CMR
INSTITUTE OF
TECHNOLOGY

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CMR

Improvement Test

Sub:	**ENGINEERING MATHEMATICS III**							Code:	15MAT31
Date:	18 / 11 / 2016	Duration:	90 mins	Max Marks:	50	Sem:	III	Branch:	**CSE:A,B,C**

Answer**SIX** questions,choosing either **Q1**or **Q2** and any **FIVE** questions from **Q3-Q9**

Marks | OBE: CO | RBT

1.(a) Derive Euler's equation in the standard form viz., $\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0$. [10] CO6 L4

(b) Find the curve on which the functional $\int_0^1 (y^2+x^2y')\,dx$ with $y(0)=0$ and $y(1)=1$can be extremised. [05] CO6 L3

OR

2.(a) A heavy cable hangs freely under gravity between two fixed points. Show that the shape of the cable is a catenary. [08] CO6 L4

(b) Find the geodesics on a surface, given that the arc length on the surface is $s=\int_{x_1}^{x_2}\sqrt{x(1+(y')^2}dx.$ [07] CO6 L3

3. Find the Z-transform of $\cos n\theta$ and $\sin n\theta$. Hence evaluate the Z-transform of $\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)$. [07] CO2 L3

4. Obtain the inverse Z-transform of $\frac{4z^2-2z}{(z-1)(z-2)^2}$. [07] CO2 L3

5. Using Z transform, solve $y_{n+2}-4y_n=0$, given $y_0=0$ and$y_1=2$. [07] CO2 L3

6. Find a second degree parabola $y=ax^2+bx+c$ that best fits the given data, by the method of least squares: [07] CO4 L3

x	1	2	3	4	5
y	10	12	13	16	19

7. If θ is the acute angle between the lines of regression, show that $\tan\theta=\frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}\left(\frac{1-r^2}{r}\right)$. Explain the significance of $\tan\theta$ when $r=0$ & $r=\pm1$. [07] CO3 L4

8. Find a real root of the equation $x\,sin\,x+\cos x=0$ near $x=\pi$, correct to four decimal places, using Newton-Raphson method. Carry out three iterations. [07] CO4 L3

9. Use Weddle's rule to evaluate $\int_0^{\frac{\pi}{2}}\sqrt{\cos\theta}\;d\theta$, dividing $[0,\frac{\pi}{2}]$ into six equal parts. [07] CO4 L3

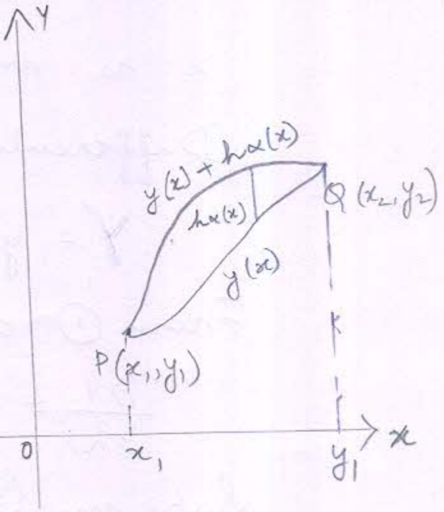
(2)(2)

Course Outcomes		PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12
CO1:	Evaluate the real form of the Fourier series for standard periodic and finite waveforms as half-range series which has its applications in finding the sum of infinite series using Dirichlet's conditions.	1	1	0	0	0	0	0	0	1	0	0	0
CO2:	Apply integral expressions for the forward and inverse Fourier transform to a range of non-periodic waveforms such as rectangular, unit-step, sinusoidal and exponential decay functions and solve second order difference equations using Z transform and inverse Z transform.	2	2	0	0	0	0	0	0	1	0	0	0
CO3:	Estimate the strength of the relationship between the variables using correlation coefficients and express the relationship in the form of an equation using regression analysis.	0	2	0	1	1	0	0	0	1	0	0	0
CO4:	Apply numerical techniques to perform various mathematical tasks such as solving equations, interpolation, integration and curve fitting.	0	2	1	1	1	0	0	0	1	0	0	0
CO5:	Evaluate line and surface integrals using Green's, Stokes' and Gauss divergence theorems which have its application in computing the amount of work done, area and volume.	0	2	0	0	0	0	0	0	1	0	0	0
CO6:	Solve Brachistochrone, shortest distance, minimal surface area and hanging chain problems and find the geodesics of known surfaces using Euler-Lagrange method.	0	0	0	0	0	0	0	0	1	0	0	0

