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INSTITUTE OF
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Internal Assesment Test - II

Sub	ENGINEERING MATHEMATICS III (REGULAR)							Code	15MAT31
Date	02 / 11 / 2016	Duration	90 mins	Max Marks	50	Sem	3	Branch	EEE

First Question(8 marks) is compulsory. Answer any six from the rest

Question	Marks	CO	RBT
1.Find the Fourier transform of the function $f(x)=\begin{cases}1, & \lvert x\rvert \le a\\ 0 & \lvert x\rvert > a\end{cases}$ where a is a positive constant. Hence evaluate $\int_{-\infty}^{\infty}\frac{\sin(ax)\cos(\alpha x)}{x}\,dx$ e^{-ax}	[8]	CO2	L3
2.Using the Fourier sine transform of e , a>0,x>0 show that $\int_0^{\infty}\frac{x\sin mx}{a^2+x^2}dx=\frac{\pi}{2}e^{-am}, m>0$. Hence find the Fourier sine transform of $\frac{x}{a^2+x^2}$.	[7]	CO2	L3
3. Find the Z transform of $\cosh n\theta$, $\sinh n\theta$ and n^2a^{-n}.	[7]	CO2	L3
4.Find the inverse Z transforms of the following $a)\frac{1}{z}e^{1/z}$ b) $\frac{18z^2}{(2z-1)(4z+1)}$	[7]	CO2	L3
5.Solve the difference equation $u_{n+2}-2u_{n+1}+u_n=2^n, u_0=2, u_1=1$.	[7]	CO2	L3
6.In a partially destroyed laboratory record of correlation data, the following results only are available: Variance of x is 9. The regression equations are 8x-10y+66=0, 40x-18y=214. Find a) the mean values of x and y b) standard deviation of y c) the co-efficient of correlation between x and y.	[7]	CO4	L3
7.If θ is the acute angle between the two regression lines relating the variables x and y, show that $\tan\theta=\frac{1-r^2}{r}\frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}$ Indicate the significance of the cases $r=0$ and $r=\pm1$	[7]	CO4	L4
8.If $\vec{f}=(x^2-27)\hat{i}-6yz\hat{j}+8xz^2\hat{k}$ evaluate $\int_C \vec{f}.d\vec{r}$ from O(0,0,0) to P(1,1,1) along the straight line from O to A(1,0,0), A to B(1,1,0) and B to P.	[7]	CO5	L3
9. Verify Green's theorem for $\int_C (x^2-xy^3)dx+(y^2-2xy)dy$ where C is the square with vertices (0,0),(2,0),(2,2),(0,2).	[7]	CO5	L4

Cognitive level	KEYWORDS
L1	List, define, tell, describe, identify, show, label, collect, examine, tabulate, quote, name, who, when, where, etc.
L2	summarize, describe, interpret, contrast, predict, associate, distinguish, estimate, differentiate, discuss, extend
L3	Apply, demonstrate, calculate, complete, illustrate, show, solve, examine, modify, relate, change, classify, experiment, discover.
L4	Analyze, separate, order, explain, connect, classify, arrange, divide, compare, select, explain, infer.
L5	Assess, decide, rank, grade, test, measure, recommend, convince, select, judge, explain, discriminate, support, conclude, compare, summarize.

Course Outcomes		PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12
CO1	Evaluate the real form of the Fourier series for standard periodic and finite waveforms which has its applications in finding out the sum of infinite series using Dirichlet's conditions.	3	0	0	0	0	0	1	0	0	0	0	0
CO2	Apply integral expressions for the forward and inverse Fourier transform to a range of non-periodic waveforms such as rectangular, unit-step, sinusoidal and exponential decay functions and solve second order difference equations using Z transform and inverse Z transform.	3	0	0	0	1	0	1	0	0	0	0	0
CO3	Solve Brachistochrone, shortest distance, minimal surface area and hanging chain problems and to find the geodesics of known surfaces using Euler-Lagrange method.	3	0	0	0	0	0	1	0	0	0	0	0
CO4	Estimate the strength of the relationship between the variables using correlation coefficients and to express the relationship in the form of an equation using regression analysis.	3	0	0	0	1	0	1	0	0	0	0	0
CO5	Evaluate line and surface integrals using Green's, Stoke's and Gauss divergence theorems which have its application in computing the amount of work done, area and volume	3	0	0	0	1	0	1	0	0	0	0	0
CO6	Apply numerical techniques to perform various mathematical tasks such as solving equations, interpolation, integration and curve fitting.	3	0	0	0	1	0	1	0	0	0	0	0

