CMR INSTITUTE OF TECHNOLOGY

USN | | | | | | | | | |

Improvement Test									
Sub:	ENGINEERING ELECTROMAGNETICS							Code:	15EC36
Date:	20/11 / 2017	Duration:	90 mins	Max Marks:	50	Sem:	3rd	Branch:	ECE
Answer FIVE FULL Questions									

		Marks	OBE CO	OBE RBT
1.	Starting from Maxwell's equations obtain the general expression for electric and magnetic fields for TEM waves.	[10]	CO4	L3
2.	Explain TEM wave propagation in a good conductor. Define skin depth.	[10]	CO4	L3
3.	State and explain Poynting's theorem. Derive the expression for Poynting vector.	[10]	CO4	L3
4.(a)	A 100 MHz uniform plane wave propagates in a lossless medium for which $\varepsilon_r = 5$ and $\mu_r = 1$. Find: (i) v_p; (ii) β; (iii) λ; (iv) **Es**; (v)**Hs**; (vi) <**S**> .	[7]	CO4	L3
(b)	Define wavelength and phase velocity.	[3]	CO4	L1
5.	Derive point form of Maxwell's equation of electrostatics.	[10]	CO1	L3
6.(a)	Given $\mathbf{E} = 4xyz\ \mathbf{a_x} + 4x^2(z+1)\ \mathbf{a_y} + 4xy\ \mathbf{a_z}$ .Compute Electric flux density and volume charge density at point (1,1,1).	[6]	CO2	L3
(b)	Derive an expression for the work done in moving a point charge Q in the presence of an electric field **E**.	[4]	CO1	L3
7.	Calculate the work done in moving a 2 μC charge from A (2,1,-1) to B (8,2,1) in electric field $\mathbf{E} = y\ \mathbf{a_x} + x\ \mathbf{a_y}$: i. Along parabola $x = 2y^2$. ii. Along a straight line $x = 6y - 4$.	[10]	CO1	L3
8.(a)	Define current and current density. Derive the equation of continuity of current.	[7]	CO3	L1
(b)	Derive the expression for potential difference between two points at radial distances r_a and r_b from a point charge placed at the origin.	[3]	CO1	L3
9.	Find the total charge in a volume defined by the six planes for which $1 \le x \le 2$, $2 \le y \le 3$, $3 \le z \le 4$ if $\mathbf{D} = 4x\ \mathbf{a_x} + 3y^2\ \mathbf{a_y} + 2z^3\ \mathbf{a_z}$ C/m^2. Verify the result using Gauss's divergence theorem.	[10]	CO1	L3

Scheme of evaluation
Improvement Test, Nov. 2017

1. Maxwell's equations: 3 m
Expression for electric field: 5 m
Expression for magnetic field: 2 m.

2. Explanation and derivation: 8 m.
Definition of skin depth: 2m.

3. Statement and explanation: 3 m
Derivation for Poynting vector: 7m

4. (a) Phase velocity: 1m
β : 2m
λ : 1m.
E_x : 1m
H_y : 1m
$\langle S \rangle$: 1m -

4. (b) Definition of wavelength: $1\frac{1}{2}$ m.
Definition of phase velocity: $1\frac{1}{2}$ m

5. Derivation: 8m.
Final expression: 2m.

6. (a) Electric flux density : 2m
Volume charge density : 3 m.

6. (b) Derivation : 4m.
Final expression : 1m.

7. (i) Approach : 3m.
Final answer : 2m.
(ii) Approach : 3m
Final answer : 2m.

8. (a) Definition of current : $1\frac{1}{2}$ m
Definition of current-density : $1\frac{1}{2}$ m.
Derivation of eqn. of continuity of current : 4m.

8. (b) Derivation and final expression : 3m.

9. Surface integral : 5m.
Volume integral : 5m.

1) General Wave equation for TEM waves:

General Wave equation (for the medium considered) of UPW.

Maxwell's equations:

$$\vec{\nabla}\cdot\vec{D}=0 \longrightarrow \vec{\nabla}\cdot\vec{E}=0 \rightarrow ①$$

$$\vec{\nabla}\times\vec{E}=-\mu\frac{\partial \vec{H}}{\partial t} \Longrightarrow \vec{\nabla}\times\vec{E}=-\mu\frac{\partial \vec{H}}{\partial t} \rightarrow ②$$

$$\vec{\nabla}\cdot\vec{B}=0 \longrightarrow \vec{\nabla}\cdot\vec{H}=0 \rightarrow ③$$

$$\vec{\nabla}\times\vec{H}=\sigma\vec{E}+\varepsilon\frac{\partial \vec{E}}{\partial t} \Longrightarrow \vec{\nabla}\times\vec{H}=\sigma\vec{E}+\varepsilon\frac{\partial \vec{E}}{\partial t} \rightarrow ④$$