Cognitive level	KEYWORDS
L1	List, define, tell, describe, identify, show, label, collect, examine, tabulate, quote, name, who, when, where, etc.
L2	summarize, describe, interpret, contrast, predict, associate, distinguish, estimate, differentiate, discuss, extend
L3	Apply, demonstrate, calculate, complete, illustrate, show, solve, examine, modify, relate, change, classify, experiment, discover.
L4	Analyze, separate, order, explain, connect, classify, arrange, divide, compare, select, explain, infer.
L5	Assess, decide, rank, grade, test, measure, recommend, convince, select, judge, explain, discriminate, support, conclude, compare, summarize.

PO1 - *Engineering knowledge*; PO2 - *Problem analysis*; PO3 - *Design/development of solutions*; PO4 - *Conduct investigations of complex problems*; PO5 - *Modern tool usage*; PO6 - *The Engineer and society*; PO7- *Environment and sustainability*; PO8 – *Ethics*; PO9 - *Individual and team work*; PO10 - *Communication*; PO11 - *Project management and finance*; PO12 - *Life-long learning*

Improvement Test
Nov, 2016 (CSE A,B,C)



1 a) $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ [Euler's Equation]

A necessary condition for $I = \int_{x_1}^{x_2} f(x,y,y')$ ~~to~~where $y(x_1) = y_1$ and $y(x_2) = y_2$ to be extremum is that ~~Eulers~~ equation is satisfied —— (1M)

Proof: Let I be an extremum along the curve $y = y(x)$ passing through $P(x_1,y_1)$ and $Q(x_2,y_2)$. —— ①

Let $Y = y(x) + h\alpha(x)$ be a neighbouring curve, with $\alpha(x_1) = 0 = \alpha(x_2)$ —— ② so that $Y(x)$ passes through P and Q. —— (2M)

When $h = 0$, $Y(x) = y(x)$, the extremal

Consider $I = \int_{x_1}^{x_2} f(x, Y, Y')\,dx$

$$= \int_{x_1}^{x_2} f(x, y(x)+h\alpha(x), y'(x)+h\alpha'(x))\,dx \quad \text{—— (1M)}$$

I is a function of h. A necessary condition for I to be extremum is that $\frac{dI}{dh} = 0$ —— (1M)

$$\frac{dI}{dh} = \frac{d}{dh}\int_{x_1}^{x_2} f(x,Y,Y')\,dx$$

$$= \int_{x_1}^{x_2} \frac{\partial}{\partial h} f(x,Y,Y')\,dx$$ (by Leibnitz rule for differentiation under the integral sign)

$$= \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial x}\frac{\partial x}{\partial h} + \frac{\partial f}{\partial Y}\frac{\partial Y}{\partial h} + \frac{\partial f}{\partial Y'}\frac{\partial Y'}{\partial h}\right] dx \quad \text{—— ③}$$

(applying chain rule for p.ds)

b) $\int_0^1 (y^2 + x^2 y') \, dx$ with $y(0) = 0, \; y(1) = 1$

$f = y^2 + x^2 y'$

Euler's formula : $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ —— ①

—— (1M)

$\frac{\partial f}{\partial y} = 2y$.

$\frac{\partial f}{\partial y'} = x^2$

$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 2x$.

—— (2M)

Sub in ①,

~~y~~ $2y - 2x = 0$

$y - x = 0$

$y = x$.

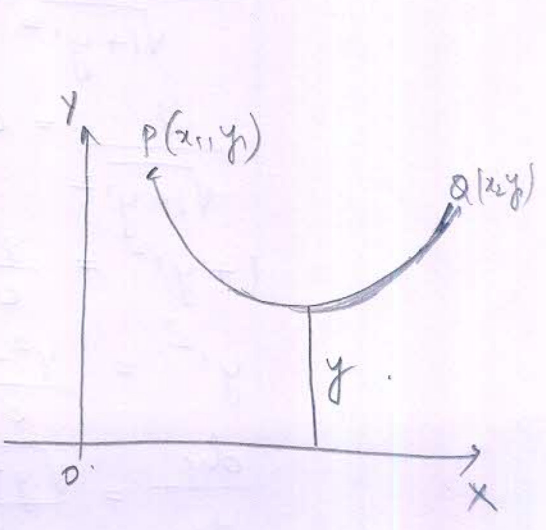
—— (2M)

∴ the given functional is extremised on the straight line $\underline{\underline{y = x}}$.

