CMR
INSTITUTE OF
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Internal Assesment Test – II

Sub:	Information Theory and Coding		Sec	ECE 5C & 5D ;TCE 5A & 5B				Code:	15EC54
Date:	09 / 11 /17	Duration:	90 mins	Max Marks:	50	Sem:	V	Branch:	ECE/TCE

	ANSWER ANY FIVE FULL QUESTIONS	MARKS	OBE CO	OBE RBT
1	For the joint probability matrix given, calculate $H(X), H(Y), H(X,Y), H(X\|Y), H(Y\|X)$ and $I(X;Y)$, if $P(X) = \left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}$ $$\text{P(Y\|X)} = \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0.2 & 0.8 \end{bmatrix}$$	[10]	C504.3	L3
2	Derive an expression for the channel capacity of a binary erasure channel.	[10]	C504.3	L3
3	The noise characteristic of a channel is as shown in Fig. Q3 below. Obtain the channel matrix and calculate the channel capacity.	[10]	C504.3	L3

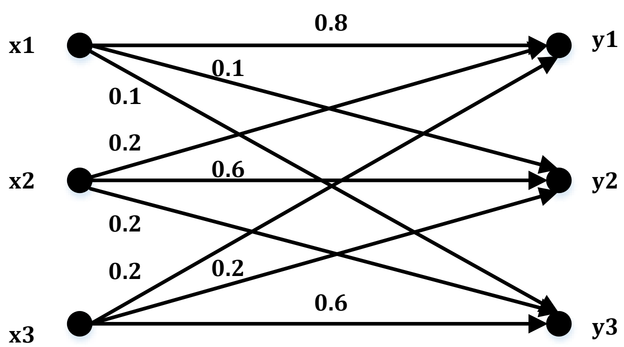


Fig. Q3

4 Prove the following properties of mutual information, — C504.3 L1

a) $I(X,Y) = I(Y,X)$. [05]

b) $I(X,Y) = H(X) + H(Y) - H(X,Y)$. [05]

5 For a systematic (6, 3) linear block code, the parity matrix is given by — C504.4 L2

$$P = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

a) Find all possible valid code vectors. [03]

b) Draw the corresponding encoding circuit. [02]

c) A single error has occurred in each of these received vectors. Detect and correct those errors. $R_A = [011111]$ and $R_B = [111001]$. [02]

d) Draw the syndrome calculation circuit. [03]

6 Construct a standard array for (6, 3) codes namely, (000000), (001110), (010011), (011101), (100101), (101011), (110110) and (111000). Let the received codeword be [011011]. Decode this codeword using this standard array and obtain the correct sequence. [10] C504.4 L2

P.T.O.

7 The parity check bits of a (7, 4) Hamming code are generated using [10] C504.4 L3

$$c_5 = d_1 + d_3 + d_4$$
$$c_6 = d_1 + d_2 + d_3$$
$$c_7 = d_2 + d_3 + d_4$$

Obtain the generator matrix $[G]$ [02]

Obtain the parity check matrix $[H]$ [02]

From the obtained matrices show that $GH^T = 0$ [02]

Find the minimum distance of the code and calculate the error detecting and error correcting capability of the code. [04]

8 Consider a linear block code whose generator matrix is C504.4 L3

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

a) Find all the code vectors [03]

b) Find all the hamming weights and minimum distance [03]

c) Obtain the parity check matrix [02]

d) Draw the encoding circuit. [02]

Solution for IAT-2

1 For the joint probability matrix calculate $H(X)$, $H(Y)$, $H(X,Y)$, $H(X|Y)$, $H(Y|X)$ and $I(X;Y)$ if $P(X) = \{1/3, 1/3, 1/3\}$

$$P(Y|X) = \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0.2 & 0.8 \end{bmatrix}$$

Sol. Given $P(X) = \{p(x_1), p(x_2), p(x_3)\} = \{1/3, 1/3, 1/3\}$

W.K.T. $P(X,Y) = P(Y|X)\,P(X)$

$$\therefore P(X,Y) = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & \frac{8}{30} & \frac{2}{30} & 0 \\ x_2 & \frac{1}{30} & \frac{8}{30} & \frac{1}{30} \\ x_3 & 0 & \frac{2}{30} & \frac{8}{30} \end{array} \Rightarrow \begin{aligned} p(y_1) &= \frac{9}{30} \\ p(y_2) &= \frac{12}{30} \\ p(y_3) &= \frac{9}{30} \end{aligned}$$