PO1 - *Engineering knowledge*; PO2 - *Problem analysis*; PO3 - *Design/development of solutions*; PO4 - *Conduct investigations of complex problems*; PO5 - *Modern tool usage*; PO6 - *The Engineer and society*; PO7- *Environment and sustainability*; PO8 – *Ethics*; PO9 - *Individual and team work*; PO10 - *Communication*; PO11 - *Project management and finance*; PO12 - *Life-long learning*

Engineering Mathematics III
II Internals

1. Find the Fourier transform of

$$f(x) = \begin{cases} 1, & |x| \le a \\ 0, & |x| > a \end{cases}$$ where 'a' is a positive constant. Hence evaluate

$$\int_{-\infty}^{\infty} \frac{\sin(ax)\cos(\alpha x)}{x}\,dx$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\alpha x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} 1 \cdot e^{i\alpha x}\,dx$$ 1 M

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{i\alpha x}}{i\alpha} \right)_{-a}^{a} = \frac{1}{\sqrt{2\pi}} \frac{1}{i\alpha} \left(e^{ia\alpha} - e^{-ia\alpha} \right)$$

$$\frac{1}{\sqrt{2\pi}} \frac{1}{i\alpha} 2i\sin a\alpha$$

$$F(\alpha) = \sqrt{\frac{2}{\pi}} \frac{\sin(a\alpha)}{\alpha}$$ 3 M

Inverse Fourier transform is

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha) e^{-i\alpha x} d\alpha$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \frac{\sin(a\alpha)}{\alpha} e^{-i\alpha x} d\alpha$$

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(a\alpha)}{\alpha} \left[\cos(\alpha x) - i\sin(\alpha x)\right] d\alpha$$

$$f(x) = \frac{1}{\pi}\left[\underbrace{\int_{-\infty}^{\infty} \frac{\sin(a\alpha)\cos(\alpha x)}{\alpha} d\alpha}_{\text{I (even)}} - i\underbrace{\int_{-\infty}^{\infty} \frac{\sin(a\alpha)\sin(\alpha x)}{\alpha} d\alpha}_{\text{II (odd)}}\right]$$

2M

I integral exist. II integral vanishes

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin(a\alpha)\cos(\alpha x)}{\alpha} d\alpha$$

$$\int_0^\infty \frac{\sin(a\alpha)\cos(\alpha x)}{\alpha}\, d\alpha = \frac{\pi}{2} f(x)$$

$$= \frac{\pi}{2} \cdot 1 \quad \text{when } |x| < a$$

1M

$$\Rightarrow \int_0^\infty \frac{\sin(a\alpha)\cos(\alpha x)}{\alpha}\, d\alpha = \frac{\pi}{2}$$

1M

2. Using the Fourier sine transform of e^{-ax}, $a > 0$, $x > 0$ S.T $\int_0^\infty \frac{x \sin(mx)}{a^2 + x^2}\, dx = \frac{\pi}{2} e^{-am}$, $m > 0$.

Hence find the Fourier sine transform of $\frac{x}{a^2+x^2}$

Ans We K.T $F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(\alpha x)\, dx$ 1M