2 a) Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the fixed points.

Let ds = elementary arc length of the cable

ρ = linear density



Then,

mass $m = \rho ds$.

$$PE = mgh$$

$$= \int_{x_1}^{x_2} \rho \, ds \, g y$$

$$= \rho g \int_{x_1}^{x_2} y ds$$

$$= \rho g \int_{x_1}^{x_2} y \sqrt{1+y'^2} \, dx$$ —— (3M)

To find the curve that minimises P.E

Let $I = \int_{x_1}^{x_2} y\sqrt{1+y'^2} \, dx$.

$f = y\sqrt{1+y'^2}$ which is independent of x

Euler's equation is

$$f - y' \frac{\partial f}{\partial y'} = c$$ —— (1M)

$$y\sqrt{1+y'^2} - y' y \frac{\cancel{2} y'}{\cancel{2}\sqrt{1+y'^2}} = c$$

$$\frac{y(1+y'^2) - yy'^2}{\sqrt{1+y'^2}} = c$$

$$\frac{y}{\sqrt{1+y'^2}} = c$$

$$1 + y'^2 = \frac{y^2}{c^2}$$

$$y'^2 = \frac{y^2 - c^2}{c^2}$$

$$\frac{dy}{dx} = \frac{\sqrt{y^2-c^2}}{c}$$

$$\frac{c}{\sqrt{y^2-c^2}}\,dy = dx$$ —— (2M)

Integrating, $\int \frac{c}{\sqrt{y^2-c^2}}\,dy = \int dx$

$$\Rightarrow c\cosh^{-1}\left(\frac{y}{c}\right) = x+b$$ —— (2M)

$y = c\underline{\underline{\cosh\left(\frac{x+b}{c}\right)}}$ which is a catenary

b) $$s = \int_{x_1}^{x_2} \sqrt{x(1+y'^2)}\,dx$$

$f = \sqrt{x(1+y'^2)}$ which is independent of y.

Eulers Equation $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

reduces to $\frac{\partial f}{\partial y'} = c$. —— (2M)

$$\frac{\partial f}{\partial y'} = \frac{1}{\cancel{2}\sqrt{x(1+y'^2)}} \cdot \cancel{2}xy' = c$$

$$\Rightarrow \frac{\sqrt{x}\,y'}{\sqrt{1+y'^2}} = c$$

$$xy'^2 = c^2(1+y'^2)$$

$$y'^2(x-c^2) = c^2$$

$$y' = \frac{c}{\sqrt{x-c^2}}$$

$$\frac{dy}{dx} = \frac{c}{\sqrt{x-c^2}}$$

$$dy = \frac{c}{\sqrt{x-c^2}}\,dx.$$ —— (3M)

$$\int dy = c \int \frac{dx}{\sqrt{x-c^2}}$$

$$y = c \cdot 2\sqrt{x-c^2} + b.$$

$(y-b)^2 = 4c^2(x-c^2)$ is the required geodesic which is a parabola. —— (2 M)

3. We know that $e^{in\theta} = \cos n\theta + i \sin n\theta$

Since $Z_T(k^n) = \frac{z}{z-k}$, we have

$$Z_T(e^{in\theta}) = Z_T((e^{i\theta})^n)$$

$$= \frac{z}{z-e^{i\theta}}$$

$$= \frac{z(z-e^{-i\theta})}{(z-e^{i\theta})(z-e^{-i\theta})}$$

$$= \frac{z(z-\cos\theta + i\sin\theta)}{z^2 - z(e^{i\theta}+e^{-i\theta})+1}$$

$$= \frac{z(z-\cos\theta) + iz\sin\theta}{z^2-2z\cos\theta+1} \quad \text{—— (3 M)}$$

or $Z_T(\cos n\theta + i\sin n\theta) = \frac{z(z-\cos\theta)+iz\sin\theta}{z^2-2z\cos\theta+1}$

or $Z_T(\cos n\theta) + i Z_T(\sin n\theta) = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1} + i\, \frac{z\sin\theta}{z^2-2z\cos\theta+1}$

$\therefore Z_T(\cos n\theta) = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$ —(i) —— (1 M)

and $Z_T(\sin n\theta) = \frac{z\sin\theta}{z^2-2z\cos\theta+1}$ —(ii) (Equating R.P and I.P) —— (1 M)

$$\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right) = \cos\frac{n\pi}{2}\cos\frac{\pi}{4} - \sin\frac{n\pi}{2}\sin\frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}}\left(\cos\frac{n\pi}{2} - \sin\frac{n\pi}{2}\right) \quad — ①$$