Take curl on both sides of ②

$$\underline{\vec{\nabla}\times\vec{\nabla}\times\vec{E}}=-\vec{\nabla}\times\left(\mu\frac{\partial \vec{H}}{\partial t}\right).$$

Vector identity $\rightarrow \vec{\nabla}\times\vec{\nabla}\times\vec{E}=\vec{\nabla}(\vec{\nabla}\cdot\vec{E})-\nabla^2\vec{E}$

$$\vec{\nabla}(\cancel{\vec{\nabla}\cdot\vec{E}})-\nabla^2\vec{E}=-\mu\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{H}).$$

apply equation ① Apply equation ④

$$-\nabla^2\vec{E}=-\mu\frac{\partial}{\partial t}\left[\sigma\vec{E}+\varepsilon\frac{\partial \vec{E}}{\partial t}\right]$$

$$-\nabla^2\vec{E}=-\mu\sigma\frac{\partial \vec{E}}{\partial t}-\mu\varepsilon\frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2\vec{E}=\mu\sigma\frac{\partial \vec{E}}{\partial t}+\mu\varepsilon\frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\boxed{\frac{\partial^2\vec{E}}{\partial z^2}=\mu\sigma\frac{\partial \vec{E}}{\partial t}+\mu\varepsilon\frac{\partial^2 \vec{E}}{\partial t^2}} \rightarrow ⑤$$

General wave equation for $\vec{E}$ field of UPW.

For UPW $\vec{E}(z,t)$ and $\vec{H}(z,t)$

$$\nabla^2\vec{E}=\frac{\partial^2\vec{E}}{\partial z^2}$$

Similarly,

Take curl on both sides of ④

$$\vec{\nabla}\times\vec{\nabla}\times\vec{H}=\vec{\nabla}\times\left(\sigma\vec{E}+\varepsilon\frac{\partial \vec{E}}{\partial t}\right)$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{H})-\nabla^2\vec{H}=\sigma(\vec{\nabla}\times\vec{E})+\varepsilon\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E})$$

applying equation ③ applying equation ②

$$-\nabla^2\vec{H}=-\sigma\cdot\mu\frac{\partial \vec{H}}{\partial t}-\varepsilon\mu\frac{\partial^2 \vec{H}}{\partial t^2}$$

$$\boxed{\nabla^2\vec{H} = \mu\sigma\frac{\partial \vec{H}}{\partial t} + \mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2}}$$

and $\vec{H}(z,t)$

$$\nabla^2\vec{H} = \frac{\partial^2\vec{H}}{\partial z^2}$$

$$\boxed{\frac{\partial^2\vec{H}}{\partial z^2} = \mu\sigma\frac{\partial\vec{H}}{\partial t} + \mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2}} \rightarrow (6)$$

$\Rightarrow$ General wave equation for $\vec{H}$ field of UPW.

For free space equations (5) and (6) becomes.

$\sigma = 0$

$\varepsilon = \varepsilon_0$

$\mu = \mu_0$

Wave equation for free space for $\vec{E}$ field

$$(5) \Rightarrow \boxed{\frac{\partial^2\vec{E}}{\partial z^2} = \mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}} \quad (\text{or}) \quad \boxed{\frac{\partial^2\vec{E}}{\partial t^2} = \frac{1}{\mu_0\varepsilon_0}\frac{\partial^2\vec{E}}{\partial z^2}}$$

Wave equation for free space for $\vec{H}$-field

$$(6) \Rightarrow \boxed{\frac{\partial^2\vec{H}}{\partial z^2} = \mu_0\varepsilon_0\frac{\partial^2\vec{H}}{\partial t^2}} \quad (\text{or}) \quad \boxed{\frac{\partial^2\vec{H}}{\partial t^2} = \frac{1}{\mu_0\varepsilon_0}\frac{\partial^2\vec{H}}{\partial z^2}}$$

where

$$v_p = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = 3\times10^8 = c$$

= velocity of light (or) EM waves in free space.

$$\frac{1}{\sqrt{\mu_0\varepsilon_0}} = \frac{1}{\sqrt{4\pi\times10^{-7}\times\frac{10^{-9}}{36\pi}}} = \frac{1}{\sqrt{\frac{10^{-16}}{9}}} = \sqrt{9\times10^{+16}} = 3\times10^8\ \text{m/s}.$$