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin(\alpha x)\, dx$$

$$F_s(\alpha) = \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{(-a)^2 + \alpha^2} \left(-a \sin(\alpha x) - \alpha \cos(\alpha x) \right) \right]_0^\infty$$

$$F_s(\alpha) = \sqrt{\frac{2}{\pi}} \frac{\cancel{e^{-a x}}\ 1}{a^2+\alpha^2} (0+\alpha)$$

$$F_s(\alpha) = \sqrt{\frac{2}{\pi}} \frac{\alpha}{a^2+\alpha^2} \quad -(1)$$ 1M

Inverse Fourier sine transform is 1M

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(\alpha) \sin(\alpha x)\, d\alpha$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{\alpha}{a^2+\alpha^2} \sin(\alpha x)\, d\alpha$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{\alpha \sin(\alpha x)}{a^2+\alpha^2}\, d\alpha$$

$$\Rightarrow \int_0^\infty \frac{\alpha \sin(\alpha x)}{a^2+\alpha^2}\, d\alpha = \frac{\pi}{2} f(x)$$

or $$\int_0^\infty \frac{\alpha \sin(\alpha x)}{a^2+\alpha^2}\, d\alpha = \frac{\pi}{2} e^{-ax}$$

same as $$\int_0^\infty \frac{x \sin(mx)}{a^2+x^2}\, dx = \frac{\pi}{2} e^{-am}, \quad m>0 \quad -(2)$$ 1M

(5)

We K.T $F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(\alpha x)\, dx$

$F_s\left[\frac{x}{a^2+x^2}\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{x}{a^2+x^2} \sin(\alpha x)\, dx$ (2 M)

From (2) $= \sqrt{\frac{2}{\pi}}\, \frac{\pi}{2} e^{-a\alpha}$

$= \sqrt{\frac{\pi}{2}}\, e^{-a\alpha}$ (1 M)

3. Find the Z transform of $\cosh n\theta$, $\sinh n\theta$ and $n^2 a^{-n}$

$\cosh n\theta = \frac{e^{n\theta}+e^{-n\theta}}{2}$

$Z(\cosh n\theta) = Z\left(\frac{1}{2}\left(e^{n\theta}+e^{-n\theta}\right)\right)$

$= \frac{1}{2}\left[Z(e^{n\theta}) + Z(e^{-n\theta})\right]$

$= \frac{1}{2}\left[\frac{z}{z-e^{\theta}} + \frac{z}{z-e^{-\theta}}\right]$

$= \frac{z}{2}\left[\frac{1}{z-e^{\theta}} + \frac{1}{z-e^{-\theta}}\right]$

$$Z(\cosh n\theta) = \frac{z}{2}\left[\frac{z-e^{-\theta}+z-e^{\theta}}{(z-e^{\theta})(z-e^{-\theta})}\right]$$

$$Z(\cosh n\theta) = \frac{z\left(2z-(e^{\theta}+e^{-\theta})\right)}{2\left(z^2-z(e^{\theta}+e^{-\theta})+1\right)}$$

$$Z(\cosh n\theta) = \frac{z}{2}\,\frac{(2z-2\cosh\theta)}{z^2-2z\cosh\theta+1}$$

$$\boxed{Z(\cosh n\theta) = \frac{z(z-\cosh\theta)}{z^2-2z\cosh\theta+1}}$$

2M

$$Z(\sinh n\theta) = Z\left(\frac{1}{2}(e^{n\theta}-e^{-n\theta})\right)$$

$$= \frac{1}{2}\left[Z(e^{n\theta}) - Z(e^{-n\theta})\right]$$

$$= \frac{1}{2}\left[\frac{z}{z-e^{\theta}} - \frac{z}{z-e^{-\theta}}\right]$$

$$= \frac{z}{2}\left[\frac{(z-e^{-\theta})-(z-e^{\theta})}{(z-e^{\theta})(z-e^{-\theta})}\right]$$

$$Z(\sinh n\theta) = \frac{z}{2}\frac{(e^{\theta}-e^{-\theta})}{z^2-2z\cosh\theta+1}$$