Putting $\theta = \frac{\pi}{2}$ in (i) and (ii),

$$Z_T\left[\cos\left(n\frac{\pi}{2}\right)\right] = \frac{z^2}{z^2+1} \text{ and } Z_T\left(\sin\frac{n\pi}{2}\right) = \frac{z}{z^2+1}$$

Sub in ①, $\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\ \frac{z^2-z}{z^2+1} = \underline{\underline{\frac{z(z-1)}{\sqrt{2}(z^2+1)}}}$ — (2M)

4) $Z_T^{-1}\left[\frac{4z^2-2z}{(z-1)(z-2)^2}\right]$

We have $Z_T^{-1}\left[\frac{z}{z-1}\right] = 1$, $Z_T^{-1}\left[\frac{z}{z-2}\right] = 2^n$, $Z_T^{-1}\left[\frac{2z}{(z-2)^2}\right] = 2^n . n$

$$\bar{u}(z) = \frac{4z^2-2z}{(z-1)(z-2)^2} = A\left[\frac{z}{z-1}\right] + B\left[\frac{z}{z-2}\right] + C\left[\frac{2z}{(z-2)^2}\right] \quad — ①$$

$$= \frac{Az(z-2)^2 + Bz(z-1)(z-2) + 2Cz(z-1)}{(z-1)(z-2)^2} \quad — (3M)$$

$$4z - 2 = A(z-2)^2 + B(z-1)(z-2) + 2C(z-1)$$

Put $z = 1$ $\quad 2 = A \quad\quad A = 2$

Put $z = 2$ $\quad 6 = 2C \quad\quad C = 3$

Equate coeff of z^2 on both sides

$0 = A + B \quad\quad B = -2.$

Sub for A, B, C in ①,

$$\bar{u}(z) = 2\ \frac{z}{z-1} + -2\ \frac{z}{z-2} + 3 \cdot \frac{2z}{(z-2)^2} \quad — (2M)$$

Taking inverse,

$$Z_T^{-1}\left[\cancel{z}\,\bar{u}(z)\right] = 2 Z_T^{-1}\left[\frac{z}{z-1}\right] - 2 Z_T^{-1}\left[\frac{z}{z-2}\right] + 3 Z_T^{-1}\left[\frac{2z}{(z-2)^2}\right]$$

$$= 2.1 - 2.2^n + 3n.2^n$$

$$u_n = 2 - 2^{n+1} + 3n.2^n \,//$$ — (2M)

5. $y_{n+2} - 4y_n = 0, \quad y_0 = 0, \quad y_1 = 2$.

Taking Z transform on both sides,

$$Z_T(y_{n+2}) - 4Z_T(y_n) = Z_T(0)$$

$$z^2\left[\bar{y}(z) - y_0 - \frac{y_1}{z}\right] - 4\bar{y}(z) = 0$$

$$z^2\left[\bar{y}(z) - \frac{2}{z}\right] - 4\bar{y}(z) = 0$$

$$\bar{y}(z)\left[z^2 - 4\right] = 2z$$

$$\bar{y}(z) = \frac{2z}{z^2-4} = \frac{2z}{(z-2)(z+2)} \quad \text{———} \quad ②M$$

$$\frac{\bar{y}(z)}{z} = \frac{2}{(z-2)(z+2)} = \frac{A}{z-2} + \frac{B}{z+2}$$

$$2 = A(z+2) + B(z-2)$$

Put $z = 2$. $\quad 2 = 4A \quad A = 1/2$

Put $z = {}^{-}2 \quad 2 = {}^{-}4B \quad B = {}^{-}1/2$. ——— ③M

$$\therefore \bar{y}(z) = \frac{1}{2}\frac{z}{z-2} - \frac{1}{2}\frac{z}{z+2}.$$

$$Z_T^{-1}\left[(\bar{y}(z)\right] = \frac{1}{2}\left(Z_T^{-1}\left[\frac{z}{z-2}\right]\right) - \frac{1}{2}\left(Z_T^{-1}\left[\frac{z}{z+2}\right]\right)$$

$$y_n = \frac{1}{2}\cdot 2^n - \frac{1}{2}(-2)^n =$$

$$= \frac{2^n}{2} + \frac{(-2)^n}{-2} = \underline{\underline{2^{n-1} + (-2)^{n-1}}} \quad \text{———} \quad ②M$$

