1) Prove that the sampling of DTFT of an N-point sequence $x[n]$ results in N-point DFT.

Solution:

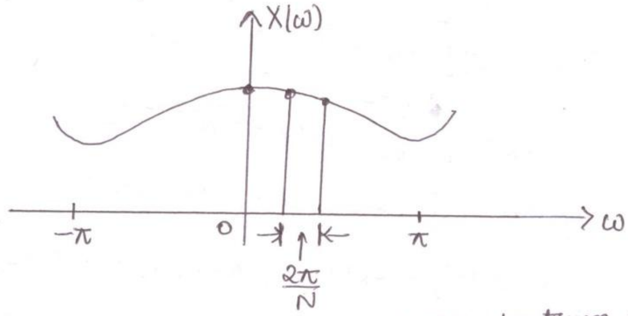

Fig: Frequency sampling of DTFT spectrum at $\omega = \frac{2\pi}{N}k$, where $k=0,1,\ldots N$

DTFT: $X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$, $|\omega| < \pi$ — ①

we sample $X(\omega)$ at intervals of ω_K. $\omega_K = \frac{2\pi}{N}k$, $k=0,1\ldots(N-1)$

$$X\left(\frac{2\pi}{N}k\right) = X(\omega)\Big|_{\omega=\frac{2\pi k}{N}}, \quad k=0,1,\ldots(N-1) \rightarrow ②$$

Let $x(n)$ be a DT sequence of length 'L'.

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\frac{2\pi}{N}kn}, \quad k=0,1,\ldots(N-1) \rightarrow ③$$

RHS of ③ can be expanded as follows:

$$X\left(\frac{2\pi}{N}k\right) = \ldots + \sum_{n=-N}^{-1} x(n)\, e^{-j\frac{2\pi}{N}kn} + \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn} + \sum_{n=N}^{2N-1} x(n)\, e^{-j\frac{2\pi}{N}kn} + \ldots$$

$$X\left(\frac{2\pi}{N}k\right) = \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}, \quad k=0,1,\ldots(N-1)$$

$$X\left(\frac{2\pi}{N}k\right) = \sum_{l=-\infty}^{\infty} \sum_{n=0}^{N-1} x(n-lN)\, e^{-j\frac{2\pi}{N}k(n-lN)}$$

∵ of periodicity of complex discrete exponential

$$X\left(\frac{2\pi}{N}k\right) = \sum_{l=-\infty}^{\infty}\left[\sum_{n=0}^{N-1} x(n-lN)\right] e^{-j\frac{2\pi}{N}kn}$$

$x_p(n)$: has fundamental period of N

where $x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN) \longrightarrow ③$

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1}\left[\sum_{l=-\infty}^{\infty} x(n-lN)\right] e^{-j\frac{2\pi}{N}kn} \longrightarrow ④$$

Eq^n ③ is periodic extension of $x(n)$ of length L. we have DTFS representation of periodic signal $x_p(n)$.

$$x_p(n) = \sum_{n=0}^{N-1} C_K \, e^{j\frac{2\pi}{N}kn} \longrightarrow ⑤$$

where, $$C_K = \frac{1}{N} \sum_{K=0}^{N-1} x_p(n) \, e^{-j\frac{2\pi}{N}kn} \longrightarrow ⑥$$

On comparing ④ & ⑥, we can write

$$\boxed{X\left(\frac{2\pi k}{N}\right) = C_k N} \quad , K = 0, 1, \ldots (N-1)$$

The samples of DTFT are equal to DTFS co-eff of $x(n)$ multiplied by fundamental period of $x_p(n)$. The result of C_K is substituted in ⑤

$$\boxed{x_p(n) = \frac{1}{N} \sum_{K=0}^{N-1} X\left(\frac{2\pi}{N}k\right) e^{j\frac{2\pi}{N}kn}}$$

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2. ⓐ Find the 6-point DFT of the sequence $x[n] = [1, 1, 2, 2, 3, 3]$. Plot the magnitude and phase spectra.