$$= \frac{z}{2}\frac{2\sinh\theta}{z^2-2z\cosh\theta+1}$$

$$= \frac{z\sinh\theta}{z^2-2z\cosh\theta+1}$$ 2M

n^2a^{-n}

We K.T $Z(a^{-n}u_n) = \bar{u}(az)$ where

$$Z(a^{-n}n^2) = \bar{u}(az)$$

$$\bar{u}(z) = Z(u_n) = Z(n^2) = \frac{z(z+1)}{(z-1)^3}$$ 3M

$$\bar{u}(az) = \frac{az(az+1)}{(az-1)^3}$$

4. Find the inverse Z transforms of

a) $\frac{1}{z}e^{1/z}$

$$e^{1/z} = 1+\frac{(1/z)}{1!}+\frac{(1/z)^2}{2!}+\frac{(1/z)^3}{3!}+\cdots$$

$$= 1+\frac{z^{-1}}{1!}+\frac{(z^{-1})^2}{2!}+\frac{(z^{-1})^3}{3!}+\cdots$$

$$= \sum_{n=0}^{\infty}\frac{(z^{-1})^n}{n!}$$

$$\frac{1}{z} e^{1/z} = \sum_{n=0}^{\infty} \frac{(z^{-1})^{n+1}}{n!}$$

$$= \sum_{n=1}^{\infty} \frac{z^{-n}}{(n-1)!}$$

$$= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} z^{-n}$$

3M

$$u_n = \frac{1}{(n-1)!}$$

b) $\dfrac{18z^2}{(2z-1)(4z+1)}$

Let $\bar{u}(z) = \dfrac{18z^2}{(2z-1)(4z+1)} = \dfrac{18z^2}{8z^2\left(1-\frac{1}{2z}\right)\left(1+\frac{1}{4z}\right)}$

$$= \frac{9}{4} \frac{1}{(1-(2z)^{-1})(1+(4z)^{-1})}$$

$$= \frac{9}{4}\left[\frac{A}{1-(2z)^{-1}} + \frac{B}{1+(4z)^{-1}}\right]$$

$$\frac{1}{(1-(2z)^{-1})(1+(4z)^{-1})} = \frac{A}{1-(2z)^{-1}} + \frac{B}{1+(4z)^{-1}}$$

$$\bar{u}(z) = \frac{2}{3}\sum_{n=0}^{\infty} 2^{-n} z^{-n} + \frac{1}{3}\sum (-1)^n 4^{-n} z^{-n}$$ (9)

Ans $u_n = \left(\frac{2}{3} 2^{-n} + \frac{1}{3}(-1)^n 4^{-n}\right)\frac{9}{4}$

4M

$$= \frac{3}{2} 2^{-n} + \frac{3}{4}(-1)^n (4^{-n})$$

(or) Let $\bar{u}(z) = \dfrac{18 z^2}{(2z-1)(4z+1)}$

$$\frac{\bar{u}(z)}{z} = \frac{18 z}{(2z-1)(4z+1)}$$

$$\frac{18 z}{(2z-1)(4z+1)} = \frac{A}{2z-1} + \frac{B}{4z+1}$$

$$18 z = A(4z+1) + B(2z-1)$$

$$A - B = 0$$

$$4A + 2B = 18 \quad \text{or} \quad 2A + B = 9$$

$$A - B = 0 \qquad 3A = 9 \quad A = 3$$

$$2A + B = 9 \qquad B = 3$$

$$1 = A\left(1+(4z)^{-1}\right) + B\left(1-(2z)^{-1}\right)$$

$$A + B = 1 \quad - 1$$

Co-eff of z^{-1} $\quad \frac{1}{4}A - \frac{1}{2}B = 0$
$\quad - 2$

$$A + B = 1 \quad - 1$$

$$A - 2B = 0 \quad - 2$$

$$1 - 2 \quad 3B = 1 \quad \boxed{B = 1/3} \quad \boxed{A = \frac{2}{3}}$$

$$\cancel{\bar{u}(z) = \frac{2}{3}\frac{1}{(1-(2z)^{-1})} + \frac{1}{3(1+4z)^{-1}}}$$

$$\cancel{(1-2z)}$$

$$= \frac{2}{3}\left(1-(2z)^{-1}\right)^{-1} + \frac{1}{3}\left(1+(4z)^{-1}\right)^{-1}$$