W.K.T. $P(X,Y) = P(X|Y)\,P(Y) \Rightarrow P(X|Y) = \dfrac{P(X,Y)}{P(Y)}$

$$\therefore P(X|Y) = \begin{bmatrix} \frac{8}{9} & \frac{2}{12} & 0 \\ \frac{1}{9} & \frac{8}{12} & \frac{1}{9} \\ 0 & \frac{2}{12} & \frac{8}{9} \end{bmatrix}$$

$$H(X) = \sum_{i=1}^{3} p(x_i)\log_2 \frac{1}{p(x_i)} = 3\left(\frac{1}{3}\log_2 3\right) = 1.585 \text{ bits/sym}$$

$$H(Y) = \sum_{j=1}^{3} p(y_j)\log_2 \frac{1}{p(y_j)} = \frac{9}{30}\log_2\frac{30}{9} + \frac{12}{30}\log\frac{30}{12} + \frac{9}{30}\log_2\frac{30}{9} = 1.571 \text{ bits/sym}$$

$$H(X,Y) = \sum_{i=1}^{3}\sum_{j=1}^{3} p(x_i,y_j)\log_2\frac{1}{p(x_i,y_j)} = 3\left(\frac{8}{30}\log_2\frac{30}{8}\right) + 2\left(\frac{2}{30}\log_2\frac{30}{2}\right) + 2\left(\frac{1}{30}\log_2 30\right) = 2.374 \text{ bits/sym}$$

$$H(X|Y) = \sum_{i=1}^{3}\sum_{j=1}^{3} p(x_i,y_j)\log_2\frac{1}{p(x_i|y_j)} = 2\left(\frac{8}{30}\log_2\frac{9}{8}\right) + 2\left(\frac{2}{30}\log_2\frac{12}{2}\right) + 2\left(\frac{1}{30}\log_2\frac{9}{1}\right) + \frac{8}{30}\log_2\frac{12}{8} = 0.803 \text{ bits/sym}$$

$$H(Y|X) = \sum_{j=1}^{3}\sum_{i=1}^{3} p(x_i,y_j)\log_2\frac{1}{p(y_j|x_i)} = 3\left(\frac{8}{30}\log_2\frac{1}{0.8}\right) + 2\left(\frac{2}{30}\log_2\frac{1}{0.2}\right) + 2\left(\frac{1}{30}\log_2\frac{1}{0.1}\right) = 0.789 \text{ bits/sym}$$

$$I(X;Y) = H(X) - H(X|Y) = 0.782 \text{ bits/sym}$$

4. Derive an expression for channel capacity of binary erasure channel

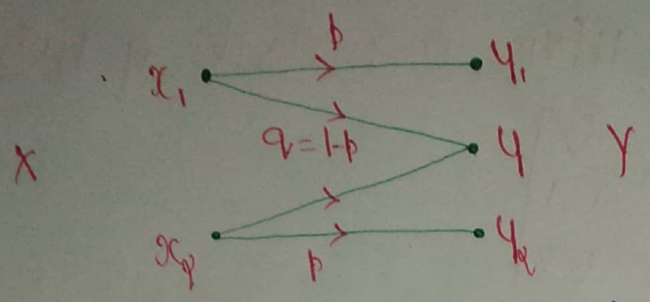


Fig: Binary Erasure channel

The channel matrix for BEC is

$$P(Y|X) = \begin{array}{c c} & \begin{array}{ccc} y_1 & y & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} p & q & 0 \\ 0 & q & p \end{bmatrix} \end{array}$$

Assuming $p(x_1) = \omega$.

$p(x_2) = \bar{\omega}$

$P(X,Y) = P(Y|X) \cdot P(X)$

we have

$$P(X,Y) = \begin{bmatrix} p\omega & q\omega & 0 \\ 0 & q\bar{\omega} & p\bar{\omega} \end{bmatrix} \Rightarrow \begin{array}{l} p(y_1) = p\omega \\ p(y) = q\omega + q\bar{\omega} = q \\ p(y_2) = p\bar{\omega} \end{array}$$

$$P(X|Y) = \frac{P(X,Y)}{P(Y)}$$

$$P(X|Y) = \begin{bmatrix} 1 & \omega & 0 \\ 0 & \bar{\omega} & 1 \end{bmatrix}$$