<u>Solution:</u> given : $x[n] = [1, 1, 2, 2, 3, 3]$

here $N = 6$

$\therefore$ 6 pt. DFT is given by $X(K) = \sum_{n=0}^{5} x(n) \, w_6^{kn}$; $k = 0, 1, \ldots 5$

Expanding over n, we get

$$X(K) = 1 + w_6^k + 2w_6^{2k} + 2w_6^{3k} + 3w_6^{4k} + 3w_6^{5k} \longrightarrow ①$$

$K = 0, 1, \ldots 5$

$w_6^0 = 1$

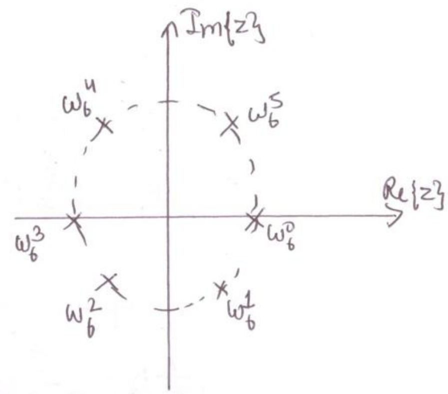
$w_6^1 = 0.5 - j0.866$

$w_6^2 = -0.5 - j0.866$

$w_6^3 = -1$

$w_6^4 = -0.5 + j0.866$

$w_6^5 = 0.5 + j0.866$



$k=0 \Rightarrow X(0) = 1 + \omega_6^0 + 2\omega_6^0 + 2\omega_6^0 + 3\omega_6^0 + 3\omega_6^0 = 12$

$K=1 \Rightarrow X(1) = 1 + \omega_6^1 + 2\omega_6^2 + 2\omega_6^3 + 3\omega_6^4 + 3\omega_6^5 = -1.5 + j2.598$

$K=2 \Rightarrow X(2) = 1 + \omega_6^2 + 2\omega_6^4 + 2\omega_6^6 + 3\omega_6^8 + 3\omega_6^{10} = -1.5 + j0.866$

$K=3 \Rightarrow X(3) = 1 + \omega_6^3 + 2\omega_6^6 + 2\omega_6^9 + 3\omega_6^{12} + 3\omega_6^{15} = 0$

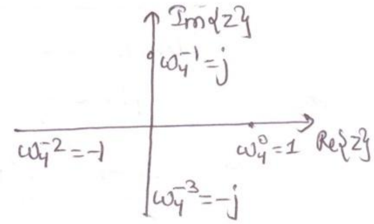
$K=4 \Rightarrow X(4) = 1 + \omega_6^4 + 2\omega_6^8 + 2\omega_6^{12} + 3\omega_6^{16} + 3\omega_6^{20} = -1.5 - j0.866$

$K=5 \Rightarrow X(5) = 1 + \omega_6^5 + 2\omega_6^{10} + 2\omega_6^{15} + 3\omega_6^{20} + 3\omega_6^{25} = -1.5 - j2.598$

$X(K) = \{12, -1.5 + j2.598, -1.5 + j0.866, 0, -1.5 - j0.866, -1.5 - j2.598\}$

(b) Compute the inverse DFT of the sequence $X[K] = [2, 1+j, 0, 1-j]$

Solution: $x(n) = \frac{1}{4}\sum_{K=0}^{3} X(K)\, \omega_4^{-kn}$ $\quad n = 0, 1, 2, 3$



$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & -j \end{bmatrix}\begin{bmatrix} 2 \\ 1+j \\ 0 \\ 1-j \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore x(n) = (1, 0, 0, 1)$

3.(a) Compute the 8-point circular convolution of $x_1[n] = \left(\frac{1}{4}\right)^n, \ 0 \le n \le 7$ and $x_2[n] = \cos\left(\frac{4\pi n}{8}\right), \ 0 \le n \le 7$

Solution: $x_1[n] = \left(\frac{1}{4}\right)^n, \ 0 \le n \le 7$

$x_1(n) = \left\{1, \frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \frac{1}{256}, \frac{1}{1024}, \frac{1}{4096}, \frac{1}{16384}\right\}$