Wave equations for free space can be written as

$$\boxed{\frac{\partial^2\vec{E}}{\partial z^2} = \frac{1}{v_p^2}\frac{\partial^2\vec{E}}{\partial t^2}} \quad (\text{or}) \quad \boxed{\frac{\partial^2\vec{E}}{\partial t^2} = v_p^2\frac{\partial^2\vec{E}}{\partial z^2}}$$

and

$$\boxed{\frac{\partial^2\vec{H}}{\partial z^2} = \frac{1}{v_p^2}\frac{\partial^2\vec{H}}{\partial t^2}} \quad (\text{or}) \quad \boxed{\frac{\partial^2\vec{H}}{\partial t^2} = v_p^2\frac{\partial^2\vec{H}}{\partial z^2}}$$

2) Wave propagation in Good conductors:

For good conductors,

$\sigma \simeq \infty$.

$\frac{\sigma}{\omega\epsilon} >> 1$

$j\gamma = j\omega\sqrt{\mu\epsilon'}\sqrt{1 - j\frac{\sigma}{\omega\epsilon'}}$

$j\gamma = j\omega\sqrt{\mu\epsilon'}\sqrt{-j\frac{\sigma}{\omega\epsilon'}}$

$j\gamma = j\omega\sqrt{\mu\epsilon'}\sqrt{\frac{\sigma}{\omega\epsilon'}}\sqrt{-j}$

$\sqrt{-j} = \sqrt{1\angle -90^\circ}$

$= 1\angle -45^\circ$

$\sqrt{-j} = \frac{1-j}{\sqrt{2}}$

$j\gamma = j\sqrt{\omega\mu\sigma} \cdot \sqrt{-j}$

$= j\sqrt{\omega\mu\sigma}\,\frac{(1-j)}{\sqrt{2}}$

$= \frac{j\sqrt{\omega\mu\sigma}}{\sqrt{2}} + \frac{\sqrt{\omega\mu\sigma}}{\sqrt{2}}$

$j\gamma = \sqrt{\frac{\omega\mu\sigma}{2}} + j\sqrt{\frac{\omega\mu\sigma}{2}} = \alpha + j\beta$

$$\boxed{\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f\mu\sigma}}$$

Solution of wave equation:

$\vec{E} = \vec{E_s}\, e^{j\omega t}$

$\vec{E} = \left(E_{x0}\, e^{-j\gamma z} e^{j\omega t} + \underbrace{E_{x0}'\, e^{j\gamma z} e^{j\omega t}}_{\text{Backward wave}}\right)\vec{a_x}$

Consider only forward wave

$\vec{E} = E_{x0}\, e^{-\alpha z} \cdot e^{-j\beta z} e^{j\omega t}$

$$\boxed{\vec{E} = E_{x0}\, e^{-\alpha z}\cos(\omega t - \beta z)\,\vec{a_x}}$$

$\vec{H} = \frac{E_{x0}}{\eta}\, e^{-\alpha z}\cos(\omega t - \beta z)\,\vec{a_y}$ $\quad \vec{H} = \frac{E_{x0}}{|\eta|\angle\theta_\eta}\, e^{-\alpha z}\cos(\omega t - \beta z)\,\vec{a_y}$.

$$\boxed{\vec{H} = \frac{E_{x0}}{\sqrt{\frac{\omega\mu}{\sigma}}}\, e^{-\alpha z}\cos(\omega t - \beta z - 45^\circ)\,\vec{a_y}}$$

As $\vec{E}$ (or $\vec{H}$) travels in a conducting medium, its amplitude is attenuated by a factor $e^{-\alpha z}$.

The distance through which the amplitude of wave decreases to a factor e^{-1} (about 37% of its original value) is called skin depth or depth of penetration of the medium.

$$E_0 e^{-\alpha z} = E_0 e^{-1}$$

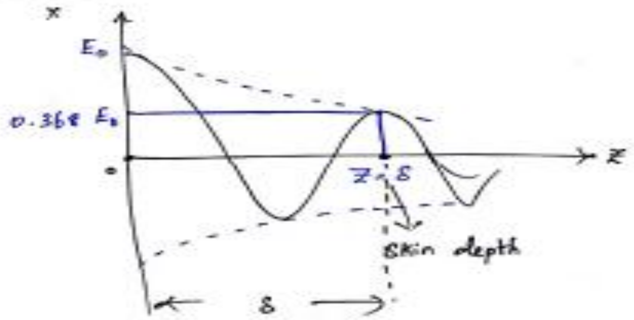
$z = \delta \leftarrow$ depth of penetration

$$e^{-\alpha \delta} = e^{-1}$$

$$\alpha \delta = 1$$

$$\boxed{\delta = \frac{1}{\alpha} = \frac{1}{\sqrt{\pi f \mu \sigma}}}$$

Skin depth is the measure of the depth to which an EM wave can penetrate the medium



3) Poynting's theorem and Wave power:

Poynting's theorem.