$$\bar{u}(z) = \frac{2}{3}\left[1+(2z)^{-1}+(2z)^{-2}+(2z)^{-3}+\cdots\right]$$

$$\bar{u}(z) = \frac{2}{3}\left[1+(2z)^{-1}+(2z)^{-2}+(2z)^{-3}+\cdots\right]$$

$$+\frac{1}{3}\left[1-(4z)^{-1}+(4z)^{-2}-(4z)^{-3}+\cdots\right]$$

$$\bar{u}(z) = \frac{2}{3}\sum_{n=0}^{\infty}(2z)^{-n} + \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n(4z)^{-n}$$

$$\frac{\bar{u}(z)}{z} = \frac{3}{2z-1} + \frac{3}{4z+1}$$

$$\bar{u}(z) = \frac{3z}{2z-1} + \frac{3z}{4z+1}$$

$$\bar{u}(z) = \frac{3}{2}\frac{z}{(z-1/2)} + \frac{3}{4}\frac{z}{(z+1/4)}$$

$$\boxed{u_n = \frac{3}{2}(1/2)^n + \frac{3}{4}(-1/4)^n}$$ (4M)

5. Solve the difference equation

$$u_{n+2} - 2u_{n+1} + u_n = 2^n, \quad u_0 = 2, \; u_1 = 1$$

$$Z(u_{n+2} - 2u_{n+1} + u_n) = Z(2^n)$$

$$Z(u_{n+2}) - 2Z(u_{n+1}) + Z(u_n) = Z(2^n)$$

$$z^2\left\{\bar{u}(z) - u_0 - \frac{u_1}{z}\right\} - 2z\left\{\bar{u}(z) - u_0\right\} + \bar{u}(z) = \frac{z}{z-2}$$

$$(z^2 - 2z + 1)\,\bar{u}(z) - z^2(2) - z^2\left(\frac{1}{z}\right) + 2z(2) = \frac{z}{z-2}$$

12

$$(z^2-2z+1)\,\bar{u}(z) - 2z^2 - z + 4z = \frac{z}{z-2}$$

$$(z^2-2z+1)\,\bar{u}(z) - 2z^2 + 3z = \frac{z}{z-2}$$

$$(z^2-2z+1)\,\bar{u}(z) = \frac{z}{z-2} + 2z^2 - 3z$$

3M

$$\frac{\bar{u}(z)}{z} = \left[\frac{1}{z-2} + (2z-3)\right]\frac{1}{(z-1)^2}$$

$$\frac{\bar{u}(z)}{z} = \frac{1}{(z-2)(z-1)^2} + \frac{2z-3}{(z-1)^2} \quad \text{— (1)}$$

$$\frac{1}{(z-2)(z-1)^2} = \frac{A}{z-2} + \frac{B}{z-1} + \frac{C}{(z-1)^2}$$

$$1 = A(z-1)^2 + B(z-2)(z-1) + C(z-2)$$

$z=2$, $\quad 1 = A \quad$ or $\boxed{A=1}$

$z=1$, $\quad 1 = -C \quad$ or $\boxed{C=-1}$

$z=0$ $\quad 1 = A + 2B - 2C$

$$1 = 1 + 2B - 2(-1)$$

$$1 = 1 + 2B + 2 \quad \text{or} \quad 1 = 3 + 2B$$

$$2B = -2 \Rightarrow \boxed{B=-1}$$

$$\frac{2z-3}{(z-1)^2} = \frac{A}{z-1} + \frac{B}{(z-1)^2}$$

$$2z-3 = A(z-1)+B$$

$A = 2$ $\quad -A+B=-3$

$-2+B=-3$ $\quad \boxed{B=-1}$

$$\frac{\bar{u}(z)}{z} = \frac{1}{z-2} - \frac{1}{z-1} - \frac{1}{(z-1)^2} + \frac{2}{z-1} - \frac{1}{(z-1)^2}$$

$$\frac{\bar{u}(z)}{z} = \frac{1}{z-2} + \frac{1}{z-1} - \frac{2}{(z-1)^2}$$

$$\bar{u}(z) = \frac{z}{z-2} + \frac{z}{z-1} - 2\frac{z}{(z-1)^2}$$ 3M

1M

Taking inverse

$$u_n = 2^n + 1 - 2n$$

6. Variance of x is 9.

The regression eqns are

$8x - 10y + 66 = 0$, $40x - 18y = 214$

Find the a) mean values of x and y

b) S.D of y c) co-eff of correlation between x and y

The regression lines pass thro' $(\bar{x}\ \bar{y})$ (14)