& $x_2[n] = \cos\left(\frac{4\pi n}{8}\right), \ 0 \le n \le 7$

$x_2(n) = \{1, 0, -1, 0, 1, 0, -1, 0\}$

$x_3(n) = x_1(n) \circledast_8 x_2(n)$

$x_3(n) = \sum_{m=0}^{7} x_1(m)\, x_2((n-m))_8, \quad n = 0, 1, \ldots 7$

$X_3(n) = \{0.941, 0.235, -0.941, -0.235, 0.941, 0.235, -0.941, -0.235\}$

$$\begin{bmatrix} X_3(0) \\ X_3(1) \\ X_3(2) \\ X_3(4) \\ X_3(5) \\ X_3(6) \\ X_3(7) \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1/4 \\ 1/16 \\ 1/64 \\ 1/256 \\ 1/1024 \\ 1/4096 \\ 1/16384 \end{bmatrix}$$

(b) Obtain the relationship between DFT & Z-Transform of an N-pt. sequence.

Solution:

$$X(z) = \sum_{n=0}^{N-1} x(n) z^{-n} \longrightarrow ①$$

we have,

$$x(n) = \text{IDFT}\left[X(K)\right]$$

$$x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{j\frac{2\pi}{N}kn} \longrightarrow ②$$

Use ② in ①

$$X(z) = \sum_{n=0}^{N-1} \left\{ \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn} \right\} z^{-n}$$

$$X(z) = \frac{1}{N} \sum_{k=0}^{N-1} X(K) \sum_{n=0}^{N-1} \left(e^{j\frac{2\pi}{N}k}\right)^n \left(z^{-1}\right)^n$$

$$X(z) = \frac{1}{N} \sum_{k=0}^{N-1} X(K) \left[\frac{1 - z^{-N}}{1 - e^{j\frac{2\pi}{N}k} z^{-1}}\right]$$

$$\therefore \boxed{X(z) = \frac{1 - z^{-N}}{N} \sum_{k=0}^{N-1} \frac{X(K)}{1 - e^{j\frac{2\pi k}{N}} z^{-1}}}$$

4.(a) Prove that $DFT\{DFT\{DFT\{DFT\{x(n)\}\}\}\} = N^2 x(n)$

Solution: $X(K) = DFT[x(n)]$ or $X(K) = F[x(n)]$

$x(n) = IDFT[X(K)]$ or $x(n) = F^{-1}[X(K)]$

where $F^{-1} = \frac{1}{N} F$

$DFT\{DFT\{DFT\{DFT\{x(n)\}\}\}\} = F\{F\{F\{F\{x(n)\}\}\}\}$

$F\{NF^{-1}\{F\{NF^{-1}\{x(n)\}\}\}\} = N^2 x(n)$

(b) Prove that, if $x(n)$ is real then $X(K) = X^*(N-K)$

Solution: $X(K) = DFT[x(n)] = \sum_{n=0}^{N-1} x(n)\, \omega_N^{kn}$, $0 \leq K \leq (N-1)$

Replace K by $N-k$

$X(N-K) = \sum_{n=0}^{N-1} x(n)\, \omega_N^{(N-k)n}$

$X(N-K) = \sum_{n=0}^{N-1} x(n)\, \cancel{\omega_N^{Nn}}^{\,1}\, \omega_N^{-kn}$

but $\omega_N^{-kn} = e^{j\frac{2\pi}{N}kn} = \left(e^{-j\frac{2\pi}{N}kn}\right)^* = \left(\omega_N^{kn}\right)^*$

$X(N-k) = \sum_{n=0}^{N-1} x(n)\, \omega_N^{-kn}$

$X(N-k) = \left[\sum_{n=0}^{N-1} x^*(n)\, (\omega_N^{kn})^*\right] = \left[\sum_{n=0}^{N-1} x(n)\, \omega_N^{kn}\right]^*$

$X(N-K) = X^*(K)$

or $X^*(N-K) = X(K)$ (Hence proved)

(c) Find the IDFT of $X^*(-K)$

Solution: Let $y(n) = x^*(n)$

$Y(K) = \sum_{n=0}^{N-1} x^*(n)\, \omega_N^{kn}$

$Y(K) = \sum_{n=0}^{N-1} x^*(n)\, (\omega_N^{-kn})^* = \left[\sum_{n=0}^{N-1} x(n)\, \omega_N^{-kn}\right]^*$

$Y(K) = [X(-K)]^*$

$Y(K) = X^*(-K)$

Take IDFT on both sides

$$y(n) = x^*(n)$$

5-(a) Determine the output of an LTI system with the impulse response $h(n) = (1, 2)$ when a long data sequence $x(n) = \{1, 2, -1, 2, 3, -2, -3, -1, 1, 1, 2, -1\}$ is applied. Use 5-point circular convolution & overlap-save method.