It states that the net power flowing out of a given volume is equal to the time rate of decrease in stored energy within the volume minus conduction losses.

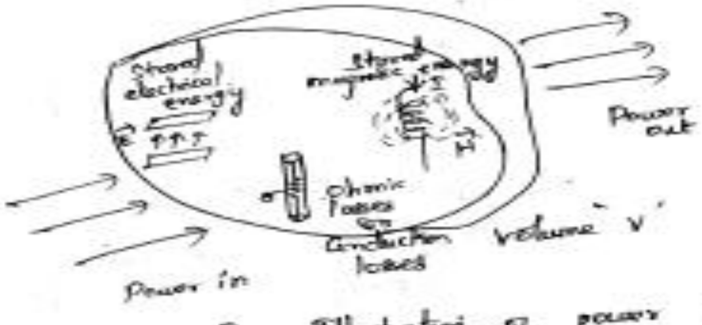


Fig. Illustration of power balance for EM fields

Proof:

Maxwell's equations

$$\vec{\nabla}\cdot\vec{D} = \rho_v$$

$$\vec{\nabla}\times\vec{E} = -\mu\frac{\partial \vec{H}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{H} = \sigma\vec{E} + \epsilon\frac{\partial \vec{E}}{\partial t}$$

Vector identity:

$$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = \vec{H}\cdot\vec{\nabla}\times\vec{E} - \vec{E}\cdot\vec{\nabla}\times\vec{H}$$

Apply Maxwell's equations into this vector identity,

$$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = -\vec{H}\cdot\mu\frac{\partial \vec{H}}{\partial t} - \vec{E}\cdot\left[\sigma\vec{E} + \epsilon\frac{\partial \vec{E}}{\partial t}\right]$$

$$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = -\mu\,\vec{H}\cdot\frac{\partial \vec{H}}{\partial t} - \sigma E^2 - \epsilon\,\vec{E}\cdot\frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = -\sigma E^2 - \epsilon\,\vec{E}\cdot\frac{\partial \vec{E}}{\partial t} - \mu\,\vec{H}\cdot\frac{\partial \vec{H}}{\partial t}$$

$\vec{E}\cdot\frac{\partial \vec{E}}{\partial t} = ?$ Let $\frac{\partial \vec{E}^2}{\partial t} = 2\vec{E}\cdot\frac{\partial \vec{E}}{\partial t}$

$\vec{H}\cdot\frac{\partial \vec{H}}{\partial t} = ?$

$$\Rightarrow \boxed{\vec{E}\cdot\frac{\partial \vec{E}}{\partial t} = \frac{1}{2}\frac{\partial \vec{E}^2}{\partial t}}$$

$||^{ly}$

$$\frac{\partial H^2}{\partial t} = 2\vec{H}\cdot\frac{\partial \vec{H}}{\partial t}$$

$$\Rightarrow \boxed{\vec{H}\cdot\frac{\partial \vec{H}}{\partial t} = \frac{1}{2}\frac{\partial H^2}{\partial t}}$$

Point form.

$$\boxed{\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = -\sigma E^2 - \frac{1}{2}\cdot\epsilon\frac{\partial E^2}{\partial t} - \frac{1}{2}\mu\frac{\partial H^2}{\partial t}}$$

Integrating over the given volume,

$$\iiint_V \vec{\nabla}\cdot(\vec{E}\times\vec{H})\,dv = -\iiint_V \sigma E^2\,dv - \frac{1}{2}\iiint_V \epsilon\frac{\partial E^2}{\partial t}\,dv - \frac{1}{2}\iiint_V \mu\frac{\partial H^2}{\partial t}\,dv$$

Apply Divergence theorem

Divergence theorem, $\iiint_V (\vec{\nabla}\cdot\vec{A})\,dv = \oint\!\!\oint_S \vec{A}\cdot\vec{ds}$

∴ The equation becomes Poynting theorem.

$$\boxed{\underbrace{\oint\!\!\oint_S (\vec{E}\times\vec{H})\cdot\vec{ds}}_{\text{net power flowing out of the volume}} = \underbrace{-\sigma\iiint_V E^2 dv}_{\text{Ohmic losses or Conduction losses}} \underbrace{-\frac{d}{dt}\left[\frac{1}{2}\iiint_V \epsilon E^2 dv\right]}_{\text{Rate of decrease of stored electric energy}} \underbrace{-\frac{d}{dt}\left[\frac{1}{2}\iiint_V \mu H^2 dv\right]}_{\text{Rate of decrease of stored magnetic energy}}}$$

Hence proved.

w.k.t.