$$8\bar{x} - 10\bar{y} = -66$$

$$40\bar{x} - 18\bar{y} = 240$$

2M

Solving $\bar{x} = 13,\ \bar{y} = 17.$

The regression eqns are rewritten as

$$10y = 8x + 66$$

$$y = \frac{8}{10}x + \frac{66}{10} \text{ or } \boxed{y = \frac{4}{5}x + \frac{33}{5}}$$

$$40x = 18y + 214$$

$$x = \frac{18}{40}y + \frac{214}{40}$$

$$\boxed{x = \frac{9}{20}y + \frac{107}{20}}$$

3M

$$b_{yx} = \frac{4}{5} \qquad b_{xy} = \frac{9}{20}$$

Co-eff of correlation between x & y is

$$r = \sqrt{b_{yx} b_{xy}} = \sqrt{\left(\frac{4}{5}\right)\left(\frac{9}{20}\right)}$$

$$\boxed{r = 0.6}$$

1M

given $\sigma_x^2 = 9 \Rightarrow \sigma_x = 3$

$$b_{yx} = r\frac{\sigma_y}{\sigma_x} = \frac{(0.6)}{3}\sigma_y$$

$$\frac{4}{5} = \frac{0.6}{3}\sigma_y$$

$$\sigma_y = \frac{4 \times 3}{5 \times 0.6} = 4$$

$$\boxed{\sigma_y = 4}$$

1M

7. If θ is the acute angle between the two regression lines relating the variables x and y, S.T $\tan\theta = \frac{1-r^2}{r} \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}$

Indicate the significance of the cases $r=0$ and $r = \pm 1$.

Ans $b_{yx} = r\frac{\sigma_y}{\sigma_x}$ and $\frac{1}{b_{xy}} = \frac{1}{r}\frac{\sigma_y}{\sigma_x}$ (2M)

are the slopes of the regression lines.

The angle θ between the regression lines is

$$\tan\theta = \frac{\frac{1}{r}\frac{\sigma_y}{\sigma_x} - r\frac{\sigma_y}{\sigma_x}}{1 + \left(\frac{1}{r}\frac{\sigma_y}{\sigma_x}\right)\left(r\frac{\sigma_y}{\sigma_x}\right)}$$

$$\tan\theta = \frac{1-r^2}{r}\frac{\sigma_y}{\sigma_x}\frac{1}{1+\sigma_y^2/\sigma_x^2}$$

$$\tan\theta = \frac{1-r^2}{r}\frac{\sigma_y}{\sigma_x}\frac{\sigma_x^2}{\sigma_x^2+\sigma_y^2}$$

(2M)

$$\tan\theta = \frac{1-r^2}{r}\frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}$$

$r=0 \Rightarrow \theta = \pi/2$ The regression lines are perpendicular to each other.

If $r = \pm 1$ we get $\theta = 0$ or π.

When $r = \pm 1$, the regression lines coincide.

The regression lines are $\perp^r$ when x & y are non-correlated and coincident when x & y are perfectly correlated. (3M)

8. If $\vec{f} = (x^2 - 27)\hat{i} - 6yz\hat{j} + 8xz^2\hat{k}$, evaluate $\int_C \vec{f} \cdot d\vec{r}$ from $O(0,0,0)$ to $P(1,1,1)$ along the st. line from O to $A(1,0,0)$, A to $B(1,1,0)$ and B to P.

$$\int_C \vec{f} \cdot d\vec{r} = \left(\int_{OA} + \int_{AB} + \int_{BP} \right) \vec{f} \cdot d\vec{r} \quad - (1)$$ (1M)

A lies on the x axis so that on the line segment OA we have $y = 0$ and $z = 0$ and $x \uparrow 0$ to 1.