Solution: Here $h(n) = (1, 2)$ i.e. $M = 2$

$N = 2^M = 2^2 = 4$. Hence, $L = N - M + 1 = 5 - 2 + 1 = 4$

$$\boxed{L = 4}$$

Using five point circular convolution, we have to append 3 zeros to $h(n)$.

i.e. $h(n) = \{1, 2, 0, 0, 0\}$

The sequences formed are as follows:

$x_1(n) = (0, 1, 2, -1, 2)$

$x_2(n) = (2, 3, -2, -3, -1)$

$x_3(n) = (-1, 1, 1, 2, -1)$

$x_4(n) = (-1, 0, 0, 0, 0)$

Now,

$y_1(n) = x_1(n) \; Ⓝ \; h(n) = \{0, 1, 2, -1, 2\} \; ⑤ \; \{1, 2, 0, 0, 0\} = \{4, 1, 4, 3, 0\}$

$$\begin{bmatrix} y_1(0) \\ y_1(1) \\ y_1(2) \\ y_1(3) \\ y_1(4) \end{bmatrix} = \begin{bmatrix} 0 & 2 & -1 & 2 & 1 \\ 1 & 0 & 2 & -1 & 2 \\ 2 & 1 & 0 & 2 & -1 \\ -1 & 2 & 1 & 0 & 2 \\ 2 & -1 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 4 \\ 3 \\ 0 \end{bmatrix}$$

$y_2(n) = x_2(n) \; Ⓝ \; h(n) = \{2, 3, -2, -3, -1\} \; ⑤ \; \{1, 2, 0, 0, 0\} = \{0, 7, 4, -7, -7\}$

$$\begin{bmatrix} y_2(0) \\ y_2(1) \\ y_2(2) \\ y_2(3) \\ y_2(4) \end{bmatrix} = \begin{bmatrix} 2 & -1 & -3 & -2 & 3 \\ 3 & 2 & -1 & -3 & -2 \\ -2 & 3 & 2 & -1 & -3 \\ -3 & -2 & 3 & 2 & -1 \\ -1 & -3 & -2 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 4 \\ -7 \\ -7 \end{bmatrix}$$

$y_3(n) = x_3(n) \circledN h(n) = \{-1, 1, 1, 2, -1\} ⑤ \{1, 2, 0, 0, 0\} = \{-3, -1, 3, 4, 3\}$

$$\begin{bmatrix} y_3(0) \\ y_3(1) \\ y_3(2) \\ y_3(3) \\ y_3(4) \end{bmatrix} = \begin{bmatrix} -1 & -1 & 2 & 1 & 1 \\ 1 & -1 & -1 & 2 & 1 \\ 1 & 1 & -1 & -1 & 2 \\ 2 & 1 & 1 & -1 & -1 \\ -1 & 2 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \\ 3 \\ 4 \\ 3 \end{bmatrix}$$

$y_4(n) = x_4(n) \circledN h(n) = \{-1, 0, 0, 0, 0\} ⑤ \{1, 2, 0, 0, 0\} = \{-1, -2, 0, 0, 0\}$

$$\begin{bmatrix} y_4(0) \\ y_4(1) \\ y_4(2) \\ y_4(3) \\ y_4(4) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

First M-1 i.e. 1 sample of each of $y_1(n)$, $y_2(n)$, $y_3(n)$ and $y_4(n)$ are discarded and remaining sample are fitted one after another i.e.