Electric potential energy $\boxed{W_E = \frac{1}{2}\iiint_V \epsilon E^2 dv}$

and

Magnetic potential energy $\boxed{W_H = \frac{1}{2}\iiint_V \mu H^2 dv}$

Power flow of an electromagnetic wave

$$\boxed{P = \oint\!\!\oint_S (\vec{E}\times\vec{H})\cdot\vec{ds}} \quad W$$

where $\boxed{\vec{E}\times\vec{H} = \vec{P}}$ = Poynting vector = Power density vector

or

$\boxed{\vec{E}\times\vec{H} = \vec{S}}$ = Poynting vector (W/m^2).

4)a)

18

$f = 100$ MHz, $\epsilon_r = 5$, $\mu_r = 1$

(a) $v_p = \frac{\omega}{\beta}$

now (b) $\beta = \omega\sqrt{\mu\epsilon'} = 2\pi\times 100\times 10^6 \sqrt{4\pi\times 10^{-7}\times 1\times 5\times 8.854\times 10^{-12}}$

$= 2\pi\times 10^8 \times 74.56\times 10^{-10}$

$= 4.68\ m^{-1}$

∴ (a) $v_p = \frac{\omega}{\beta} = \frac{2\pi\times 10^8}{4.68} = 1.34\times 10^8$ m/s.

(c) $\lambda = \frac{2\pi}{\beta} = \frac{2\times 3.14}{4.68} = 1.34$ m.

(d) $\vec{E}_x = E_0 \exp(-j\beta z)\,\hat{a}_x$

$E_0 \rightarrow$ real amplitude, forward

z - travelling wave.

Also x - polarized.

As phasor form, no time term

$$\eta = \sqrt{\frac{\mu}{\epsilon'}} = \sqrt{\frac{\mu_0 \mu_r}{\epsilon_0 \epsilon_r'}} = \frac{\eta_0}{\sqrt{\epsilon_r'}} \quad [\because \mu_r = 1]$$

$$= \frac{377}{\sqrt{5}} = 169\ \Omega.$$

(c) $\vec{H}_s = \frac{E_0}{\eta} \exp(-j\beta z)\hat{a}_y = \frac{E_0}{169} \exp(-j4.69z)\hat{a}_y$ A/m

(d) $\langle S \rangle = \frac{1}{2} \mathrm{Re}\{\vec{E}_s \times H_s^*\}$

$$= \frac{1}{2} \mathrm{Re} \frac{E_0^2}{337} \hat{a}_z \ \mathrm{W/m^2}.$$

4)b)

Phase Velocity:

The phase velocity of a wave is the rate at which the phase of the wave propagates in space.

Wavelength:

Wavelength is the distance over which the wave's shape repeats.

5) Derive point form of Maxwell's equation of electrostatics.

Application of Gauss' law: (31)

Differential volume element.

No symmetry.

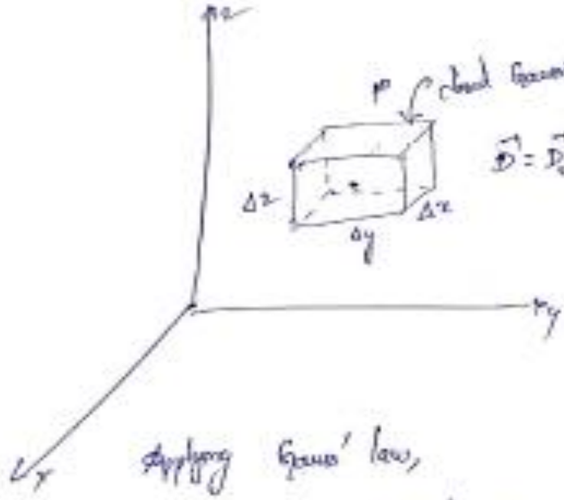
Choose a small gaussian surface → almost $\vec{D}$ is constant over that surface.

Result becomes correct only when volume $\Delta v \to 0$ (shrinks).

We will not obtain $\vec{D}$, obtain the valuable information about the way $\vec{D}$ varies in the region

⇓

one of Maxwell's four equations (base to all electromagnetic theory).



$\vec{D} = \vec{D}_0 = D_{x0}\vec{a}_x + D_{y0}\vec{a}_y + D_{z0}\vec{a}_z$.