$$H(X) = \sum_{i=1}^{2} p(x_i) \log_2 \frac{1}{p(x_i)} \text{ bits/sym}$$

$$H(X) = \omega \log_2 \frac{1}{\omega} + \bar{\omega} \log_2 \frac{1}{\bar{\omega}} \text{ bits/sym}$$

$$H(X|Y) = \sum_{i=1}^{3} \sum_{\forall j} p(x_i, y_j) \log_2 \frac{1}{p(x_i|y_j)}$$

$$H(X|Y) = p\omega \log_2 1 + q\omega \log_2 \frac{1}{\omega} + q\bar{\omega} \log_2 \frac{1}{\bar{\omega}} + p\bar{\omega} \log_2 1$$

$$= q\left(\omega \log_2 \frac{1}{\omega} + \bar{\omega} \log_2 \frac{1}{\bar{\omega}}\right)$$

$H(X|Y) = q\,H(X)$ bits/sym

$$I(X;Y) = H(X) - H(X|Y) = \left[\omega \log_2 \frac{1}{\omega} + \bar{\omega} \log_2 \frac{1}{\bar{\omega}}\right] - q\left[\omega \log_2 \frac{1}{\omega} + \bar{\omega} \log_2 \frac{1}{\bar{\omega}}\right]$$

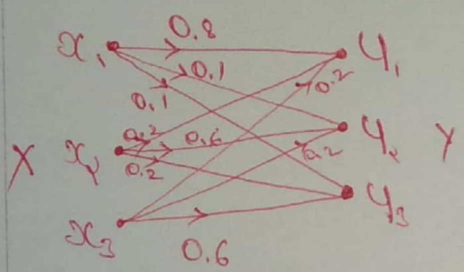
$$= (1-q)\,H(X)$$

$I(X;Y) = p(H(X))$ bits/sym

∴ The channel capacity of BEC is

$$C = \max\{I(X;Y)\}$$
$$= \max\{p\,H(X)\}$$
$$= p\,H(X)_{max}$$
$$\boxed{C = p \text{ bits/sec}} \qquad \because H(X)_{max} = \log_2 m = \log_2 2 = 1$$

3. The noise characteristic of a channel is as shown in Fig Q3. Obtain the channel matrix and calculate capacity of the channel.



$$\Rightarrow \quad P(Y|X) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}\begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = \begin{bmatrix} 0.8\log_2 1/0.8 + 0.1\log_2 1/0.1 + 0.1\log_2 1/0.1 \\ 0.2\log_2 1/0.2 + 0.6\log_2 1/0.6 + 0.2\log_2 1/0.2 \\ 0.2\log_2 1/0.2 + 0.2\log_2 1/0.2 + 0.6\log_2 1/0.6 \end{bmatrix}$$

$$0.8Q_1 + 0.1Q_2 + 0.1Q_3 = 0.922$$
$$0.2Q_1 + 0.6Q_2 + 0.2Q_3 = 1.371$$
$$0.2Q_1 + 0.2Q_2 + 0.6Q_3 = 1.371$$

Solving for Q_1, Q_2 and Q_3

we have $Q_1 = 0.7723$, $Q_2 = 1.5207$, $Q_3 = 1.5207$

$$C = \log_2\left[2^{-Q_1} + 2^{-Q_2} + 2^{-Q_3}\right] r_s$$
$$= \log_2\left[2^{-0.7723} + 2^{-1.5207} + 2^{-1.5207}\right] r_s$$
$$\boxed{C = 0.3598 \text{ bits/sec}}$$

4a. Prove the properties of mutual information

$I(X;Y) = I(Y;X)$

Proof: W.k.t. $H(X) = \sum_{i=1}^{n} p(x_i) \log_2 \frac{1}{p(x_i)}$ bits/sym $\rightarrow$ (1)

$$H(X|Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} p(x_i, y_j) \log_2 \frac{1}{p(x_i|y_j)} \text{ bits/sym} \rightarrow (2)$$

$$\sum_{j=1}^{m} p(y_j|x_i) = 1 \rightarrow (3)$$

$$I(X;Y) = H(X) - H(X|Y) \rightarrow (4)$$

Since R.H.S. of equation 3 is unity, eq (1) can be written as

$$H(X) = \sum_{i=1}^{n} p(x_i) \log_2 \frac{1}{p(x_i)} \cdot \sum_{j=1}^{m} p(y_j|x_i)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} p(y_j|x_i)\, p(x_i) \log_2 \frac{1}{p(x_i)}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} p(x_i, y_j) \log_2 \frac{1}{p(x_i)} \rightarrow (5)$$