$y(n) = \{1, 4, 3, 0, 7, 4, -7, -7, -1, 3, 4, 3, -2, 0, 0, 0\}$

(b) Find the circular convolution of $x[n] = [1, 2, 3, 1]$ and $h[n] = [4, 3, 2, 2]$

<u>Solution:</u> $x[n] = [1, 2, 3, 1]$ & $h[n] = [4, 3, 2, 2]$

$y(n) = x[n] ④ h[n]$

$$y(n) = \sum_{m=0}^{3} x(m) \, h((n-m))_4 \qquad n = 0, 1, 2, 3$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} x(0) & x(3) & x(2) & x(1) \\ x(1) & x(0) & x(3) & x(2) \\ x(2) & x(1) & x(0) & x(3) \\ x(3) & x(2) & x(1) & x(0) \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \\ h(2) \\ h(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 & 2 \\ 2 & 1 & 1 & 3 \\ 3 & 2 & 1 & 1 \\ 1 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 17 \\ 19 \\ 22 \\ 19 \end{bmatrix}$$

6(a) Prove the circular frequency-shift property of DFT.

Solution: If $x((n))_N \xleftrightarrow[DFT]{N\,pt.} X((k))_N$ then

$$W_N^{-pn}\, x((n))_N \xleftrightarrow{N\,pt.\,DFT} X((k-p))_N$$

Proof: $X(k) = DFT[x(n)] = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$ — (1)

Let $s(n) = W_N^{-pn}\, x(n)$

$$S(k) = DFT[s(n)]$$

$$S(k) = \sum_{n=0}^{N-1} s(n)\, W_N^{pn}$$

$$S(k) = \sum_{n=0}^{N-1} W_N^{-pn}\, x(n)\, W_N^{kn}$$

$$S(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{(k-p)n} \rightarrow (2)$$

Comparing (1) & (2)

$$S(k) = X(k)\Big|_{k \to k-p}$$

$$\boxed{S(k) = X((k-p))_N}$$

(b) If DFT of $x[n]$ is $X[k]$, show that

i) $DFT[x(-n)_N] = X[(-k)_N]$

ii) $DFT[x^*(n)] = X^*[N-k]$

Solution i) $DFT[x(-n)_N] = X[(-k)_N]$

$X(k) = DFT[x((n))_N] = \sum_{n=0}^{N-1} x[n]\, W_N^{kn}$; $k = 0, 1, 2, \ldots (N-1)$

Let $y(n) = x(N-n)$

$\therefore Y(k) = \sum_{n=0}^{N-1} x(N-n)\, W_N^{kn}$

Let $N-n = m$

then, $m = N$ & $UL \Rightarrow N-(N-1) = m = 1$

$$Y(k) = \sum_{m=N}^{1} x(m)\, W_N^{k(N-m)}$$

Let $m = n$

$$Y(k) = \sum_{n=N}^{1} x(n)\, \cancel{W_N^{kN}}^{\,1} \left(W_N^{-k}\right)^n$$

$$Y(K) = \sum_{n=1}^{N} x(n)\, w_N^{-kn}$$

The range 1 to N is same as 0 to N-1.

$\therefore$ By periodicity,

$$Y(K) = \sum_{n=0}^{N-1} x(n)\, w_N^{-kn} \quad \text{---} (2)$$

Comparing RHS of (1) & (2)

$$Y(K) = X(K)\big|_{K \to -K}$$

$$Y(K) = X((-K))_N = X((-K+N))_N$$

$$\therefore x((-n))_N = x((N-n))_N \xleftrightarrow[\text{DFT}]{N \text{ pt.}} X((-K))_N = X((N-K))_N$$

ii> $\text{DFT}[x^*(n)] = X^*[N-K]$

By definition DFT of $x^*(n)$ will be given as,

$$\text{DFT}\{x^*(n)\} = \sum_{n=0}^{N-1} x^*(n)\, e^{-j2\pi kn/N} \quad \text{---} (1)$$

We know that,

$$e^{+j2\pi Nn/N} = e^{j2\pi n} = \cos(2\pi n) + j\sin(2\pi n) = 1 \text{ always}$$

Hence multiplying eqn (1) by $e^{j2\pi nN/N}$

Hence,

$$\text{DFT}\{x^*(n)\} = \sum_{n=0}^{N-1} x^*(n)\, e^{-j2\pi kn/N} \cdot e^{j2\pi nN/N}$$

$$= \sum_{n=0}^{N-1} x^*(n)\, e^{j2\pi n(N-K)/N}$$

$$= \left[\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi n(N-K)/N}\right]^*$$

$$= [X(N-K)]^* = X^*(N-K)$$

7. (a) Show that the multiplication of Two DFTs results in the circular convolution of their time domain sequences.