Applying Gauss' law,

$$\oint_S \vec{D} \cdot d\vec{s} = Q$$

$$\int_{front} + \int_{back} + \int_{left} + \int_{Right} + \int_{top} + \int_{bottom} \vec{D}\cdot\vec{ds} = q.$$

Surface element is very small → $\vec{D}$ is constant over the surface.

$$\int_{front} \vec{D}\cdot\vec{ds} = \vec{D}_{front} \cdot \vec{\Delta S}_{front}$$

$$= \vec{D}_{front} \cdot \Delta y \Delta z \, \vec{a_x}$$

$$= D_{x\,front} \cdot \Delta y \Delta z$$

front face → distance of $\frac{\Delta x}{2}$ from P.

$$\boxed{D_{x,front} = D_{x0} + \frac{\Delta x}{2} \cdot \frac{\partial D_x}{\partial x}}$$

Rate of change of D_x with x

∴ D_x varies with y & z.

value of D_x at P.

↓

Expansion of Taylor series.

$$\boxed{\int_{front} = \left(D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x}\right) \Delta y \Delta z}$$

Integral over the back surface

$$\int_{back} = \vec{D}_{back} \cdot \Delta \vec{S}_{back}$$

$$= \vec{D}_{back} \cdot (-\Delta y \Delta z \, \vec{a_x})$$

$$\int_{back} = -D_{x,back} \, \Delta y \Delta z.$$

$$\int_{back} = -\left[D_{x0} + \left(-\frac{\Delta x}{2}\right)\frac{\partial D_x}{\partial x}\right] \Delta y \Delta z$$

$$\boxed{\int_{back} = \left(-D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x}\right) \Delta y \Delta z}$$

$$\boxed{\int_{front} + \int_{back} = \frac{\partial D_x}{\partial x} \Delta x \Delta y \Delta z}$$

Exactly in the same way

$$\int_{right} + \int_{left} = \frac{\partial D_y}{\partial y} \Delta x \Delta y \Delta z$$

$$\int_{top} + \int_{bottom} = \frac{\partial D_z}{\partial z} \Delta x \Delta y \Delta z$$

$\therefore \oint_S \vec{D}\cdot d\vec{s} = \frac{\partial D_x}{\partial x}\Delta x\Delta y\Delta z + \frac{\partial D_y}{\partial y}\Delta x\Delta y\Delta z + \frac{\partial D_z}{\partial z}\Delta x\Delta y\Delta z$ (5)

$$\boxed{\oint_S \vec{D}\cdot d\vec{s} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)\Delta x\Delta y\Delta z = Q}$$

It is an approximation which is better when Δv becomes smaller

$\therefore \Delta v \to 0$. where $\Delta v = \Delta x\Delta y\Delta z$

$$\boxed{\text{Charge enclosed in volume } \Delta v = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)\Delta v}$$

We can obtain the exact relationship by $\underline{\Delta v \to 0}$.

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) = \frac{\oint \vec{D}\cdot d\vec{s}}{\Delta v} = \frac{Q}{\Delta v}$$

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) = \lim_{\Delta v\to 0}\frac{\oint_S \vec{D}\cdot d\vec{s}}{\Delta v} = \left(\lim_{\Delta v\to 0}\frac{Q}{\Delta v}\right) \to \text{volume charge density}$$

$$\boxed{\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \lim_{\Delta v\to 0}\frac{\oint \vec{D}\cdot d\vec{s}}{\Delta v}} \to (1)$$

&

$$\boxed{\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho_v} \to (2)$$

Divergence of $\vec{D}$ = $\operatorname{div}\vec{D} = \lim_{\Delta v\to 0}\frac{\oint_S \vec{D}\cdot d\vec{s}}{\Delta v}$

> Divergence of the vector flux density $\vec{A}$ is the outflow of flux from a small closed surface per unit volume as the volume shrinks to zero.

(7) MAXWELL'S FIRST EQUATION OF ELECTROSTATICS:

1) $\operatorname{div}\vec{D} = \lim_{\Delta v\to 0}\frac{\oint \vec{D}\cdot d\vec{s}}{\Delta v}$ (Definition of divergence)

2) $\operatorname{div}\vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$ (Result of applying the definition to a differential volume element (In Rectangular co-ordinates)

3) $\operatorname{div}\vec{D} = \rho_v$

Gauss's law, flux leaving any closed surface

$$\boxed{\oint_S \vec{D}\cdot d\vec{s} = Q} \leftarrow \text{charge enclosed}$$

Gauss' law per unit volume, $\frac{\oint_S \vec{D}\cdot d\vec{s}}{\Delta v} = \frac{Q}{\Delta v}$

As the volume shrinks to zero,

$$\lim_{\Delta v \to 0} \frac{\oint \vec{D}\cdot d\vec{s}}{\Delta v} = \lim_{\Delta v \to 0} \frac{Q'}{\Delta v}$$ ← volume charge density

→ divergence

$$\boxed{\operatorname{div} \vec{D} = \rho_v.}$$

⇓ First of Maxwell's four equations

Point form of Gauss's law (or) Maxwell's first equation

Statement:

The electric flux per unit volume leaving a vanishingly small volume unit is exactly equal to the volume charge density there.