5. $y = ax^2 + bx + c.$

The normal equations are:

$\Sigma y = a \Sigma x^2 + b \Sigma x + nc \quad , \quad n = 5$

$\Sigma xy = a \Sigma x^3 + b \Sigma x^2 + c \Sigma x$

$\Sigma x^2 y = a \Sigma x^4 + b \Sigma x^3 + c \Sigma x^2$ —— 2 M

x	y	xy	x^2y	x^2	x^3	x^4
1	10	10	10	1	1	1
2	12	24	48	4	8	16
3	13	39	117	9	27	81
4	16	64	256	16	64	256
5	19	95	475	25	125	625
15	70	232	906	55	225	979

—— 2 M

$\therefore$ We have $\quad 55a + 15b + 5c = 70$

$225a + 55b + 15c = 232$

$979a + 225b + 55c = 906$

Solving, $a = 0.2857 \approx 0.29$

$b = 0.4857 \approx 0.49$

$c = 9.4.$ —— 2 M

$\therefore$ the required parabola is

$y = 0.29x^2 + 0.49x + 9.4$ —— 1 M

7. If θ is acute, the angle b/w the lines $y = m_1 x + c_1$ and $y = m_2 x + c_2$ is given by

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}.$$

The lines of regression are:

$$y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x}) \quad \text{——} (1)$$

and $$x - \bar{x} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y}). \quad \text{——} (2)$$

$$(2) \rightarrow \quad y - \bar{y} = \frac{\sigma_y}{r\sigma_x}(x - \bar{x})$$

The slopes of the 2 lines are

$$m_1 = \frac{r\sigma_y}{\sigma_x} \quad \text{and} \quad m_2 = \frac{\sigma_y}{r\sigma_x}$$

$$\therefore \tan\theta = \frac{\frac{\sigma_y}{r\sigma_x} - \frac{r\sigma_y}{\sigma_x}}{1 + \frac{\cancel{r}\sigma_y}{\sigma_x} \cdot \frac{\sigma_y}{\cancel{r}\sigma_x}}$$

$$= \frac{\frac{\sigma_y}{\sigma_x}\left(\frac{1}{r} - r\right)}{1 + \frac{\sigma_y^2}{\sigma_x^2}}$$

$$= \underline{\underline{\frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}\left(\frac{1 - r^2}{r}\right)}} \quad \text{——} ⑤ \text{M}$$

If $r = \pm 1$, $\tan\theta = 0$. $\therefore \theta = 0$, which implies that the 2 regression lines coincide. $\therefore$ there is perfect correlation b/w the variables.

If $r = 0$, $\tan\theta = \infty$ or $\theta = \pi/2$. the regression lines are perpendicular and the variables are uncorrelated. —— (2 M)

8. $x \sin x + \cos x = 0$ near $x = \pi$

$f(x) = x \sin x + \cos x$

$f'(x) = x \cos x + \sin x - \sin x = x \cos x.$ — (1M)

$x_0 = \pi$

I) $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = \pi - \dfrac{f(\pi)}{f'(\pi)} = \pi - \dfrac{\pi \sin \pi + \cos \pi}{\pi \cos \pi}$

$= \pi - \dfrac{1}{\pi} = 2.82328$ — (2M)

II) $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = 2.82328 - \dfrac{f(2.82328)}{f'(2.82328)}$

$= 2.7986$ — (2M)

III) $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)} = 2.7986 - \dfrac{f(2.7986)}{f'(2.7986)}$

$= 2.79838$

$\approx \underline{2.7984}$ — (2M)

9. $\int_0^{\pi/2} \sqrt{\cos \theta}\, d\theta$, $n = 6$

$h = \dfrac{\pi/2 - 0}{6} = \pi/12$ — (1M)

θ	0	$\pi/12$	$\pi/6$	$\pi/4$	$\pi/3$	$5\pi/12$	$\pi/2$
$\sqrt{\cos \theta}$	1	0.9828	0.9306	0.8409	0.7071	0.5087	0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

— (3M)

Weddle's rule : $\int_{x_0}^{x_0+6h} f(x)\,dx = \dfrac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right]$ — (1M)

$\therefore \int_0^{\pi/2} \sqrt{\cos \theta}\, d\theta = \dfrac{3.\pi}{120}\left[1 + (5 \times 0.9828) + 0.9306 + (6 \times 0.8409) + 0.7071 + (5 \times 0.5087) + 0\right]$

$= \underline{1.1891}$ — (2M)