Substituting eq 2 and 5 in 4

$$I(X;Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} p(x_i, y_j) \log_2 \frac{p(x_i|y_j)}{p(x_i)} \rightarrow (6)$$

W.k.t. $\frac{p(x_i|y_j)}{p(x_i)} = \frac{p(y_j|x_i)}{p(y_j)} \rightarrow (7)$

Substituting eq^n 7 in 6

$$I(X;Y) = \sum_{j=1}^{m} \sum_{i=1}^{n} p(x_i, y_j) \log_2 \frac{p(y_j|x_i)}{p(y_j)}$$

$$\boxed{I(X;Y) = I(Y;X)}$$

6. $I(X;Y) = H(X) + H(Y) - H(X,Y)$

w.k.t

Proof $H(X,Y) = \sum_{i=1}^{n}\sum_{j=1}^{m} p(x_i,y_j) \log_2 \frac{1}{p(x_i,y_j)} \rightarrow ①$

multiply and divide eq^n ① throughout by $p(x_i) \cdot p(y_j)$

$$H(X,Y) = \sum_{i=1}^{n}\sum_{j=1}^{m} p(x_i,y_j) \log \frac{p(x_i)\cdot p(y_j)}{p(x_i,y_j)} + \sum_{i=1}^{n}\sum_{j=1}^{m} p(x_i,y_j) \log_2 \frac{1}{p(x_i)p(y_j)}$$

where

$$\sum_{i=1}^{n}\sum_{j=1}^{m} p(x_i,y_j) \log_2 \frac{1}{p(x_i)p(y_j)} = \sum_{j=1}^{m} p(x_i,y_j) \sum_{i=1}^{n} \log_2 \frac{1}{p(x_i)} + \sum_{j=1}^{m} p(x_i,y_j) \sum_{j=1}^{m} \log_2 \frac{1}{p(y_j)}$$

$$= \sum_{i=1}^{n} p(x_i) \log_2 \frac{1}{p(x_i)} + \sum_{j=1}^{m} p(y_j) \log_2 \frac{1}{p(y_j)}$$

$$= H(X) + H(Y)$$

$\therefore\ H(X,Y) = -I(X;Y) + H(X) + H(Y)$

$$\Rightarrow \boxed{I(X;Y) = H(X) + H(Y) - H(X,Y)}$$

5. For a systematic (6,3) linear block code, the parity matrix is given by

$$P = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

a. The codewords are given by

$[C] = [D][G]$

$$[C] = [d_1\ d_2\ d_3]_{1\times3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]_{3\times6}$$

$[C] = [d_1\ d_2\ d_3\ (d_2+d_3)\ (d_1+d_3)\ (d_1+d_2)]$

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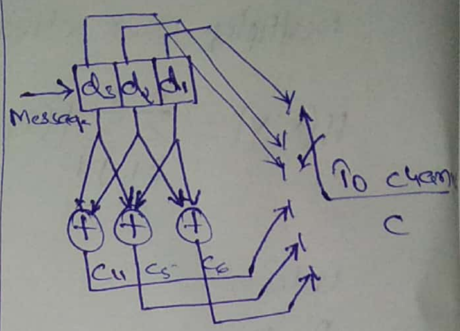
a.

Message d_1	d_2	d_3	Codeword d_1 (c_1)	d_2 (c_2)	d_3 (c_3)	(d_2+d_3) (c_4)	(d_1+d_3) (c_5)	(d_1+d_2) (c_6)
0	0	0	0	0	0	0	0	0
0	0	1	0	0	1	1	1	0
0	1	0	0	1	0	1	0	1
0	1	1	0	1	1	0	1	1
1	0	0	1	0	0	0	1	1
1	0	1	1	0	1	1	0	1
1	1	0	1	1	0	1	1	0
1	1	1	1	1	1	0	0	0

b. Encoding circuit



c. $R_A = [011111]$

$S_A = R_A H^T$

$$S_A = [011111]\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\\1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$S_A = [100]$