$\vec{r} = x\hat{i}$ $\quad d\vec{r} = \hat{i}\,dx$

$$\vec{f} = (x^2 - 27)\hat{i}$$

$$\vec{f} \cdot d\vec{r} = (x^2 - 27)$$

$$\int_{OA} \vec{f} \cdot d\vec{r} = \int_{x=0}^{1} (x^2 - 27)\,dx = \left(\frac{x^3}{3} - 27(x)\right)_0^1$$

$$= -\frac{80}{3}$$

On AB, $x=1, z=0$ $y \uparrow 0$ to 1.

$\vec{f} = (1-27)\hat{i}$ $\quad \vec{r} = \hat{i} + y\hat{j}$ $\cancel{= dy}$

$d\vec{r} = dy\,\hat{j}$

$$\int_{AB} \vec{f}\cdot d\vec{r} = \int_{y=0}^{1} -26\,\hat{i}\cdot\hat{j}\,dy = 0$$

Along BP $\quad x=1, y=1 \quad z \uparrow 0$ to 1

$\vec{r} = \hat{i} + \hat{j} + z\hat{k} \quad d\vec{r} = \hat{k}\,dz$

$$\int_{BP} \vec{f}\cdot d\vec{r} = \int_{z=0}^{1} \left\{(1-27)\hat{i} - 6z\hat{j} + 8z^2\hat{k}\right\}\cdot\left\{\hat{k}\,dz\right\}$$

$$= \int_{z=0}^{1} 8z^2\,dz = 8/3$$ 5M

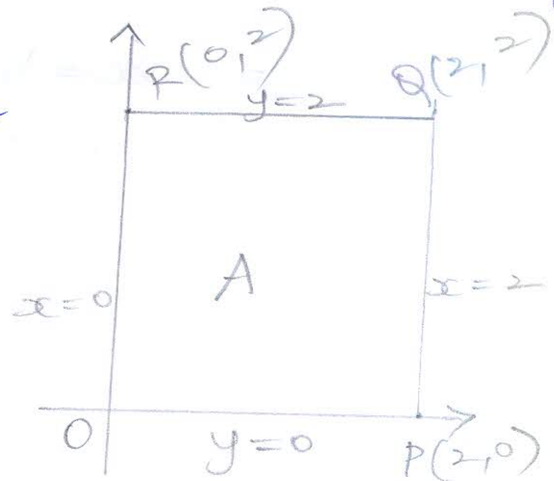
$$\int_C \vec{f}\cdot d\vec{r} = -\frac{80}{3} + 0 + \frac{8}{3} = -24$$ 1M

9. Verify Green's theorem for $\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy$

where C is the square with vertices

O(0,0) P(2,0) Q(2,2) R(0,2)

The region A is bounded by the given square.



$y =$ constant along OA and QR $dy=0$

$dy = 0$ $x =$ constant along OR and PQ $dx = 0$

2M

Taking $M = x^2 - xy^3$, $N = y^2 - 2xy$

$$\int_C M\,dx + N\,dy = \int_{OP} M\,dx + \int_{PQ} N\,dy + \int_{QR} M\,dx + \int_{RO} N\,dy$$

$$= \int_0^2 x^2\,dx + \int_0^2 (y^2 - 4y)\,dy + \int_2^0 (x^2 - 8x)\,dx + \int_2^0 y^2\,dy$$

$$= \left(\frac{x^3}{3}\right)_0^2 + \left(\frac{y^3}{3} - 4\frac{y^2}{2}\right)_0^2 + \left(\frac{x^3}{3} - 8\frac{x^2}{2}\right)_2^0 + \left(\frac{y^3}{3}\right)_2^0$$

$$= \frac{8}{3} + \left(\frac{8}{3} - 4 \times \frac{4}{2}\right) + \left(-\frac{8}{3} + 8 \times \frac{4}{2}\right) - \frac{8}{3} = 8$$

3M

$$\iint_A \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_A (-2y + 3xy^2)\,dx\,dy$$

$$\int_{x=0}^{2} \left(-y^2 + xy^3\right)_0^2 dx$$

$$= \int_{x=0}^{2} \left(-4 + 8x\right) dx$$

$$= -8 + 8(2) = 8$$

2M

Green's Theorem is verified.