Solution: If $x_1(n) \xleftrightarrow[\text{DFT}]{N\text{ pt.}} X_1(K)$ &

$x_2(n) \xleftrightarrow[\text{DFT}]{N\text{ pt.}} X_2(K)$ then

$$x_3(n) = x_1(n) \circledN x_2(n) \xleftrightarrow[\text{DFT}]{N\text{ pt.}} X_3(K) = X_1(K) \cdot X_2(K)$$

where,

$$x_3(n) = x_1(n) \circledN x_2(n) = \sum_{m=0}^{N-1} x_1(m)\, x_2((n-m))_N$$

$$X_1(K) = \text{DFT}[x_1(n)] = \sum_{m=0}^{N-1} x_1(m)\, W_N^{km} \quad \text{— (1)}$$

$$X_2(K) = \text{DFT}[x_2(n)] = \sum_{l=0}^{N-1} x(l)\, W_N^{kl} \quad \text{— (2)}$$

$$X_3(K) = \text{DFT}[x_3(n)]$$

$$x_3(n) = \text{IDFT}[X_3(K)]$$

$$x_3(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_3(K)\, W_N^{-kn} \quad \text{— (3)}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X_1(K) \cdot X_2(K)\, W_N^{-kn}$$

Use (1) & (2) in above eq^n:

$$x_3(n) = \frac{1}{N}\sum_{k=0}^{N-1} \left\{\sum_{m=0}^{N-1} x_1(m)\, W_N^{km}\right\} \left\{\sum_{l=0}^{N-1} x_2(l)\, W_N^{kl}\right\} W_N^{-kn}$$

Rearranging the summation,

$$x_3(n) = \frac{1}{N}\sum_{m=0}^{N-1} x_1(m) \sum_{l=0}^{N-1} x_2(l) \sum_{k=0}^{N-1} W_N^{km}\, W_N^{kl}\, W_N^{-kn}$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} x_1(m) \sum_{l=0}^{N-1} x_2(l) \sum_{k=0}^{N-1} W_N^{-k(n-l-m)} \quad \text{— (3)}$$

let $S = \sum_{k=0}^{N-1} \left\{W_N^{-(n-m-l)}\right\}^k = \begin{cases} \dfrac{\left[1-W_N^{-(n-m-l)}\right]^N}{1-W_N^{-(n-m-l)}} = 0, & W_N^{-(n-m-l)} \neq 1 \\ N, & W_N^{-(n-m-l)} = 1 \\ & n-m-l = 0 \end{cases}$

$$\because W_N^{-(n-m-l)N} = e^{j\frac{2\pi}{N}(n-m-l)N} = e^{j2\pi(n-m-l)} = 1 \quad [\because n-m-l = \text{integer}]$$

$$S = \begin{cases} 0 \;; & n-m-l \neq 0 \\ N \;; & n-m-l = 0 \end{cases}$$

eq^n (3) $\Rightarrow$ $x_3(n) = \frac{1}{\cancel{N}}\sum_{m=0}^{N-1} x_1(m) \sum_{l=0}^{N-1} x_2(l) \cancel{N}$

Substituting $l = n-m$, $\therefore x_3(n) = \sum_{m=0}^{N-1} x_1(m) \cdot x_2(n-m) = x_1(n) \circledN x_2(n)$

(b) Let $x[n]$ be a finite length sequence with $X[K] = [0, 1+j, 1, 1-j]$. Find DFT of $z[n] = \cos\left(\frac{\pi}{2}n\right) x[n]$.