$E_A = 000100$

$R_A = 011111$

$C_A = 011011$

$R_B = [111001]$

$S_B = R_B H^T$

$$S_B = [111001]\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\\1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$S_B = [001]$

$E_B = 000001$

$R_B = 111001$

$C_B = 111000$

Syndrome calculation circuit

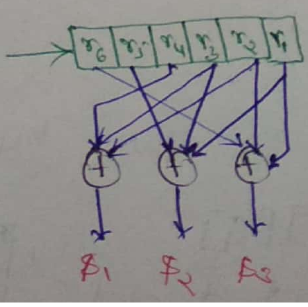
$R = [r_1\ r_2\ r_3\ r_4\ r_5\ r_6]$

$S = [R][H]^T$

$$= [r_1\ r_2\ r_3\ r_4\ r_5\ r_6]\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\\1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$= \Big[\underset{s_1}{(r_2+r_3+r_4)}\ \underset{s_2}{(r_1+r_3+r_5)}\ \underset{s_3}{(r_1+r_2+r_6)}\Big]$$

Received vector



6. Construct a standard array lookuptable

000000	001110	010011	011101	100101	101011	110110	111000
100000	101110	110011	111101	000101	001011	010110	011000
010000	011110	000011	001101	110101	111011	100110	101000
001000	000110	011011	010101	101101	100011	111110	110000
000100	001010	010111	011001	100001	100011	110010	111100
000010	001100	010001	011111	100111	101001	110100	111010
000001	001111	010010	011100	100100	101010	110111	111001

Given $R = 011011 \Rightarrow E = 001000$

$\therefore$ $C = 010011$ which is a valid codeword for $\underline{D = 010}$

7. The parity check bits of a (7,4) Hamming code are generated using

$$c_5 = d_1 + d_3 + d_4$$

$$c_6 = d_1 + d_2 + d_3$$

$$c_7 = d_2 + d_3 + d_4$$

The generator matrix

$$G = \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]_{4\times7}$$

Parity check matrix

$$H^T = \left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]_{7\times3}$$

$$GH^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{4\times3}$$

Code words for all 16 combination

$d_{min} = 3$

error detection capability $= d_{min} - 1 = 2$

Error correction capability $= \frac{d_{min} - 1}{2} = 1$

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8 Consider a linear block code whose generator matrix is

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Sol a. From the given G we know that $(n,k) = (6,3)$

$C = DG$

$= \underset{c_1}{d_1} \; \underset{c_2}{d_2} \; \underset{c_3}{d_3} \; \underset{c_4}{(d_1+d_2)} \; \underset{c_5}{(d_2+d_3)} \; \underset{c_6}{(d_1+d_3)}$

Msg i/p			Code word						
d_1	d_2	d_3	c_1	c_2	c_3	c_4	c_5	c_6	Hw
0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	1	0	1	1	3
0	1	0	0	1	0	1	1	0	3
0	1	1	0	1	1	1	0	1	4
1	0	0	1	0	0	1	0	1	3
1	0	1	1	0	1	1	1	0	4
1	1	0	1	1	0	0	1	1	4
1	1	1	1	1	1	0	0	0	3

d.

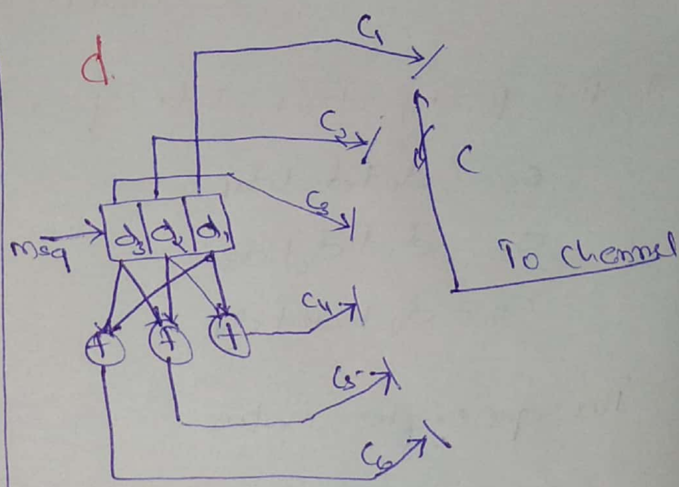

Fig: Encoding Circuit

b Minimum distance = $Hw_{min} = 3$

c $$H^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$