Gauss law → $\boxed{\oint_s \vec{D}\cdot d\vec{s} = Q = \int_v \rho_v \, dv}$

Integral form of Maxwell's first equation

6)a)
Problem:

$$\vec{D} = \varepsilon_0 \vec{E} = \varepsilon_0 \left(4xyz\,\vec{a_x} + 4x^2(z+1)\,\vec{a_y} + 4xy\,\vec{a_z} \right)$$

(i) $\boxed{\vec{D}\Big|_{P(1,1,1)} = \varepsilon_0 \left[4\vec{a_x} + 8\vec{a_y} + 4\vec{a_z} \right]}$ C/m² — $\boxed{\vec{D} = 35.416\,\vec{a_x} + 70.832\,\vec{a_y} + 35.416\,\vec{a_z}}$ pC/m²

$$\rho_v = \vec{\nabla}\cdot\vec{D} = \varepsilon_0 \left[\frac{\partial}{\partial x}(4xyz) + \frac{\partial}{\partial y}\left(4x^2(z+1)\right) + \frac{\partial}{\partial z}(4xy) \right]$$

$$\boxed{\rho_v = \varepsilon_0 \cdot 4yz} \text{ C/m}^3$$

(ii) $\boxed{\rho_v \text{ at } P(1,1,1) = 35.416 \text{ pC/m}^3}$

6)b)

Energy expended in moving a point charge in an electric field:

$\vec{E}$ ← force per unit test charge at P.

→ Move the test charge against $\vec{E}$, we require force equal & opposite to the field.

↓

requires expend energy (or) do work.

If we wish to Move a charge Q, a distance $\vec{dL}$ in an electric field $\vec{E}$.

Force on Q due to $\vec{E}$, $\boxed{\vec{F_E} = Q\vec{E}}$

force arises from electric field.

The component of this force in the direction $\vec{dL}$,

$F_{EL} = \vec{F_E} \cdot \vec{a_L}$ ← unit vector in the direction of $\vec{dL}$

$$\boxed{F_{EL} = Q\vec{E} \cdot \vec{a_L}}$$

The force we must apply is equal and opposite to F_{EL}.

$$\boxed{F_{appl} = -Q\vec{E} \cdot \vec{a_L}}$$

Energy expended = Force × distance

Differential work done by an external source moving charge Q is

$$dW = -Q\vec{E} \cdot \vec{a_L} \cdot dL$$

$$\boxed{dW = -Q\vec{E} \cdot \vec{dL}}$$

Work required to move a charge a finite distance,

$$\boxed{W = -Q\int_{initial}^{final} \vec{E} \cdot \vec{dL}}$$

← Path must be specified

Charge is assumed at rest at both its initial & final position

7)

Determine work done in carrying a 2-μc charge from $A(2,1,-1)$ to $B(8,2,-1)$ in the field $\vec{E} = y\,\vec{a_x} + x\,\vec{a_y}$ along

a) parabola $x = 2y^2$

b) Hyperbola $x = \frac{8}{7-3y}$

c) straight line $x = 6y - 4$

Solution:

a) $W = -Q\int_A^B \vec{E}\cdot\vec{dL} \Rightarrow W = -2\mu\left\{\int_2^8 y\,dx + \int_1^2 x\,dy\right\}$

$x = 2y^2$

$y = \sqrt{\frac{x}{2}}$

$W = -2\mu\left[\int_2^8 \sqrt{\frac{x}{2}}\,dx + \int_1^2 2y^2 dy\right]$

$\boxed{W = -28\,\mu J}$

b) $W = -Q\int_A^B \vec{E}\cdot\vec{dL} \Rightarrow W = -2\mu\left\{\int_2^8 \left(\frac{7}{3} - \frac{8}{3x}\right)dx + \int_1^2 \left(\frac{8}{7-3y}\right)dy\right\}$

$x = \frac{8}{7-3y}$

$7 - 3y = 8/x$

$y = \frac{7}{3} - \frac{8}{3x}$

$\boxed{W = -28\,\mu J}$

c) $W = -Q\int_A^B \vec{E}\cdot\vec{dL} \Rightarrow W = -2\mu\left\{\int_1^2 (6y-4)\,dy + \int_2^8 \left(\frac{x+4}{6}\right)dx\right\}$

$x = 6y - 4$

$y = \frac{x+4}{6}$

$\boxed{W = -28\,\mu J}$

8)a)

Current and Current Density:

Electric charges in motion $\rightarrow$ current (Ampere (A)).