Solution:

$$Z[n] = \cos\left(\frac{\pi n}{2}\right) x[n] = \left(\frac{e^{j\frac{\pi n}{2}} + e^{-j\frac{\pi n}{2}}}{2}\right) x[n]$$

$$= \frac{e^{j\frac{\pi n}{2}} x(n)}{2} + \frac{e^{-j\frac{\pi n}{2}} x(n)}{2}$$

$$\left[\because 1 \xleftrightarrow{DFT} N\delta(K), \quad \frac{1}{2} \xleftrightarrow{DFT} \frac{N}{2}\delta(K)\right]$$

$$= \frac{\omega_4^{-n} x(n)}{2} + \frac{\omega_4^{n} x(n)}{2}$$

$$Z[K] = \frac{1}{2} X(K-1) + \frac{1}{2} X(K+1)$$

$$Z[K] = \frac{1}{2}\left\{[(1-j), 0, (1+j), 1] + [(1+j), 1, (1-j), 0]\right\}$$

$$= \frac{1}{2}\{2, 1, 2, 1\} = \{1, 1/2, 1, 1/2\}$$

8.(a) Let $x[n] = \begin{cases} 1, & n \text{ even} \\ 0, & \text{otherwise} \end{cases}$

where n, $0 \le n \le N-1$. Find N-point DFT of $x[n]$.

Solution:

$$X(K) = \sum_{n=0,2,4}^{N-2} x(n)\, \omega_N^{kn}$$

$$X(K) = \sum_{n=0,2,4}^{N-2} 1 \cdot \omega_N^{kn}$$

$$X(K) = \sum_{n=0}^{N-1} (\omega_N^{kn})$$

$n = 0, 2, 4 \ldots (N-2)$ Let $r = \frac{n}{2} \Rightarrow r = 0, 1, 2 \ldots \left(\frac{N}{2} - 1\right)$

$$\Rightarrow X(K) = \sum_{r=0}^{\frac{N}{2}-1} (\omega_N^{k})^{2n}$$

$$X(K) = \begin{cases} \dfrac{1 - (\omega_N^{2k})^{\frac{N}{2}}}{1 - \omega_N^{4k}} & \omega_N^{2k} \ne 1 \text{ or } K \ne 0 \\[2ex] \left(\frac{N}{2} - 1\right) - 0 + 1 = \frac{N}{2}, & \omega_N^{2k} = 1 \text{ or } 2K = 0 \end{cases}$$

$$\omega_N^{kN} = e^{-j\frac{2\pi k N}{N}} = e^{-j2\pi k} = 1$$

$$X(K) = \begin{cases} 0 & K \ne 0 \text{ or } K = 1, 2, 3, \ldots (N-1) \\ \frac{N}{2}, & K = 0 \end{cases}$$

$$\boxed{X(K) = \frac{N}{2}\delta(K)}$$

(b) Compute the 8-point circular convolution of two sequences, $x(n) = [2, 1, 1, 1, 2, 1, 1, 1]$ and $h(n) = [2, 1, 2, -3, 2, 1, 2, -3]$.

Solution:

$$y(n) = x(n) \circledast_8 h(n)$$

$$= \sum_{m=0}^{7} x(m)\, h((n-m))_8 \quad ; \; n = 0,1,2,3,4,5,6,7$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \\ y(5) \\ y(6) \\ y(7) \end{bmatrix} = \begin{bmatrix} x(0) & x(7) & x(6) & x(5) & x(4) & x(3) & x(2) & x(1) \\ x(1) & x(0) & x(7) & x(6) & x(5) & x(4) & x(3) & x(2) \\ x(2) & x(1) & x(0) & x(7) & x(6) & x(5) & x(4) & x(3) \\ x(3) & x(2) & x(1) & x(0) & x(7) & x(6) & x(5) & x(4) \\ x(4) & x(3) & x(2) & x(1) & x(0) & x(7) & x(6) & x(5) \\ x(5) & x(4) & x(3) & x(2) & x(1) & x(0) & x(7) & x(6) \\ x(6) & x(5) & x(4) & x(3) & x(2) & x(1) & x(0) & x(7) \\ x(7) & x(6) & x(5) & x(4) & x(3) & x(2) & x(1) & x(0) \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \\ h(2) \\ h(3) \\ h(4) \\ h(5) \\ h(6) \\ h(7) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \\ -3 \\ 2 \\ 1 \\ 2 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ 6 \\ 8 \\ -2 \\ 8 \\ 6 \\ 8 \\ -2 \end{bmatrix}$$