H(S^2)}{L_2} = \frac{2.722}{2.78} = 0.97914$$

Thus the efficiency of second extension of source is 97.914 %

0.5M each for H(S^2) & L_2; 1M each for Table 8.2 & η_c^2; 3M for Table 8.3 = 6M

iii. According to Shannon's Coding Theorem,

Average length of the code reaches close to entropy if coding is carried out for n^{th} extension of source rather than source itself.

Here,

For source S, efficiency is 90.73 %,

while for second extension of source S, i.e. S^2, efficiency is 97.914%

1M for inference = 1M

Total – 10 Marks