Rate of movement of charge passing a given reference point (or crossing a reference plane).

$\Downarrow$

$1\ c/s = 1$ Ampere.

$\boxed{I = \frac{dQ}{dt}}$ $\leftarrow$ movement of positive charges

Current density → vector → $\vec{J}$ A/m²

Increment of current ΔI, crossing on incremental surface ΔS, normal to the current density

$$\Delta I = J_N \cdot \Delta S$$

If $\vec{J}$ is not ⊥ to surface

$$\Delta I = \vec{J} \cdot \vec{\Delta S}$$

Total current

$$\boxed{I = \int_S \vec{J} \cdot \vec{ds}}$$

8)b)

Potential difference between points A and B at radial distance r_A and r_B from a point charge Q at origin.

$$\vec{E} = E_r \vec{a_r} = \frac{Q}{4\pi\epsilon_0 r^2} \vec{a_r}$$

$$\vec{dL} = dr\, \vec{a_r}$$

$$V_{AB} = -\int_B^A \vec{E} \cdot \vec{dL} = -\int_{r=r_B}^{r_A} \frac{Q}{4\pi\epsilon_0 r^2} \cdot dr$$

$$= -\frac{Q}{4\pi\epsilon_0} \left[-\frac{1}{r} \right]_{r_B}^{r_A}$$

$$\boxed{V_{AB} = \frac{Q}{4\pi\epsilon_0} \left[\frac{1}{r_A} - \frac{1}{r_B} \right]} \text{ volts}$$

if $r_B > r_A \Rightarrow V_{AB}$ is +ve

↓

Energy is expended by the external source in bringing the positive charge from r_B to r_A.

Potential at A, $V_A = \dfrac{Q}{4\pi\epsilon_0 r_A}$

at B, $V_B = \dfrac{Q}{4\pi\epsilon_0 r_B}$

$$\boxed{V_{AB} = V_A - V_B = \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{r_A} - \frac{1}{r_B} \right) V}$$

9)

Find the total charge in the volume defined by six planes

$1 \le x \le 2$
$2 \le y \le 3$ if $\vec{D} = 4x\vec{a_x} + 3y^2\vec{a_y} + 2z^3\vec{a_z}$ C/m^2
$3 \le z \le 4$

$$\vec{\nabla}\cdot\vec{D} = \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(3y^2) + \frac{\partial}{\partial z}(2z^3)$$

$\rho_v = \vec{\nabla}\cdot\vec{D}$.

$Q_{enc} = \int_v (\vec{\nabla}\cdot\vec{D})\,dv$.

$$\rho_v = 4 + 6y + 6z^2$$

$$Q_{enc} = \iiint (4 + 6y + 6z^2)\,dx\,dy\,dz$$

$$= \iiint_v 4\,dx\,dy\,dz + \iiint_v 6y\,dx\,dy\,dz + \iiint 6z^2\,dx\,dy\,dz$$

$$= 4\times1\times1\times1 + \frac{6}{2}(1)(9-4)(1) + \frac{6}{3}(1)(1)(4^3-3^3)$$

$$= 4 + 15 + 74$$

$$\boxed{Q_{enc} = 93\text{ C}}$$

18) Surface integral: $\vec{D} = 4x\vec{a_x} + 3y^2\vec{a_y} + 2z^3\vec{a_z}$ C/m^2

$1 \le x \le 2,\ 2 \le y \le 3,\ 3 \le z \le 4.$

$$\oint_S \vec{D}\cdot\vec{ds} = -\iint_S 4x\,dy\,dz\Big|_{x=1} + \iint_S 4x\,dy\,dz\Big|_{x=2} - \iint_S 3y^2\,dx\,dz\Big|_{y=2} + \iint_S 3y^2\,dx\,dz\Big|_{y=3}$$

$$- \iint_S 2z^3\,dx\,dy\Big|_{z=3} + \iint_S 2z^3\,dx\,dy\Big|_{z=4}$$

$$= -4[y]_2^3[z]_3^4 + 4[2][y]_2^3[z]_3^4 - 3(2)^2[x]_1^2[z]_3^4 + 3(3)^2[x]_1^2[z]_3^4$$

$$- 2(3)^3[x]_1^2[y]_2^3 + 2(4)^3[x]_1^2[y]_2^3$$

$$= -4 + 8 - 12 + 27 - 54 + 128$$

$$\boxed{\oint_S \vec{D}\cdot\vec{ds} = 93\text{ C}}$$